2

The framework for modelling

2.1 RELATIONSHIPS BETWEEN MECHANICS AND TECHNOLOGY

The word '*mechanics*' is understood traditionally to mean a part of physics. However, the basic laws of mechanics formulated in the past were often discovered by mathematicians who, during their study of mechanics, originated new branches of mathematics so that differential and integral calculus as well as variational calculus were established.

Here we will not treat mechanics as part of physics or as a source of new mathematical concepts. Our aim is to develop for the reader the ability to model complex mechanical systems. This, however, requires a new way of thinking about mechanics, namely as of one of the fundamental sections of technology. An argument for such a view is the fact that it would be difficult to find a branch of technological science where one did not come across the notions of mechanics, although in a new rendition. For the sake of brevity we shall focus our interest on the development of only one branch of mechanics, namely *solid mechanics*.

In the nineteenth century solid mechanics was represented mainly by the *theory of elasticity*, which in turn was treated as a part of mathematical physics. Parallel to this theory, its technological applications were developed, i.e. *strength of materials*, *theory of plates and shells*, and *structural mechanics*. In the period between the world wars new sections of solid mechanics were developed, such as *theories of plasticity* and *viscoplasticity*. After the Second World War there was a very rapid development of *coupled field theory*. This term means the union of two or even more branches of physics which had previously been treated separately.

A typical example of such a theory is *thermoelasticity*, which was established by combining *heat conduction theory* and elasticity on the basis of *thermodynamics of irreversible processes*. Its main subject of study is the effect of temperature changes on the deformation of bodies and vice versa. Other good examples for coupled field theories are provided by *piezoelectricity* and *magnetoelasticity* (generation of electromagnetic effects in deformable bodes). It is worth mentioning that the impulse for research on coupled fields has come from engineering in connection with the development of aircraft structures (such as wings for supersonic planes), gas and steam turbines, chemical installations and finally nuclear power plant constructions. More and more frequently the elements of constructions are subject to high temperatures and pressures; they work in conditions of radiation, diffusion and strong magnetic fields. Concentrating now on the subject of thermodiffusion in the solid body, since this phenomenon is sometimes not even noticed within mechanics in its classical understanding, there are numerous examples of penetration of gases and liquids into solid bodies. This penetration causes swelling and shrinking of the body, as in the case of hydrogen, which, on penetration into steel, causes significant deformation. Similarly important effects are brought about by heating a body during the diffusion process. It is known that humidity in a porous medium is changed due to changes in the temperature field. In order to accelerate liberation of a gas from a metallic body it is heated. A process such as carbonization of steel takes place in the presence of a changeable temperature field. Thus a new branch of solid mechanics appears, somewhere at the boundary of elasticity theory and *physical chemistry*.

Further examples could be introduced, but we think that the ones quoted are sufficient for proper evaluation of the role of mechanics in present-day technology. In order to further this understanding we shall shed some light in section 2.2 on the most important notions of classical mechanics. We shall then generalize them in section 2.3 in the framework of integrated mechanics, so that they become helpful in the mathematical description of various physical phenomena given in the later chapters. Even if not all of the coupled phenomena get modelled, we shall nevertheless try to show that the methodology elaborated is capable of describing them.

2.2 THE FUNDAMENTAL NOTIONS OF CLASSICAL MECHANICS

2.2.1 The mechanical system

One of the fundamental notions of classical mechanics is the notion of a **mechanical system** (sometimes the notion of **material system** is encountered as well). Any set (finite or infinite) of mass particles, treated as a whole, will be referred to as a mechanical system. The 'whole' should be understood in such a way that motion of every mass particle in it depends upon the motion of the other particles. The definition of the mechanical system given above encompasses, of course, not only rigid bodies, but also deformable bodies and fluids. However, in classical mechanics it is traditionally understood that a mechanical system is either a system of particles or a single rigid body only. Since these notions are known from elementary physics, we shall give only some comments. First it should be pointed out that by 'mechanical system' we usually mean a model of the system, not the system itself. A model of mass particles may be used to describe those motions of real bodies in which the dimensions of the moving bodies can be neglected relative to the distances characterizing these motions. A classical example is provided by the Earth's motion around the sun. Both these celestial bodies are treated as mass particles, although the radius of Earth is approximately 6.4×10^6m, and that of the Sun approximately 7×10^8m. The essential fact, however, is that these dimensions are small compared with the distance between the two bodies, namely 1.5×10^{11}m. On the other hand it is nonsensical to describe the motion of the Earth about its own axis while treating the Earth as a mass particle! In some cases even a single atom cannot be treated as a mass particle, when its spin is considered.

Thus, we cannot always apply the model of mass particles to describe the behaviour of an object under consideration, and we may be obliged to apply a more realistic model.

The subsequent possibility is to consider dimensions of the object, but to neglect all the changes of these dimensions. That is how the model of a rigid body is formed. Even this model may sometimes prove unacceptable, and the possibility of deformations must then be taken into account. This leads to a approximation of real bodies in which we treat these bodies as a certain region filled continuously with matter. This is the so-called continuum postulate, applied one could say, in spite of the discrete nature of the matter. This type of idealization will be referred to as a continuous medium (see also section 1.5). The whole of Volume 2 will be devoted to just such model.

Mechanical systems are classified into unrestrained and constrained. A system is said to be **unrestrained** when all the particles of the system may at any time take any position and have any velocity. In this case the motion of a particle belonging to an unrestrained system is connected with the motion of other particles belonging to it only because the force applied to a given particle depends upon the positions and/or velocities of other particles belonging to this system. Thus, for instance, three celestial bodies, Earth, Moon and Sun, of which we know only that they attract each other according to Newton's law of gravitation, form an unrestrained mechanical system.

When positions or velocities of particles of a mechanical system cannot be arbitrary at any time, then this system is called **constrained**.

2.2.2 Constraints and their classification

In technical problems we usually deal with constrained systems. A system loses certain freedoms due to imposition of constraints. Limitations which are imposed on the motion of a system are called **constraints**. The notion of constraints was introduced in 1795 by **Jean B. J. de Fourier** (1768–1830). It should be emphasized that two things can be understood within this notion. In Newtonian mechanics constraints are *constituted by bodies* which limit the freedom of the motion of mechanical systems; typical examples are: bearings, joints, supports, pulley wheels, etc. On the other hand, in Lagrangian mechanics constraints are the *analytical description* of limitations imposed on the motion of the bodies, without considering the physical nature of these limitations. This latter definition of constraints will be considered here. Both these meanings of constraints can be put together in one notion of **material constraints**, as opposed to the so-called **programme constraints**, which we discuss in section 2.3, since we treat them in the framework of integrated mechanics.

Before we pass on to classification of material constraints, we shall mention the difficulties in distinguishing between **external** and **internal constraints**. Internal ones limit the freedom of motion of particles with regard to other points of the same body, while the external ones set limitations due to the action of some other body. For example: a rigid body can be represented as a mechanical system in which all points are fixed by distance relative to all others. Invariability of distances limits, of course, the possibility of relative translocations and constitutes, therefore, constraints within the system, so we are dealing with a constrained system. This, however, does not limit the freedom of motion of the body as a whole. It is only the imposition the external constraints, e.g. fixation of one or more points of a body by an external agency, that will cause the body to take on a specific kind of motion.

We have mentioned that only analytic constraints will be dealt with here. Such constraints are classifiable according to four criteria, namely, whether or not

(1) they are expressible as equalities,
(2) they are integrable forms,
(3) they are explicitly velocity-dependent, or
(4) they are explicitly time-dependent.

The first criterion defines **bilateral** (in the case of equalities) or **unilateral** (in the case of inequalities) **constraints**, the second **holonomic** or **nonholonomic constraints**, the third one defines **geometric** and **kinematic constraints**, and the fourth one **rheonomic** or **scleronomic constraints**. These names come from the Greek. The word 'holonomic' means 'altogether lawful', the word 'scleronomic' means 'rigid', the word 'rheonomic' means 'flowing'. The second division is sometimes (not properly) identified with the third one. A difference between kinematic (also called differential) and nonholonomic constraints or between geometric (alternatively called finite) will be explained below. When dealing with unilateral constraints rather sophisticated and nonstandard methods have to be applied. This is why we shall not consider them further, but will deal only with the classification into holonomic and nonholonomic constraints.

The distinction between holonomic and nonholonomic constraints can be concluded in the following way: geometric constraints which are not explicitly velocity-dependent prevent the system from taking some positions, and this is obvious. Differentiation of constraint equations proves that geometric constraints also impose certain limits on the velocities of the system. Is the converse true: do kinematic constraints which are explicitly velocity-dependent impose certain limits on the position of the system? The answer is sometimes yes, depending on the specific form of the constraints. It may happen that the differential constraints can be integrated. We then call them integrable constraints. Geometric and integrable kinematic constraints are also called holonomic, while kinematic non-integrable constraints are called nonholonomic. The crucial difference between the two cases is that holonomic constraints prevent the system from taking some positions, whereas nonholonomic constraints, involving certain limitations on the velocities, nevertheless allow the system to reach every position.

A mechanical system subject to holonomic (nonholonomic) constraints is called a **holonomic (nonholonomic)** system. These notions were introduced in 1894 by **Heinrich R. Hertz** (1857–1894). Because the difference between these two is essential in the analytical studies of motion, we shall illustrate this difference by means of the simple example of rolling of a rigid wheel without slipping, such that the roll axis remains horizontal.

First consider a wheel rolling without slipping along a specified straight line, as shown in Fig. 2.1. The velocity of the wheel centre is

$$\dot{x}_\mathrm{c} = r\dot{\varphi}, \tag{2.1}$$

which can be integrated to yield

$$x_\mathrm{c} = r\varphi + \mathrm{const.} \tag{2.2}$$

The system is hence holonomic.

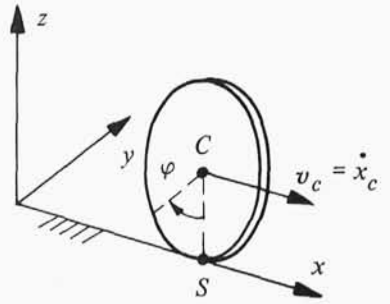

Fig. 2.1

Now consider the same wheel rolling without slipping, but not constrained to follow any particular curve. The position of the wheel is given by the coordinates (x, y) of the point of contact S, the angle ψ giving the orientation of the plane of the wheel, and the angle φ as in the previous case (see Fig. 2.2). The conditions for rolling are

$$\begin{aligned} \dot{x}_c &= r\dot{\varphi}\cos\psi, \\ \dot{y}_c &= r\dot{\varphi}\sin\psi. \end{aligned} \tag{2.3}$$

In this case, the relationship between the velocities is found to be nonintegrable, and hence the system must be classified as nonholonomic.

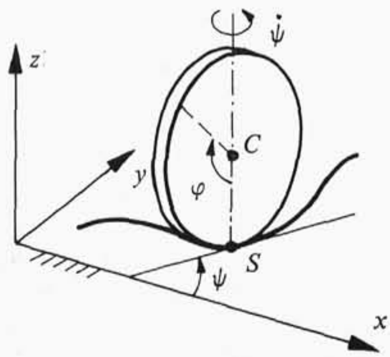

Fig. 2.2

Let us notice that for $\psi = 0$, i.e. for motion along the straight line (parallel to the x-axis), we obtain from (2.3)

$$\begin{aligned} \dot{x}_c &= r\dot{\varphi}, \\ \dot{y}_c &= 0, \end{aligned} \tag{2.4}$$

thus $y_c = const$, and $x_c = r\varphi + const$ which is equivalent to the previous case.

More generally, holonomic constraints have the form

$$f_\alpha(t, x_1, y_1, z_1, \ldots, x_n, y_n, z_n) = 0, \qquad \alpha = 1, \ldots a, \tag{2.5}$$

where a is the number of holonomic constraints, n is the number of particles. Denoting the position vector of the νth particle by the symbol $\mathbf{r}_\nu$ ($\nu = 1, \ldots, n$), the relation (2.5) can be also written in the abbreviated form

$$f_\alpha(t, \mathbf{r}_\nu) = 0, \qquad \alpha = 1, \ldots, a \le 3n, \quad \nu = 1, \ldots, n. \tag{2.6}$$

If, for instance, $a = 1$ and $n = 1$, then equation of constraints is

$$f(t, x, y, z) = 0, \tag{2.7}$$

which will often be referred in examples, since equation (2.7) describes a moving surface defined by $f = 0$, on which a mass particle has to remain during the whole time of the motion.

Also, more generally, all constraints which depend on time t, on position vectors $\mathbf{r}_\nu$, and on their derivatives, and which are not integrable to constraints of type (2.6), are nonholonomic, and can be written as

$$\varphi_\beta(t, \mathbf{r}_\nu, \dot{\mathbf{r}}_\nu) = 0, \qquad \beta = 1, \ldots, b, \quad \nu = 1, \ldots, n, \tag{2.8}$$

where b is the number of nonholonomic constraints, and n is the number of particles. A typical nonholonomic constraint is linear in velocity and can be expressed in the form

$$\varphi_\beta(t, \mathbf{r}_\nu, \dot{\mathbf{r}}_\nu) = \sum_{\nu=1}^{n} \Phi_{\beta\nu}(t, \mathbf{r}_\nu)\dot{\mathbf{r}}_\nu + \Phi_\beta(t, \mathbf{r}_\nu) = 0, \qquad \beta = 1, \ldots, b. \tag{2.9}$$

Such constraints can also be written in the equivalent differential form

$$\sum_{\nu=1}^{n} \Phi_{\beta\nu}(t, \mathbf{r}_\nu)\, \mathrm{d}\mathbf{r}_\nu + \Phi_\beta(t, \mathbf{r}_\nu)\, \mathrm{d}t = 0. \tag{2.10}$$

These constraints are called **catastatic** if the functions Φ_β are not present; alternatively they are defined as **acatastatic constraints**. In practical technical applications acatastatic constraints are encountered quite rarely.

We should here emphasize that holonomic constraints limit the allowed velocities and accelerations within this system. An instance of such a consequence is provided by the well-known theorem on the motion of a rigid rod which states: the velocity components along a rigid rod must be equal. On the other hand, nonholonomic constraints, while limiting the allowed velocities of particles, do not set any restrictions on the position of the system. For instance, let us impose on the system of particles constraints of the form

$$\sum_{\nu=1}^{n} m_\nu(x_\nu \dot{y}_\nu - y_\nu \dot{x}_\nu) = f(t), \tag{2.11}$$

which means that the angular momentum of this system about the axis z is a given function of time. The constraints considered are nonholonomic, since they are nonintegrable. We know, though, that particles of the system can take any position.

2.2.3 Generalized coordinates

Since this notion is both important and difficult, we would like to start with an example known from school—for are there any pupils who did not encounter the simple pendulum? Consider the motion of such a pendulum in the plane (x, y) (see Fig. 2.3a). Nothing seems to hinder the description of the position taken by the bob with the help of coordinates x and y. Why, then, is an entirely different variable applied, as a rule, in the description of the pendulum motion, i.e. the inclination angle φ? The answer usually is that the bob moves along an arc, and it is easier to describe such a motion in the system of polar coordinates (r, φ). The value of the first coordinate, r, is constant, because it is equal to the length of the pendulum, l, and it is only the value of the second coordinate, φ, that changes.

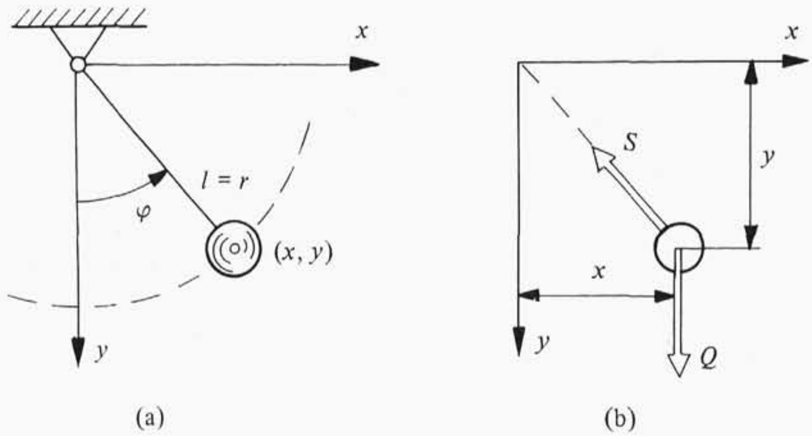

Fig. 2.3.

It turns out, however, that the problem is much more serious than that. We think that it would be proper, for didactic reasons, to justify this statement. For this purpose let us express the equations of motion in the coordinates (x, y). We shall make use, of course, of Newton's second law, which requires considering the bob in isolation, thereby entailing introduction of the new unknown, S, i.e. reaction in the thread (see Fig. 2.3b). Equations of motion shall then take form

$$
\begin{aligned}
m\ddot{x} &= -S\frac{x}{l} \\
m\ddot{y} &= Q - S\frac{y}{l}
\end{aligned}
\tag{2.12}
$$

and we attempt to solve two equations to determine three unknowns: x, y, and S. The missing equation will be provided by the fact that the motion of the bob is constrained, since it must move along a circle. This way we obtain the constraint equation

$$
x^2 + y^2 - l^2 = 0. \tag{2.13}
$$

Having eliminated the unknown S we obtain only one equation of motion

$$x\ddot{y} - y\ddot{x} = gx, \tag{2.14}$$

but it contains two unknowns: x and y. These unknowns are also related through the equation of constraints (2.13), which may be transformed to yield

$$x\dot{y} + y\dot{x} = 0 \tag{2.15}$$

The final system of equations, (2.14) and (2.15), is quite hard to handle without a change of variables, since the equations involved are nonlinear.

Let us now describe this apparently known problem, in the spirit of classical mechanics. The very first step does not consist in writing down of the equations of motion, but the expression of the fact that the bob is constrained, which, of course, leads to equation (2.13). Then, we analyse the equation, reaching the conclusion that the coordinates (x, y) are interdependent and that in fact the bob has only one independent variable, say y, while the value of the second variable can be represented, on the basis of equation (2.13), in the form $x = \pm\sqrt{(l^2 - y^2)}$, i.e. by a bivalued function. This means that with the help of this equation one cannot uniquely define the position of the bob! It is therefore much simpler to specify its position by the angle φ, which is not dependent and is free of any constraint equation. We then have

$$x = l\sin\varphi, \qquad y = l\cos\varphi, \tag{2.16}$$

and after substituting relations (2.16) into equation (2.14), we get

$$-l^2\sin\varphi(\ddot{\varphi}\sin\varphi + \dot{\varphi}^2\cos\varphi) - l^2\cos\varphi(\ddot{\varphi}\cos\varphi - \dot{\varphi}^2\sin\varphi) = gl\sin\varphi,$$

whence

$$\ddot{\varphi} + \frac{g}{l}\sin\varphi = 0, \tag{2.17}$$

i.e. the well-known equation of the simple pendulum.

But what happened to the other equation, that is (2.15)? Let us perform the same operation as before, substituting (2.16) into it. The result is

$$l^2\dot{\varphi}\sin\varphi\cos\varphi - l^2\dot{\varphi}\cos\varphi\sin\varphi = 0, \tag{2.18}$$

that is $0 = 0$, which constitutes an extremely important result. This actually means that the independent coordinate φ identically satisfies the constraints. Speaking metaphorically we can say that this coordinate abolishes or eliminates the constraint.

Naturally, when all this is known from some other source, then equation (2.17) can be obtained much more quickly. In fact, we can write the equation expressing Newton's second law in the direction of the tangent to the trajectory along which the bob moves (this direction corresponds to the change of the value of coordinate φ):

$$m\frac{\mathrm{d}v}{\mathrm{d}t} = -mg\sin\varphi. \tag{2.19}$$

Now, taking into account that $v = l\dot{\varphi}$, we obtain at once equation (2.17). This example should be studied closely, because it explains the role of so-called **generalized coordinates**. The inclination angle φ is just such a coordinate.

We shall now bring in a broader view of the question. Let the mechanical system be composed of n mass particles, of which every one is described with the position vector $\mathbf{r}_\nu$ $(\nu = 1,\ldots, n)$. Let this system be subject to holonomic constraints of the form (2.6). Note that in this case $3n$ Cartesian coordinates (x_ν, y_ν, z_ν) are interrelated via a constraint equations. Thus, there are only

$$s = 3n - a \geq 0 \tag{2.20}$$

independent variables. Some of the Cartesian coordinates can be chosen as such variables. But there is also another way of proceeding, as in the pendulum example. Namely, one can introduce more convenient, mutually independent parameters $q_1, q_2,\ldots, q_s$, whose number is defined by (2.20). Such independent coordinates are often called generalized coordinates. The concept of generalized coordinates refers to the fact that we are not obliged to choose our coordinates according to some preconceived scheme (e.g. we might have chosen polar coordinates r, θ, φ instead of x, y, z, or any other set of three variables suitable for a free point in space).

The choice of the generalized independent coordinates is guided by two requirements:

(1) position vectors $\mathbf{r}_\nu$ $(\nu = 1,\ldots, n)$ of all points should be, at every instant, uniquely expressible by the generalized coordinates, that is

$$\mathbf{r}_\nu = \mathbf{r}_\nu(t, q_\sigma), \qquad \nu = 1,\ldots, n, \quad \sigma = 1,\ldots, s; \tag{2.21}$$

(2) constraint equations (2.21) are satisfied as identities for all values q_σ $(\sigma = 1,\ldots, s)$, that is

$$f_\alpha(t, \mathbf{r}_\nu(t, q_\sigma)) \equiv 0, \qquad \nu = 1,\ldots, n, \quad \alpha = 1,\ldots, a. \tag{2.22}$$

Notice that the example of the simple pendulum illustrates well both these requirements, see (2.16) and (2.18), and that is why this model example should be deeply thought out.

Following the important property (2.22), the question to be asked is how do the generalized coordinates model the nonholonomic constraints (2.9)? In order to answer this question let us differentiate relation (2.21) with respect to time, thereby getting

$$\dot{\mathbf{r}}_\nu = \sum_{\sigma=1}^{s} \frac{\partial \mathbf{r}_\nu}{\partial q_\sigma}\dot{q}_\sigma + \frac{\partial \mathbf{r}_\nu}{\partial t}, \qquad \nu = 1,\ldots, n. \tag{2.23}$$

Having introduced (2.23) to (2.9), taking account of (2.21), we obtain

$$\sum_{\sigma=1}^{s} B_{\beta\sigma}(t, q_\sigma)\dot{q}_\sigma + B_\beta = 0, \qquad \beta = 1,\ldots, b, \tag{2.24}$$

where

$$B_{\beta\sigma} = \sum_{\nu=1}^{n} \Phi_{\beta\nu}(t, \mathbf{r}_\nu(t, q_\sigma)) \frac{\partial \mathbf{r}_\nu}{\partial q_\sigma}, \qquad \sigma = 1, \ldots, s \tag{2.25}$$

and

$$B_\beta = \sum_{\nu=1}^{n} \Phi_{\beta\nu}(t, \mathbf{r}_\nu(t, q_\sigma)) \frac{\partial \mathbf{r}_\nu}{\partial t} + \Phi_\beta\big(t, \mathbf{r}_\nu(t, q_\sigma)\big). \tag{2.26}$$

Thus, in the case of a nonholonomic system the generalized coordinates $q_\sigma (\sigma = 1, \ldots, s)$ may assume arbitrary values, while **generalized velocities** $\dot{q}_\sigma$ cannot take any values but those satisfying the relations (2.24).

There is usually no need to use formulae (2.25) and (2.26), since generalized coordinates are being chosen directly, i.e. without referring to (2.21). That is how we proceeded with formulating the nonholonomic constraints (2.24). Still, it is of interest to illustrate this fact with another example. Consider the mechanical system consisting of two balls connected by a weightless rod (a popular model of a satellite or a multiatom molecule). Assume that they are constrained to move in the vertical plane, and that the velocity of the mass centre be always directed along the connecting rod (Fig. 2.4a).

Equations of the holonomic constraints have the form

$$\begin{aligned} &(z_2 - z_1)^2 + (y_2 - y_1)^2 - l^2 = 0 \\ &x_1 = 0 \\ &x_2 = 0. \end{aligned} \tag{2.27}$$

On the basis of formula (2.20) we have ($n = 2$, $a = 3$):

$$s = 3n - a = 3 \times 2 - 3 = 3,$$

which means that we can determine three independent coordinates. Let these be coordinates of the mass centre (y_c, z_c) of the system, and the angle φ between a predefined direction, say of axis y, and the rod (Fig. 2.4b). Simultaneously, we can check whether property (2.22) is satisfied. For this purpose let us write the requirement (2.22) in the proper form for our example:

$$\begin{aligned} y_1 &= z_c - \frac{l}{2}\cos\varphi, \qquad & y_2 &= y_c + \frac{l}{2}\cos\varphi \\ z_1 &= z_c - \frac{l}{2}\sin\varphi, \qquad & z_2 &= z_c + \frac{l}{2}\sin\varphi. \end{aligned} \tag{2.28}$$

Now substituting (2.28) into (2.27) we get

$$(l\sin\varphi)^2 + (l\cos\varphi)^2 - l^2 = l^2 - l^2 \equiv 0.$$

The main purpose of this example was, however, illustration of the capacity of direct formulation of nonholonomic constraints in the form of (2.24). Let us, then, write the condition formulated in the problem (compare Fig. 2.4b):

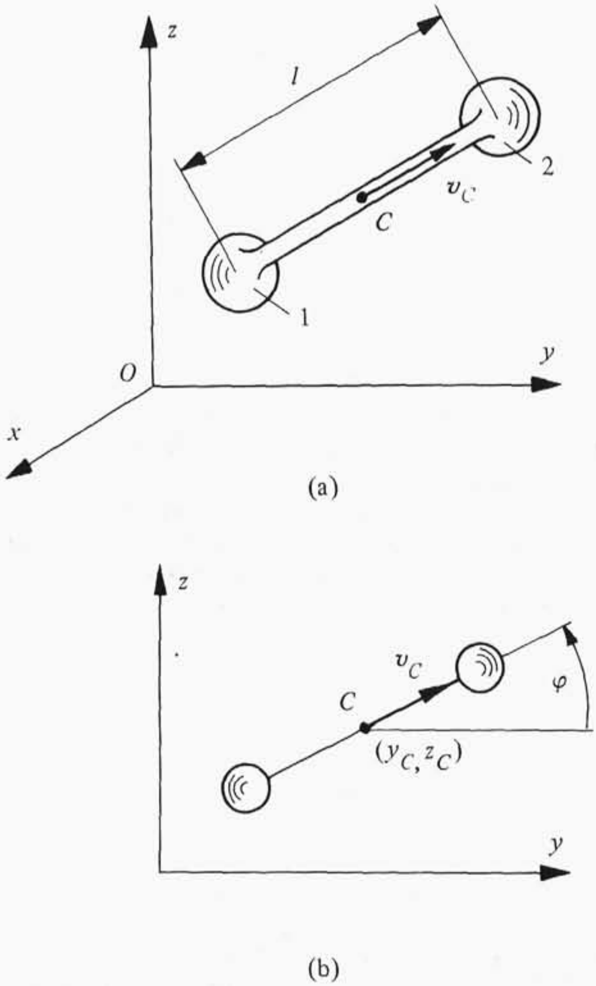

Fig. 2.4.

$$\begin{aligned} \mathbf{v}_C &= (0, \dot{y}_C, \dot{z}_C) \\ \frac{\dot{z}_C}{\dot{y}_C} &= \frac{z_2 - z_1}{y_2 - y_1} = \tan\varphi, \end{aligned} \tag{2.29}$$

from where, immediately,

$$\dot{z}_C = \dot{y}_C \tan\varphi, \tag{2.30}$$

i.e., an equation of the type of (2.24), in which ($b = 1$, $s = 3$)

$$\sum_{\sigma=1}^{s} B_{\beta\sigma}\dot{q}_\sigma + B_\beta = B_{11}\dot{q}_1 + B_{12}\dot{q}_2 + B_{13}\dot{q}_3 + B_1 = 0, \tag{2.31}$$

whence, having compared (2.30) and (2.31), we obtain

$$B_{11} = 0, \qquad B_{12} = -\tan\varphi, \qquad B_{13} = 1, \qquad B_1 = 0. \tag{2.32}$$

Thus, we have shown that there is no need to use directly formulae (2.25), (2.26)—they are only of a formal value.

It is possible that some readers have already noticed the emerging need to introduce the notion which would replace the number of independent coordinates. Such a notion exists and bears the name of *number of degrees of freedom of a system without nonholonomic constraints*, often improperly abbreviated to *number of degrees of freedom*. Such an abbreviation would be valid only for holonomic systems (see 2.2.4). Then, the number of degrees of freedom is simply defined by the formula (2.20). Hence, the dumb-bell of the example recently considered will have three degrees of freedom if we neglect constraints (2.30).

Alas, in the problems in which mass particles cannot be accepted as models, the advantage of having definition (2.20) is small. In such cases the numbers of degrees of freedom of holonomic systems are determined by the selection of independent parameters, uniquely defining the position of the mechanical system—that is, by the definition of the independent generalized coordinates. But then the number of degrees of freedom is less important.

In practical problems it is often more convenient to introduce more generalized coordinates into the model than would result from the number of degrees of freedom of the system. In this case, these coordinates will be mutually dependent. For the sake of illustration let us consider the case of the satellite transmission gear. This system has just one degree of freedom, because it is sufficient to take one generalized coordinate in the description of this system. However, in order, for instance, to calculate kinetic energy, it turns out more convenient to use two coordinates, namely the rotation angles: φ_1 of the crank, and φ_2 of the satellite (Fig. 2.5). Using these two coordinates the kinetic energy of the system may be expressed as

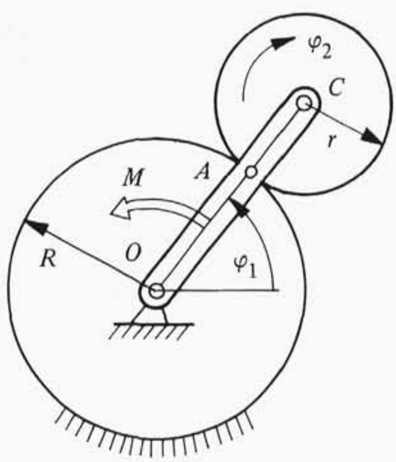

Fig. 2.5.

$$T = T_c + T_s \tag{2.33}$$

where $T_c = \frac{1}{2} I_0 \dot{\varphi}_1^2$, and $T_s = \frac{1}{2} m v_C^2 + \frac{1}{2} I_C \dot{\varphi}_2^2$ are the kinetic energies of the crank and satellite, respectively. Because, however, rolling takes place without slipping, the point of contact, A, of the two circles is the instantaneous centre of velocity of the satellite. Hence

$$v_C = (R + r)\dot{\varphi}_1 = \dot{\varphi}_2 r \tag{2.34}$$

so that

$$\dot{\varphi}_2 = \left(1 + \frac{R}{r}\right)\dot{\varphi}_1. \tag{2.35}$$

After integration with null initial conditions this yields $\varphi_2 = (1 + R/r)\varphi_1$. This result means that generalized coordinates φ_1 and φ_2 are interdependent. Such coordinates can be called *redundant coordinates*. We emphasize this in view of the fact that the notion of generalized coordinates is used to designate independent variables. In reality, the 'generality' of these coordinates concerns rather their 'universality'—namely, it is not only the usual Cartesian coordinates that can be used as generalized coordinates.

The question of completeness and independence of generalized coordinates cannot, anyway, be resolved on elementary grounds; a deeper treatment of this question requires introduction of the notion of variation, which will be done in section 4.2.2.1.

It seems to us that above-mentioned 'universality' of the notion of generalized coordinates should be illustrated by means of the concrete example, and we shall consider electric charge treated as a generalized coordinate.

Consider therefore an electric system whose basic elements RLC are connected in series (Fig. 2.6). For these elements relations between voltage $u(t)$ and current $i(t)$ have the following form

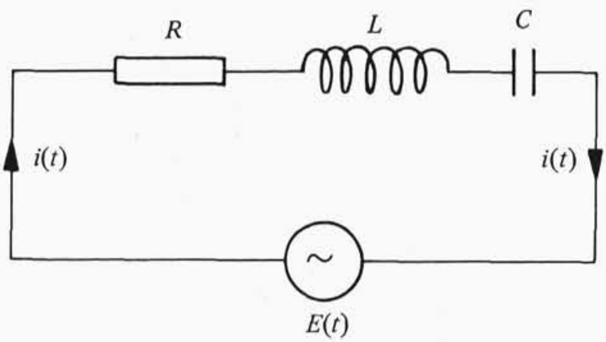

Fig. 2.6.

$$u_R = Ri, \qquad u_L = L\frac{di}{dt}, \qquad u_C = \frac{1}{C}\int i\, dt, \tag{2.36}$$

where

$$i = \frac{\mathrm{d}e}{\mathrm{d}t}, \tag{2.37}$$

with e being an electric charge. To avoid confusion we do not use the popular notation q, because q is traditionally used for the mechanical generalized coordinate.

If a source of electromotive force $E(t)$ is yet introduced into the circuit considered then, on the basis of the *second Kirchhoff law* we obtain

$$Ri = L\frac{\mathrm{d}i}{\mathrm{d}t} + \frac{1}{C}\int i\,\mathrm{d}t = E(t), \tag{2.38}$$

from which, taking into account (2.37), we get

$$L\ddot{e} + R\dot{e} + \frac{1}{C}e = E(t). \tag{2.39}$$

Let us refer not to equation (1.25) of damped oscillator with excitation, i.e.

$$m\ddot{x} + b\dot{x} + kx = F(t). \tag{2.40}$$

We now see the correspondence of parameters, and more importantly that the electric charge e corresponds to coordinate x, which, in this case, is a generalized coordinate. This means that an electric charge can also be treated analogously as a generalized coordinate.

2.2.4 The number of degrees of freedom

According to the traditional definition (see, e.g. Thompson (1961)) *the number of degrees of freedom of a body corresponds to the minimum number of independent coordinates required to define its position.* Resulting from this definition, the number of degrees of freedom is given by the formula (2.20). Some comments on nonholonomic systems will now follow. Considerations on continuous system will be contained in section 2.3.4.

The notion of the number of degrees of freedom was introduced on the basis of independent generalized coordinates, which are determined with the holonomic constraint equations, assuming there are no nonholonomic constraints. When there are nonholonomic constraints present along with the holonomic ones, the number of degrees of freedom, l, of a system, is defined as the difference between the minimum number of independent coordinates, s (i.e. the number of degrees of freedom of the holonomic system), and the number b of equations of nonholonomic constraints, that is

$$l = s - b. \tag{2.41}$$

Thus, the nonholonomic system of Fig. 2.2 has $5 - 2 = 3$ degrees of freedom, and not 5, as would be suggested by the standard definition, quoted before. This has serious consequences for modelling, since a proper model is obtained when the number of equations of motion is equal to the number of degrees of freedom. It is worth mentioning at this point that the various texts on analytical mechanics often contain the definition of the number of degrees of freedom for nonholonomic systems, which involves the notion of virtual displacement, namely *the number of degrees of freedom of a mechanical system is the number of virtual displacements of this system* (see, e.g. Neimark & Fufaev (1972)). This