# 5

# Modelling by means of graphs

The dynamical behaviour of any physical system is a function of three factors: (1) the external excitation, (2) the characteristics of each of the system elements, and (3) how they are connected together, that is, their topology. It is the latter factor that brings graph theory into the picture.

## 5.1 BASIC NOTIONS AND CONCEPTS OF GRAPH THEORY

### 5.1.1 What is a graph?

A **linear graph** (or simply a **graph**) $G = (V, E)$ consists of a set of objects $V = \{s_1, s_2, \ldots\}$ called **vertices**, and another set $E = \{e_1, e_2, \ldots\}$, whose elements are called **edges**, such that each edge $e_k$ is identified with an unordered pair $(s_i, s_j)$ of vertices of $e_k$. The vertices $s_i$, $s_j$ associated with edge $e_k$ are called the **endpoints of** $e_k$. The most common representation of a graph is by means of a diagram, in which the vertices are represented as points and each edge a line segment joining its end vertices. This diagram itself is referred to as the graph. The object shown in Fig. 5.1, for instance, is a graph.

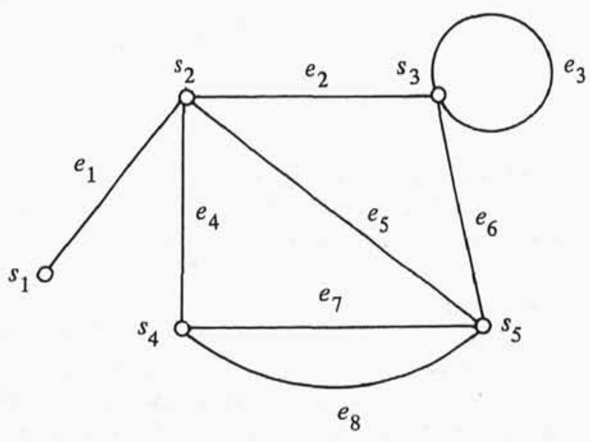



Fig. 5.1.

Observe that this definition permits an edge to be associated with a vertex pair $(s_i, s_i)$. An edge having the same vertex as both its endpoints is called a **self-loop.** Edge $e_3$ in Fig. 5.1 is a self-loop. Also note that the definition allows more than one edge associated with a given pair of vertices, for example, edges $e_7$ and $e_8$ in Fig. 5.1. Such edges are referred to as **parallel edges**.

When a vertex $s_i$ is an end vertex of some edge $e_j$, $s_i$ and $e_j$ are said to be **incident with (on or to)** each other. In Fig. 5.1, for example, edges $e_1$, $e_2$, $e_4$ and $e_5$ are incident with vertex $s_2$. Two or more nonparallel edges are said to be **adjacent** if they are incident on a common vertex. For example, $e_1$ and $e_2$ in Fig. 5.1 are adjacent.

A graph that has neither self-loops nor parallel edges is called a **simple graph**. In some graph-theory literature a graph is defined to be only a simple graph, but in most engineering applications it is necessary that parallel edges and self-loops be allowed; this is why our definition includes graph with self-loops and/or parallel edges.

It should be noted that, in drawing a graph, it is immaterial whether the lines are drawn straight or curved, long or short; what is important is the incidence between the edges and vertices. For example, the two graphs drawn in Fig. 5.2 are the same, except for their labelling because incidence between edges and vertices is the same in both cases. It is enough to note the vertex correspondence $1 \leftrightarrow a$, $2 \leftrightarrow b$, $3 \leftrightarrow c$, $4 \leftrightarrow d$. Such graphs are also called *isomorphic*; a more precise definition of isomorphic graphs is given in section 5.1.2.

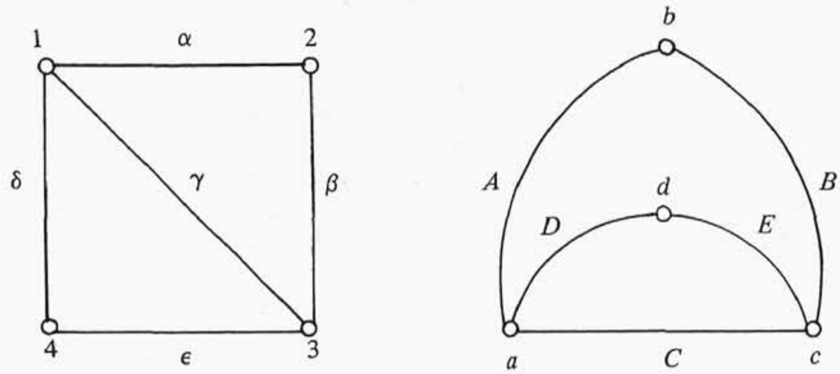



Fig. 5.2.

In the diagram of a graph, two edges may seem to intersect at a point that does not represent a vertex, for example, edges $e$ and $f$ in Fig. 5.3. Such edges should be thought of as being in different planes and thus having no common point.

### 5.1.2 Different types of graph

Two graphs are equivalent and said to be **isomorphic** if they have identical behaviour in terms of graph-theoretic properties. More precisely: two graphs $G$ and $G'$ are said to be isomorphic (to each other) if there is a one-to-one correspondence between their vertices and between their edges such that the incidence relationship is preserved. In other words, suppose that edge $e$ is incident on vertices $s_1$ and $s_2$ in $G$; then the corresponding edge $e'$

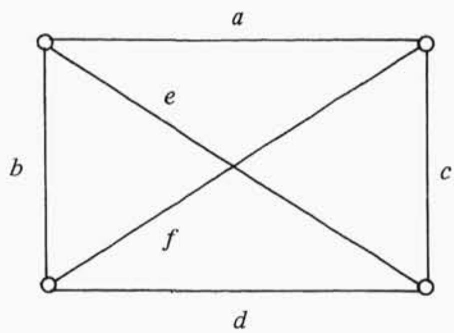



Fig. 5.3.

in $G'$ must be incident on the vertices $s'_1$ and $s'_2$ that correspond to $s_1$ and $s_2$, respectively. An example is shown in Fig. 5.2, where the vertex correspondence $1 \leftrightarrow a$, $2 \leftrightarrow b$, $3 \leftrightarrow c$, $4 \leftrightarrow d$, and the edge correspondence $\alpha \leftrightarrow A$, $\beta \leftrightarrow B$, $\gamma \leftrightarrow C$, $\delta \leftrightarrow D$, $\varepsilon \leftrightarrow E$, establish that two graphs are isomorphic.

Except for the labels (i.e. names) of their vertices and edges, isomorphic graphs are the same graph, although they may be drawn differently.

A graph $G_s(V_s, E_s)$ is said to be a **subgraph** of a graph $G(V, E)$ if all the vertices and all the edges of $G_s$ are in $G$, and each edge of $G_s$ has the same end vertices in $G_s$ and in $G$.

If $V_s = V$, the subgraph is referred to as a **spanning subgraph** of $G$. (Note that $E_s$ may differ from $E$.)

Figure 5.4 presents different subgraphs of the graph shown in Fig. 5.1.

The number of edges incident to/with a vertex $s_i$, with self-loops counted twice, is called the **degree**, $d(s_i)$, of a vertex $s_i$. In Fig. 5.1, for example, $d(s_1) = 1$, $d(s_2) = d(s_3) = d(s_5) = 4$, $d(s_4) = 3$.

A vertex having no incident edge is called an **isolated vertex**. In other words, isolated vertices are vertices with zero degree. A vertex of degree one is called a **pendant vertex** or an **end vertex**.

We say that two subgraphs are **edge-disjoint** if they have no edges in common, and **vertex-disjoint** if they have no vertices in common.

A **walk** is defined as a finite alternating sequence of vertices and edges, beginning and ending with vertices, such that each edge is incident with the vertices preceding and following it.

No edge appears more than once in a walk. A vertex, however, may appear more than once. In Fig. 5.1, for instance $s_1e_1s_2e_5s_5e_6s_3e_3s_3e_2s_2$ is a walk. A walk which begins and ends at the same vertex is called a **closed walk**. A walk that is not closed is called an **open walk**.

An open walk in which no vertex appears more than once is called a **path**. In Fig. 5.1 $s_1e_1s_2e_5s_5e_7s_4$ is a path. The number of edges in a path is called the **length of a path**.

A closed walk in which no vertex (except the initial and the final vertex) appears more than once is called a **circuit**. That is, a circuit is a closed, nonintersecting walk. In Fig. 5.1 $s_2e_4s_4e_7s_5e_5s_2$ is, for example, a circuit.

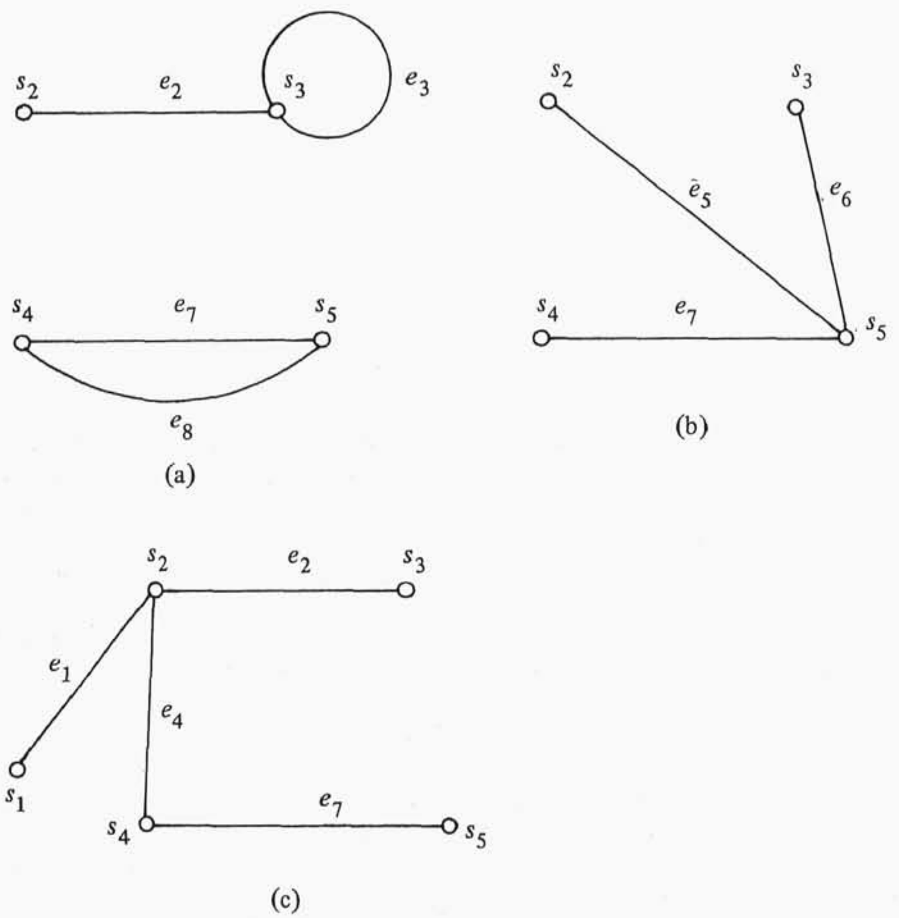



Fig. 5.4.

Intuitively, the concept of **connectedness** is obvious. A graph is **connected** if we can reach any vertex from any other vertex by travelling along the edges. More formally:

A graph $G$ is said to be connected if there is at least one path between every pair of vertices in $G$. Otherwise, $G$ is **disconnected**. For instance the graph in Fig. 5.4a is disconnected. A disconnected graph consists of two or more connected graphs. Each of these connected subgraphs is called a **component**. The graph in Fig. 5.4a consists of two components. A graph $G$ is said to be **planar** if its geometric diagram can be drawn on a plane such that no two edges have an intersection that is not a vertex.

The concept of a **tree** is probably the most important in graph theory, especially for those interested in applications of graphs. Therefore apart from the definition of a tree, some of its properties will be listed. A **tree** is a connected graph without any circuits. The graphs in Figs. 5.4b, c, for instance, are trees. A tree with $n$ vertices has $n-1$ edges. There is exactly one path between every pair of vertices in a tree.

A tree in which one vertex (called the **root**) is distinguished from all the others is called a **rooted tree**.

The tree and its properties may be discussed when it occurs as a graph by itself. The tree may be also studied as a subgraph of another graph. A given graph has numerous subgraphs—from $e$ edges, $2^e$ distinct combinations are possible. Obviously, some of these subgraphs will be trees. Out of these trees we are particularly interested in certain types of trees, called **spanning trees**, which are defined next.

A tree $G_t$ is said to be a **spanning tree** of a connected graph $G$ if $G_t$ is a subgraph of $G$ and $G_t$ contains all vertices of $G$. For instance, the subgraph in Fig. 5.4c is a spanning tree of a graph shown in Fig. 5.1.

An edge in a spanning tree $G_t$ is called a **branch** of $G_t$. An edge of $G$ that is not in a given spanning tree $G_t$ is called a **chord**. For instance, edges $e_1$, $e_2$, $e_4$, $e_7$ are branches of a spanning tree shown in Fig. 5.4c, while edges $e_3$, $e_5$, $e_6$, $e_8$ are chords of a graph shown in Fig. 5.1 with respect to the spanning tree shown in Fig. 5.4c. It must be kept in mind that branches and chords are defined only with respect to a given spanning tree. The collection of chords is called a **cotree**.

It is to be noted that a spanning tree is defined only for a connected graph, because a tree is always connected, and in a disconnected graph of $n$ vertices we cannot find a connected subgraph with $n$ vertices. Each component (which by definition is connected) of a disconnected graph, however, does have a spanning tree. Thus a disconnected graph with $k$ components has a **spanning forest** consisting of $k$ spanning tress. A collection of trees is called a **forest**.

Let us now consider a spanning tree $G_t$ in a connected graph $G$. Adding any one chord to $G_t$ will create exactly one circuit. Such a circuit, formed by adding a chord to a spanning tree, is called a **fundamental circuit**.

How many fundamental circuits does a connected graph have? Exactly as many as the number of chords, i.e. $m = e - n + 1$.

**Rank** and **nullity**: When we specify a graph $G$, the first thing we are most likely to mention is $n$, the number of vertices in $G$. Immediately follows the number of edges $e$ and then the number of components $k$. If $k = 1$, $G$ is connected. These three numbers of a graph are related as follows: since every component of a graph must have at least one vertex, $n \geq k$. Moreover, the number of edges in a component cannot be less than the number of vertices in that component minus the number of components $k$, i.e. $e \geq n - k$. Apart from the constraints $n - k \geq 0$ and $e \geq n - k$, these three numbers $n$, $e$, and $k$ are independent, and they are the fundamental numbers in graphs. (Needless to mention, these numbers alone are not enough to specify a graph, except for trivial cases.)

From these three numbers are derived two other important numbers called **rank** and **nullity**, defined as

$$\textbf{rank} \quad r = n - k,$$
$$\textbf{nullity} \quad m = e - n + k.$$

The rank of a connected graph is $n - 1$, and the nullity, $e - n + 1$. It may be observed that:

(1) *rank of G = number of branches in any spanning tree* (or *forest*) *of G*;
(2) *nullity of G = number of chords in G*; and

(3) *rank + nullity = number of edges in G*. The nullity of a graph is also referred to as its **cyclomatic number**, or **first Betti number**.

In a connected graph $G$, a **cut-set** is a set of edges whose removal from $G$ leaves $G$ disconnected, provided removal of no proper subset of these edges disconnects $G$. For instance, in Fig. 5.5a the two sets of edges $\{a, b, c\}$, $\{a, d, f, h\}$ form two distinct cut-sets. Edge $\{g\}$ alone is also a cut-set. The set of edges $\{a, d, f, i\}$, on the other hand, is not a cut-set, because one of its proper subsets $\{a, d, i\}$ is a cut-set.

A cut-set always 'cuts' a graph into two. Therefore, a cut-set can also be defined as a minimal set of edges in a connected graph whose removal reduces the rank of the graph

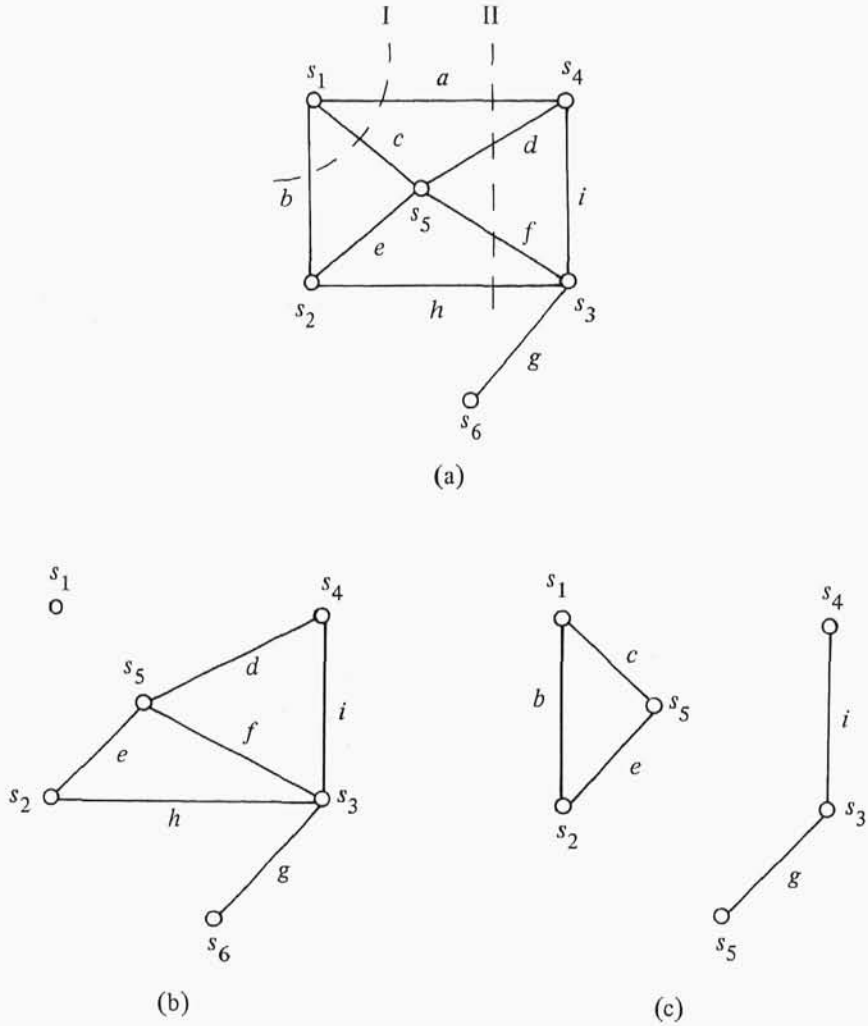



Fig. 5.5.

by one. The ranks of the graphs in Figs. 5.5b, c, for example are four, one less than that of the graph in Fig. 5.5a. Another way of looking at a cut-set is this: if we partition all the vertices of a connected graph $G$ into two mutually exclusive subsets, a cut-set is a minimal number of edges whose removal from $G$ destroys all paths between these two sets of vertices. For instance, in Fig. 5.5(a) cut-set $\{a, d, f, h\}$ connects vertex set $\{s_1, s_2, s_5\}$ with $\{s_3, s_4, s_6\}$.

Since removal of any edge from a tree breaks the tree into two parts, every edge of a tree is a cut-set.

Cut-sets are of great importance in studying properties of communication and transportation networks. Suppose, for example, that the six vertices in Fig. 5.5a represent six cities connected by telephone lines (edges). We wish to find out if there are any weak spots in the network that need strengthening by means of additional telephone lines. We look at all cut-sets of the graph, and the one with the smallest number of edges is the most vulnerable. In Fig. 5.5a, the city represented by vertex $s_6$ can be severed from the rest of the network by the destruction of just one edge.

Just as a spanning tree is essential for defining a set of fundamental circuits, so is a spanning tree essential for a set of **fundamental cut-sets**. Consider a spanning tree $G_t$ of the connected graph $G$ shown in Fig. 5.6. Take any branch $b$ in $G_t$. Since $\{b\}$ is a cut-set in $G_t$, $\{b\}$ partitions all vertices of $G_t$ into two disjoint sets—one at each end of $b$. Consider the same partition of vertices in $G$, and the cut-set $C$ in $G$ that corresponds to this partition. Cut-set $C$ will contain only one branch $b$ of $G_t$, and the rest (if any) of edges in $C$ are chords with respect to $G_t$. Such a cut-set $C$ containing exactly one branch of a spanning tree $G_t$ is called a **fundamental cut-set** with respect to $G_t$.

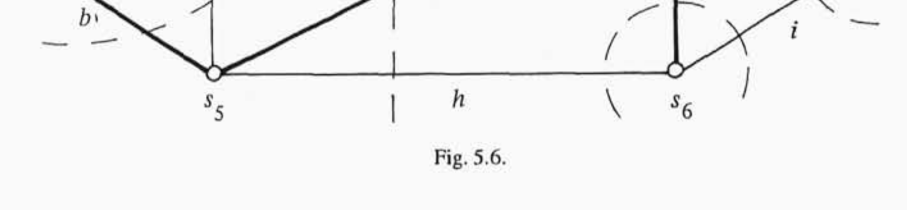



Fig. 5.6.

In Fig. 5.6, a spanning tree $G_t$ (in heavy lines) and all five of the fundamental cut-sets with respect to $G_t$ are shown (broken lines 'cutting' through each cut-set).

Just as every chord of a spanning tree defines a unique fundamental circuit, every branch of a spanning tree defines a unique fundamental cut-set. It must also be kept in mind that the term fundamental cut-set has meaning only with respect to a given spanning tree.

Many physical situations require directed topological representations. The one-way streets in the map of a city, flow networks with valves in the pipes, and electrical networks are examples of a system having some unilateral property. A unilateral property of a system element can be represented by a directed graph as follows:

A **directed graph** (or simply **digraph**) $G$ consists of a set of vertices $V = \{s_1, s_2, \dots\}$, a set of edges $E = \{e_1, e_2, \dots\}$, and a mapping $\Psi$ that maps every edge onto some *ordered* pair of vertices $(s_i, s_j)$. As in the case of unordered graphs, a vertex is represented by a point and an edge by a line segment between $s_i$ and $s_j$ with an arrow directed from $s_i$ to $s_j$. For example, Fig. 5.7 shows a digraph with five vertices and nine edges. A digraph is also referred to as an **oriented graph**. The oriented edges are also called **arcs**.

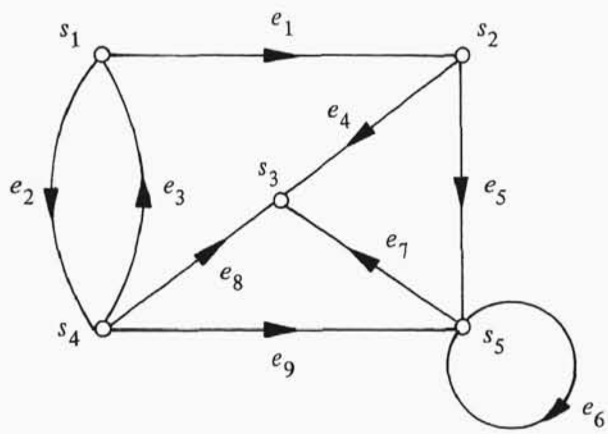



Fig. 5.7.

### 5.1.3 Matrix representation of a directed graph

Although a pictorial representation of a graph is very convenient for visual study, other representations are better for computational purposes. A matrix is a convenient and useful way of represented a graph on a computer. In this section we shall consider three most frequently used matrix representations of a graph.

Thus let us suppose that a graph $G$ has $n$ vertices and $e$ arcs, and no self-loops.

**Incidence matrix**. The incidence matrix of a digraph $G$ is an $n \times n$ matrix $\hat{\mathsf{S}} = [S_{ij}]$, whose rows correspond to vertices and columns correspond to edges, such that:

$$S_{ij} = \begin{cases} 1 & \text{if arc } j \text{ is pointing away from vertex } s_i, \\ -1 & \text{if arc } j \text{ is point toward vertex } s_i, \\ 0 & \text{otherwise.} \end{cases}$$

**Circuit matrix**. Let $G$ be a digraph with $e$ arcs and $b$ circuits. An arbitrary orientation (clockwise or counterclockwise) is assigned to each of $b$ circuits. Then a circuit matrix $\hat{\mathsf{B}} = [B_{ij}]$ of $G$ is a $b \times e$ matrix defined as

$$B_{ij} = \begin{cases} 1 & \text{if the } i\text{th circuit includes the } j\text{th arc, and the orientations of the arc} \\ & \text{and circuit coincide,} \\ -1 & \text{if the } i\text{th circuit includes the } j\text{th arc, but the orientations of the arc} \\ & \text{and circuit are opposite,} \\ 0 & \text{if the } i\text{th circuit does not include the } j\text{th arc.} \end{cases}$$

To define a cut-set matrix, a cut orientation has to be defined first.

**Cut orientation**. For a directed graph $G$, let $V_1$ and $V_2$ be sets of vertices partitioned by a cut $C$ of $G$. The cut $C$ is said to be oriented if the sets $V_1$ and $V_2$ are ordered either as $(V_1, V_2)$ or as $(V_2, V_1)$.

In most cases, the orientation of a cut may be represented by an arrow. For example, we can place an arrow near the broken line defining the cut. In Fig. 5.8 the orientation of the cut of $G$ is as indicated. However, the cut-set consisting of the arcs $e_1, e_2, e_3, e_4$ cannot be represented in this way unless we redraw $G$ such as by interchanging the positions of the vertices $s_3$ and $s_4$. Let a cut $C$ of $G$ be ordered as $(V_1, V_2)$. We shall say that the orientations of the edge $(s_i, s_j)$ and the cut $C$ coincide if $s_i$ is in $V_1$ and $s_j$ in $V_2$. Otherwise, they are opposite. For example in Fig. 5.8 the orientations of the edge $e_2 = (s_1, s_2$ and the cut-set 1 are opposite.

**Cut-set matrix**. Let $G$ be a digraph with $e$ arcs and $q$ nonempty cut-sets. An arbitrary orientation is assigned to each of $q$ cut-sets. Then a cut-set matrix $\hat{\mathsf{Q}} = [Q_{ij}]$ of $G$ is a $q \times e$ matrix defined as

$$Q_{ij} = \begin{cases} 1 & \text{if the } j\text{th arc is in the } i\text{th cut-set and the orientations of the arc} \\ & \text{and cut-set coincide,} \\ -1 & \text{if the } j\text{th arc is in the } i\text{th cut-set and the orientations of the arc} \\ & \text{and cut-set are opposite,} \\ 0 & \text{if the } j\text{th arc is not in the } i\text{th cut-set.} \end{cases}$$

In the matrices $\hat{\mathsf{S}}, \hat{\mathsf{B}}$, and $\hat{\mathsf{Q}}$ there are redundant (or linearly dependent) rows. This means that the rank of each matrix $\hat{\mathsf{S}}, \hat{\mathsf{B}}$, and $\hat{\mathsf{Q}}$ is less than the number of rows appearing in them. Thus, for a connected graph $G$ with $n$ vertices and $e$ edges, the rank of the incidence matrix $\hat{\mathsf{S}}$ is $n - 1$, the rank of the circuit matrix $\hat{\mathsf{B}}$ is $m = e - n + 1$, and the rank of the cut-set matrix $\hat{\mathsf{Q}}$ is $n - 1$. Thus not the matrices $\hat{\mathsf{S}}, \hat{\mathsf{B}}$, and $\hat{\mathsf{Q}}$, but their submatrices with the ranks $n - 1$, $m$, and $n - 1$ respectively, ought to be used, since such submatrices will be sufficient in any calculations.

If we remove any one row from the incidence matrix, $\hat{\mathsf{S}}$, the remaining $(n - 1) \times e$ submatrix is of rank $n - 1$. In other words, the remaining $n - 1$ row vectors are linearly independent. Such an $(n - 1) \times e$ submatrix $\mathsf{S}$ of $\hat{\mathsf{S}}$ is called a **reduced incidence matrix**. The vertex corresponding to the deleted row in $\hat{\mathsf{S}}$ is called the **reference vertex**. Clearly, any vertex of a connected graph can be made the reference vertex.

A submatrix of a circuit matrix in which all rows correspond to a set of fundamental circuits is called a **fundamental circuit matrix** $\mathsf{B}$. It is an $m \times e$ matrix, whose rank is $m$.

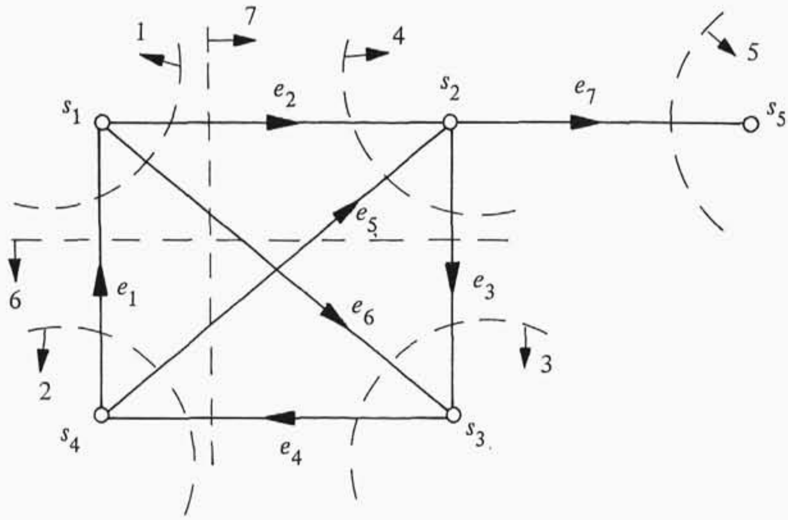



Fig. 5.8.

As in the case of a circuit matrix, the cut-set matrix generally has many redundant rows. Therefore, it is convenient to define a **fundamental cut-set matrix**, Q, as an $(n-1) \times e$ submatrix of $\hat{\mathsf{Q}}$ such that the rows correspond to the set of fundamental cut-sets with respect to some spanning tree.

In our further considerations only the matrices S, B, and Q will be used. Therefore to save space we shall refer them to as incidence, circuit and cut-set matrix, respectively.

## 5.2 A BRIEF HISTORY OF GRAPH THEORY

The origins of graph theory are humble, even frivolous. Whereas many branches of mathematics were motivated by fundamental problems of calculation, measurement, science and technology, the problems which led to the development of graph theory were often little more than puzzles. But despite the apparent triviality of such puzzles, they captured the interest of mathematicians, with the result that graph theory has become a subject rich in theoretical results of a surprising variety and depth.

**Euler** became the father graph theory as well as topology when in 1736 he settled a famous unsolved problem of his day called the *Königsberg Bridge Problem* (Euler (1736)). There were two islands linked to each other and to the banks of the Pregel River by seven bridges as shown in Fig. 5.9. The problem was to begin at any of the four land areas, walk across each bridge exactly once and return to the starting point.

Euler modelled this situation by means of graph, as shown in Fig. 5.10. The vertices represent the land areas and the edges represent the bridges. Showing that the problem is unsolvable is equivalent to showing that the graph of Fig. 5.10 cannot be traversed in a certain way.

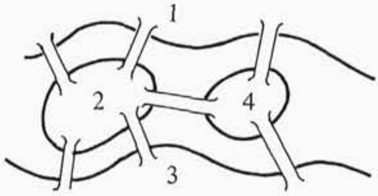



Fig. 5.9.

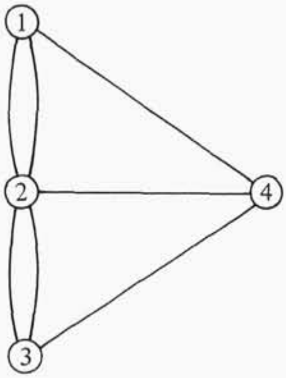



Fig. 5.10.

Rather than treating this specific situation, Euler generalized the problem and developed a criterion for a given graph to be so traversable; namely, that it is connected and every vertex is incident with an even number of edges.

For the next 100 years nothing more was done in the field.

In 1847, **Gustav R. Kirchhoff** (1824–87) developed the theory of trees for their applications in electrical networks (Kirchhoff (1847)). He abstracted an electrical network with its resistances, conductances, inductances, etc., and replaced it by its corresponding combinatorial structure consisting of vertices and edges without any indication of the type of electrical element represented by individual edges. Thus in effect, Kirchhoff replaced each electrical network by its underlying graph and showed that it is not necessary to consider every circuit in the graph of an electric network separately in order to solve the system of equations. Instead, he pointed out by a simple but powerful construction, which has since become standard procedure, that the independent circuits of a graph determined by any of its spanning trees will suffice. A contrived electrical network $N$, its underlying graph $G$, and a spanning tree $G_t$ are shown in Fig. 5.11.

In 1857, **Arthur Cayley** (1821–95) rediscovered trees while he was trying to count the number of structural isomers of the saturated hydrocarbons (or paraffin series) $C_kH_{2k+2}$ (Cayley (1857)). He used a connected graph to represent the $C_kH_{2k+2}$ molecule. Corresponding to their chemical valences, a carbon atom was modelled by a vertex of degree four and a hydrogen atom by a vertex of degree one (pendant vertices). The total number of vertices in such a graph is $n = 3k + 2$, and the total number of edges is

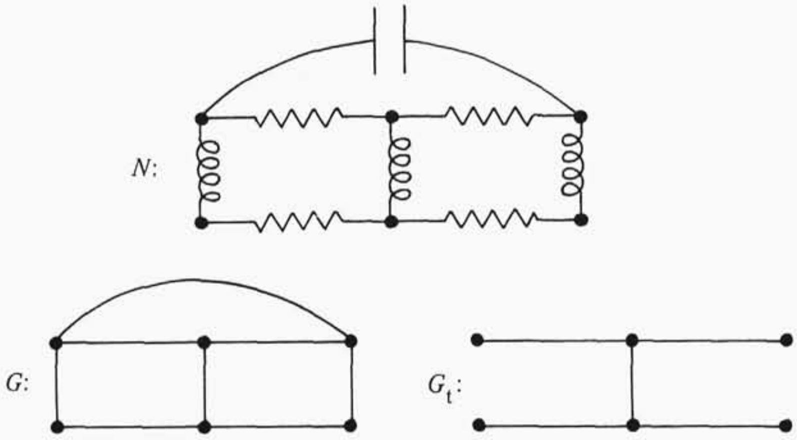



Fig. 5.11.

$$e = \tfrac{1}{2}(\textit{sum of degrees}) = \tfrac{1}{2}(4k + 2k + 2) = 3k + 1.$$

Since the graph is connected and the number of edges is one less than the number of vertices, it is a tree. Thus the problem of counting structural isomers of a given hydrocarbon becomes the problem of counting trees (with certain qualifying properties).

The first question Cayley asked was: what is the number of different trees that one can construct with $n$ distinct (or labelled) vertices? If $n = 4$, for instance, we have 16 trees, as shown in Fig. 5.12. The reader can satisfy himself that there are no more trees of four vertices. (Of course, some of these trees are isomorphic.)

A graph in which each vertex is assigned a unique name or label (i.e. no two vertices have the same label), as in Fig. 5.12, is called a **labelled graph**. The distinction between a labelled and an unlabelled graph is very important when we are counting the number of different graphs. For instance, the four graphs in the first row in Fig. 5.12 are counted as four different trees (even though they are isomorphic) only because the vertices are labelled. If there were no distinction made between $A$, $B$, $C$ or $D$, these four trees would be counted as one. A careful inspection of the graphs in Fig. 5.12 reveals that the number of unlabelled trees with four vertices (no distinction made between $A$, $B$, $C$ and $D$) is only two.

The following well-known theorem for counting trees was first stated by Cayley, and is therefore called *Cayley's theorem*:

*The number of labelled trees with n vertices* ($n \geq 2$) *is* $n^{n-2}$.

In the actual counting of isomers $C_kH_{2k+2}$, Cayley's theorem is not enough. In addition to the constraints on the degree of the vertices, two observations should be made:

(1) Since the vertices representing hydrogen are pendant, they are connected to carbon atoms in only one way, and hence make no contribution to isomerism. Therefore, we need not show any hydrogen vertices.

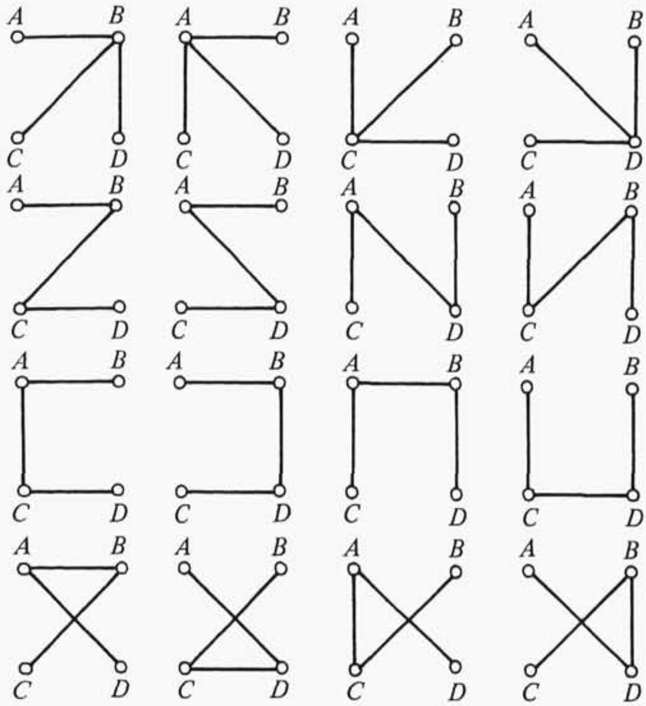



Fig. 5.12.

(2) Thus the tree representing $C_kH_{2k+2}$ reduces to one with $k$ vertices, each representing a carbon atom. In this tree no distinction can be made between vertices, and therefore it is unlabelled.

By means of these two observations Cayley enumerated the number $N$ of theoretically possible isomers of hydrocarbons with $k$ carbon atoms. The result was as follows:

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $k=1$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
| $N=1$ | 1 | 1 | 2 | 3 | 5 | 9 | 18 | 35 | 75 | 159 | 355 | 802 |

Thus, for example, for hydrocarbon $C_4H_{10}$, there are only two distinct trees, as shown in Fig. 5.13 by heavy lines. As every organic chemist knows, there are indeed exactly two different types of hydrocarbon $C_4H_{10}$: *butane* and *isobutane*.

About the time of Kirchhoff and Cayley, two other milestones in graph theory were laid. One was the **four-colour conjecture**, which states that *four colours are sufficient for colouring any atlas* (a map on a plane) such that the countries with common boundaries have different colours. The four-colour problem, which remained unsolved for more than a hundred years, has played a role of the utmost importance in the development of graph theory as we know it today. It is believed that **A. F. Möbius** (1790–1868) first presented the four-colour conjecture in one of his lectures in 1840. The first written reference to the four-colour problem occurs in a letter dated 23 October, 1852, sent to Sir

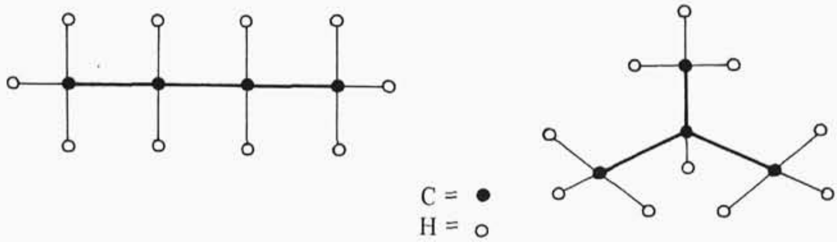
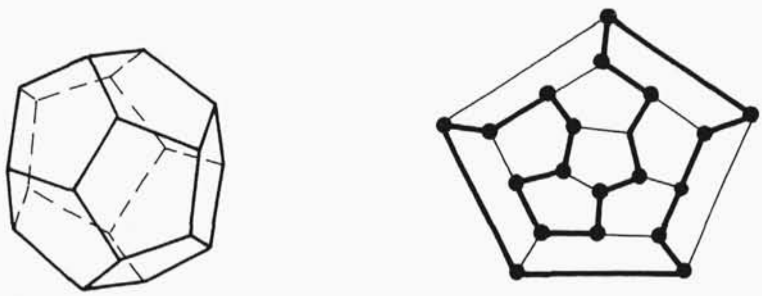



Fig. 5.13.

**W. R. Hamilton** (1805–65) by **A. De Morgan** (1806–71). The problem became well known after Cayley published it in 1879 (Cayley (1879)). Numerous attempts to solve the problem during more than 100 years contributed to obtaining many important results in several branches of mathematics. The discovery of **K. Appel** and **W. Haken** in 1976 finally established the truth of the four-colour conjecture (Appel and Haken (1976)).

The other milestone is due to Sir W. R. Hamilton. In the year 1859 he invented a puzzle and sold it for 25 guineas to a game manufacturer in Dublin. The puzzle consisted of a wooden, regular dodecahedron (a polyhedron with 12 faces and 20 corners, each face being a regular pentagon and three edges meeting at each corner; see Fig. 5.14a). The corners were marked with the names of 20 important cities. The object of the puzzle was to find a route along the edges of the dodecahedron, passing through each of the 20 cities exactly one.



(a) (b)

Fig. 5.14.

Although the solution of this specific problem is easy to obtain (as we see in Fig. 5.14b), to date no one has found a necessary and sufficient condition for the existence of such a route (called a **Hamiltonian circuit**) in an arbitrary graph.

This fertile period was followed by half a century of relative inactivity. Then a resurgence of interest in graphs started during the 1920.

In 1930 the Polish mathematician **K. Kuratowski** (1896–1980) published a remarkable theorem (Kuratowski (1930)). He proved that *if a graph is nonplanar, then it contains either* $K_5$ or $K_{3,3}$ (both graphs are shown in Fig. 5.15); in other words, these two graphs are essentially the only obstacles to planarity.

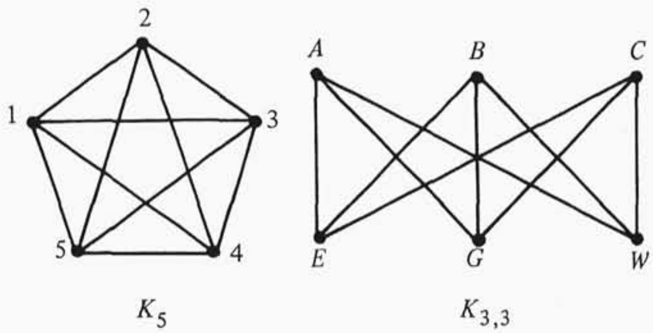



Fig. 5.15.

Like several other aspects of graph theory, the study of planarity originated from puzzles. One such puzzle is the following problem. Let us suppose that the capital cities of five neighbouring regions are to be joined by roads in such a way that no bridges or crossroads are necessary. The five cities and ten roads may be regarded as the vertices and edges of the complete graph $K_5$, and the problem requires us to draw this graph in the plane without crossings. A few experiments with pencils and paper will convince the reader that the problem is insoluble, and $K_5$ is consequently not planar.

Another puzzle involving planarity is the so-called *utilities problem*. There are three houses *A*, *B*, *C*, each to be connected to each of three utilities—electricity (*E*), gas (*G*), and water (*W*)—by means of conduits. Is it possible to make such connections without any crossovers of the conduits? As we already know, the answer to the problem is 'no'.

Since the 1950s a rough development of the engineering applications of graphs has been observed. A milestone in graph-theoretic analysis of electrical networks was achieved by **W. S. Percival**, when in 1953–55 he extended the *Kirchhoff impedance* and *Maxwell admittance* methods to networks with active elements. About the same time **S. J. Mason** developed the concept of signal flow graphs, which was originally worked out by **C. E. Shannon** in a classified report dealing with analogue computers. A few years later, in 1961, **H. Paynter** originated a new modelling technique called the **bond graph method** (Paynter (1961)). In the last three decades, the methods originally elaborated for relatively narrow classes of systems were substantially extended and adapted to the modelling and analysis of many different kinds of physical system.

The rapid development of graph theory and its applications as well as the substantial increase of interest is proved by the following fact. In the year 1936, the first comprehensive treatise on graph theory appeared, (König (1936)). The book summarized two centuries of development in the subject. Since then in just four languages—if English, French, German and Russian—nearly 200 different books concerning graph theory and its applications have been published.

## 5.3 THE LINEAR GRAPH MODELLING METHOD

### 5.3.1 System, components and terminals

A system as defined in the context of this books is a collection of interacting components,