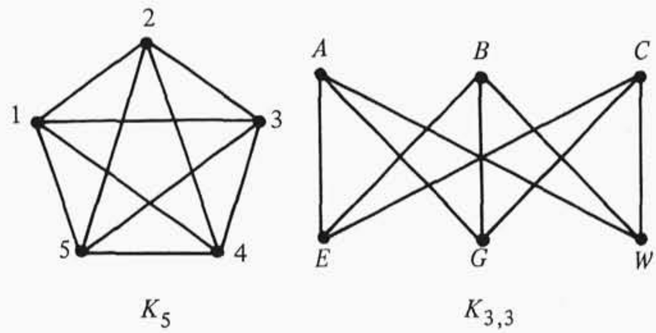

Fig. 5.15.

Like several other aspects of graph theory, the study of planarity originated from puzzles. One such puzzle is the following problem. Let us suppose that the capital cities of five neighbouring regions are to be joined by roads in such a way that no bridges or crossroads are necessary. The five cities and ten roads may be regarded as the vertices and edges of the complete graph K_5, and the problem requires us to draw this graph in the plane without crossings. A few experiments with pencils and paper will convince the reader that the problem is insoluble, and K_5 is consequently not planar.

Another puzzle involving planarity is the so-called *utilities problem*. There are three houses A, B, C, each to be connected to each of three utilities—electricity (E), gas (G), and water (W)—by means of conduits. Is it possible to make such connections without any crossovers of the conduits? As we already know, the answer to the problem is 'no'.

Since the 1950s a rough development of the engineering applications of graphs has been observed. A milestone in graph-theoretic analysis of electrical networks was achieved by **W. S. Percival**, when in 1953–55 he extended the *Kirchhoff impedance* and *Maxwell admittance* methods to networks with active elements. About the same time **S. J. Mason** developed the concept of signal flow graphs, which was originally worked out by **C. E. Shannon** in a classified report dealing with analogue computers. A few years later, in 1961, **H. Paynter** originated a new modelling technique called the **bond graph method** (Paynter (1961)). In the last three decades, the methods originally elaborated for relatively narrow classes of systems were substantially extended and adapted to the modelling and analysis of many different kinds of physical system.

The rapid development of graph theory and its applications as well as the substantial increase of interest is proved by the following fact. In the year 1936, the first comprehensive treatise on graph theory appeared, (König (1936)). The book summarized two centuries of development in the subject. Since then in just four languages—if English, French, German and Russian—nearly 200 different books concerning graph theory and its applications have been published.

5.3 THE LINEAR GRAPH MODELLING METHOD

5.3.1 System, components and terminals

A system as defined in the context of this books is a collection of interacting components, which in the most general case can be represented schematically as shown in Fig. 5.16. The closed regions represent components, and the points of contact $A, B, \ldots, F$ between the regions represent interfaces. Each component is said to have a **terminal** corresponding to each of its interfaces with other components.

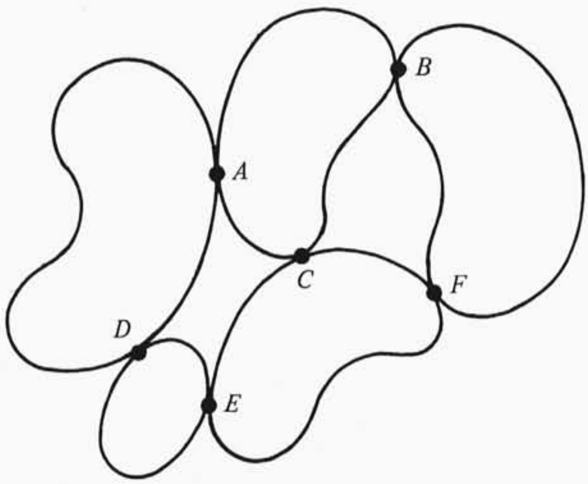

Fig. 5.16.

The modelling as well as analysis of physical systems requires;

(1) a mathematical description of each component;
(2) a mathematical description of how the components are combined to form the system.

The aggregate of these component models is called the **system model** or alternatively the **system equations**. The aggregation procedure however relies upon a fundamental assumption, which is not necessarily always fulfilled. This assumption is known as the fundamental axiom of system theory: *a mathematical model of a component characterizes the behaviour of that component of a system as an entity and independently of how the component is interconnected with other components to form a system*. This implies that the various components can be 'removed' either literally or conceptually from the remaining components and studied in 'isolation' to establish their models.

The theory of graphs is valuable as a means of achieving a simple systematic procedure for formulating the system equations. The mathematical representation of the component is required independently of the particular graph-theoretical method applied to modelling of a given system. The question now is: What constitutes a mathematical model of each identified component? The operational answer to this question forms different graph-theoretical methods, all of them, however, having one thing in common—they use a graph for the representation of the system topology.

5.3.2 Terminal representation

The motions of various elements in a mechanical system are nearly always associated with coexisting forces. In many instances we prefer to think of motion as resulting from the application of a force, whereas in other cases we may prefer to think of a certain force as resulting from a given motion. In either case, interactions involving work, energy, and power occur between mechanical elements and their surroundings.

Let us recall relationships describing the behaviour of simplest mechanical elements, i.e. linear spring, viscous damper, and mass particle. We shall assume that the motion of each element is restrained to translation along the x-axis (see Fig. 5.17).

Let us designate a fixed reference point by the letter g (the ground), the reference positions of the points 1 and 2 by symbols x_{r1}, x_{r2}, and let the displacements of points 1 and 2 be denoted by x_1 and x_2, respectively. Thus we have:

— for the linear spring

$$F_s = k(x_2 - x_1), \tag{5.1}$$

— for the viscous damper

$$F_d = b(v_2 - v_1), \tag{5.2}$$

and for the mass particle

$$F_m = ma_2. \tag{5.3}$$

Introducing the notion of relative displacement

$$x_{21} = x_2 - x_1 \tag{5.4}$$

the relations (5.1)–(5.3) may be expressed as follows:

$$F_s = kx_{21} = k\int_0^t v_{21}\,dt + F_0 \tag{5.5}$$

$$F_d = bv_{21} \tag{5.6}$$

$$F_m = ma_{21} = m\frac{dv_{21}}{dt} \tag{5.7}$$

where $v_{21} = v_2 - v_1$, $a_{21} = a_2 - a_1 = a_2$ (since $a_1 = 0$).

All three mechanical elements considered as having two terminals. In the case of a spring and a damper it is obvious where both terminals are. It is not so evident in the case of a mass. To explain the problem, let us look at Fig. 5.17c; the mass may be considered to have two terminals which describe its motion; the first is v_2 which describes the velocity of the mass itself, and the other is v_1, which describes the velocity of the non-accelerating reference frame. Usually, we shall use a reference velocity v_1 which is equal to zero.

The relations describing the motion of three basic mechanical elements have been expressed in terms of two kinds of physical variables, a **through-variable**, which has the same value at the two terminals or ends of the element, and an **across-variable**, which is specified in terms of a relative value or difference between the terminals.

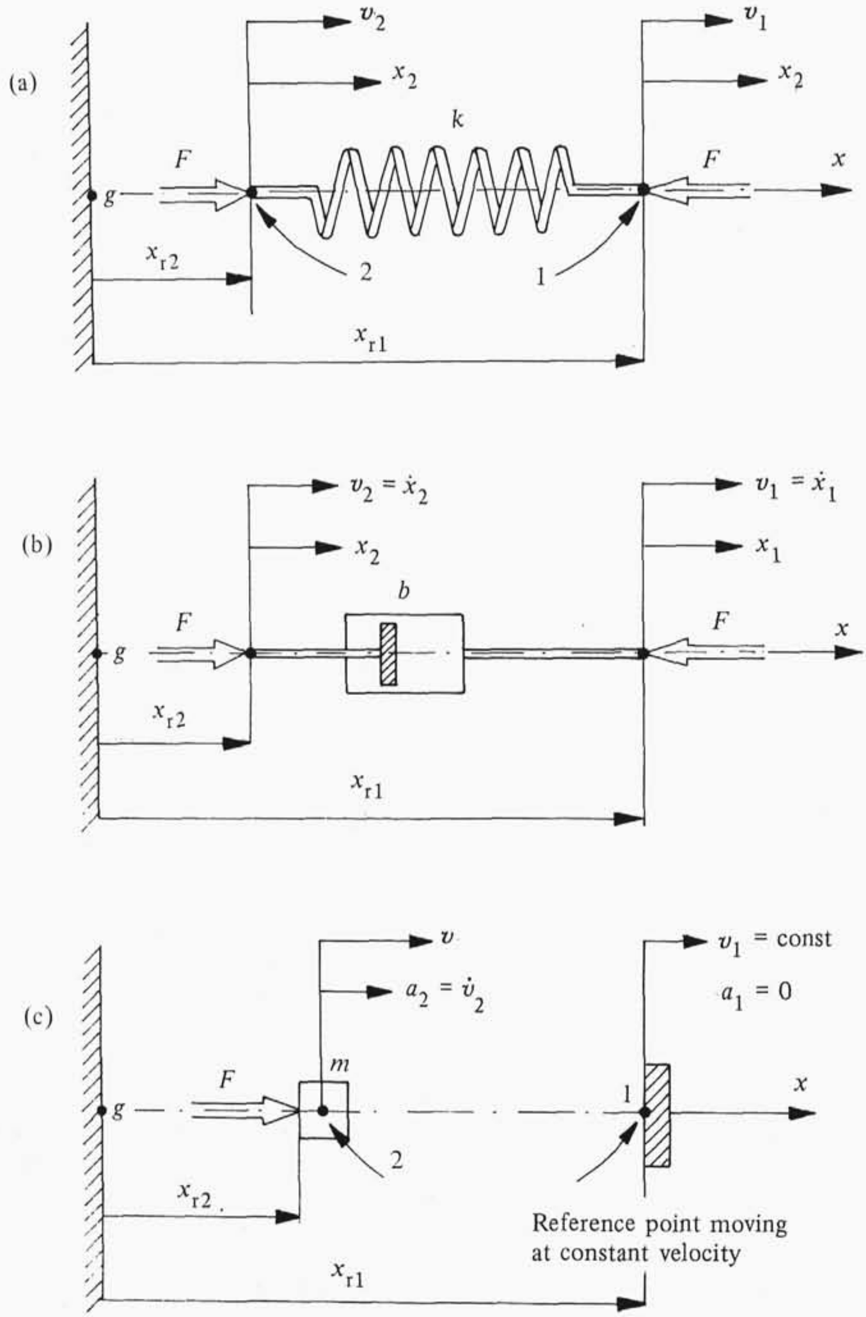
(a)
v_2
x_2
v_1
x_2
k
F
F
x
g
x_{r2}
2
1
x_{r1}
(b)
$v_2 = \dot{x}_2$
x_2
$v_1 = \dot{x}_1$
x_1
b
F
F
x
g
x_{r2}
x_{r1}
(c)
v
$a_2 = \dot{v}_2$
v_1 = const
a_1 = 0
m
F
x
g
1
2
x_{r2}
Reference point moving
at constant velocity
x_{r1}

Fig. 5.17.

Insight into the criteria for defining through- and across-variables can be gained by a consideration of the method by which a particular variable would be measured in the actual physical system. In fact, it is from this consideration that the names 'through' and 'across' arise. Forces and torques can be measured by means of a calibrated spring scale. To measure the force or torque at a point, we must sever the system at that point and insert the spring scale between the two resultant sections. Therefore, force and torque are thought of as being applied, 'through' the measuring device and hence can be described by the common name, through variables. Velocity meters (translational or angular) could be envisioned as devices which determine the rate of separation of two points within it, each of which is rigidly connected to the two points. With any of these velocity meters the measurement can be made by simply appropriately attaching the measuring device to the system; it is not necessary to break into the system. Thus velocity and angular velocity are thought of as existing across two points and can be described by the common name of across-variables. We shall use the general symbols f and v to stand for any through- and across-variables, respectively. The variables f and v associated with a pair of terminals will be called **complementary variables**.

A convenient symbol for a two-terminal element, as discussed above and depicted in Fig. 5.18a, is the linear graph shown in Fig. 5.18b. Two vertices of this graph indicate the two terminals of the element, and the labels associated with the vertices indicate the names of across-variables associated with both terminals. The term 'linear' in 'linear graph' means that the graph is defined by a line segment and should not be confused with 'linear' as used in the mathematical context.

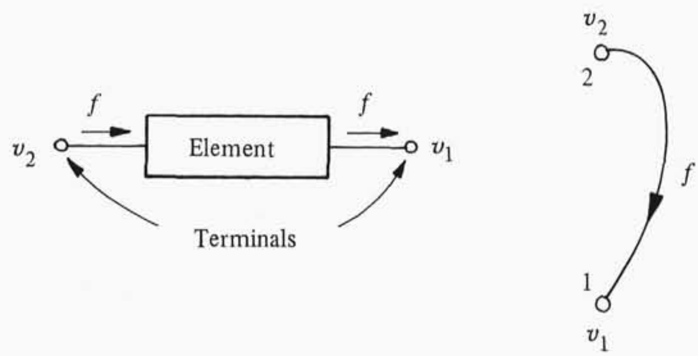

Fig. 5.18.

In many applications, it is necessary to associate with each edge of a graph an orientation or direction. In some situations, the orientation of the edges is a 'true' orientation in the sense that the system represented by the graph exhibits some unilateral property. For example, the directions of the one-way streets of a city and the orientations representing the unilateral property of a communication network are true orientations of the physical system. In other situations, the orientation used in a 'pseudo-orientation' is used in lieu of an elaborate reference system. It is this situation in which bilateral mechanical elements appear. The sign convention selected for the across-variable difference and for the through-variable can be shown by a single arrow-head drawn on the graphs in Fig. 5.18b. The graph is then said to be oriented or directed. The arrow pointing from 2 to 1 means that v_2 is algebraically larger than v_1 when v_{21} is positive. It also means that f is positive when it flows from 2 to 1, i.e. when it tends to produce a positive v_{21} in the element. These two conventions require that power flow into the element when v_{21} and f are both positive.

The complementary variables v, f may be expressed as the time derivatives of the integrated through-variable h and integrated across-variable x, respectively, i.e.

$$f = \frac{\mathrm{d}h}{\mathrm{d}t} \tag{5.8}$$

$$v = \frac{\mathrm{d}x}{\mathrm{d}t} \tag{5.9}$$

Table 5.1 lists the complementary variables f and v and their respective integrals h and x for the four physical processes

Table 5.1. Through- and across-variables for physical systems.

System	Through-variable f	Integrated through-variable h	Across-variable v	Integrated across-variable x
Mechanical–translational	Force	Translational momentum	Velocity difference	Displacement difference
Mechanical–rotational	Torque	Angular momentum	Angular velocity difference	Angular displacement difference
Electrical	Current	Charge	Voltage difference	Flux linkage
Fluid	Fluid flow	Volume	Pressure difference	Pressure–momentum

Thus it may be concluded that relationships such as (5.5)–(5.7), together with one edge linear graph, form a mathematical model of the two-terminal elements of a physical system.

Many physical systems contain components having more than two terminals. These are, for instance, triodes, transistors, transformers, gearboxes and levers. All components which serve as links between electrical and mechanical systems, called electromechanical transducer, by necessity contain at least one pair of electrical terminals and one pair of mechanical terminals. A similar statement applies to other types of transducers—electrothermal, electrohydraulic, etc. Consequently, any mathematical description of systems containing these multi-terminal components must be based on an appropriate and complete mathematical description of the terminal characteristics of the nonreducible multi-terminal components. By *nonreducible components* we mean that the component cannot be resolved into components of fewer terminals without destroying its properties.

A second type of multi-terminal component is encountered in the area of large, complex systems where, as a matter of expediency, if not by necessity, large subassemblies such as electrical or mechanical amplifiers, compensating networks, filters, and rotating machines are considered as a 'packaged unit' with two or more terminals. If each such packaged unit were to be represented by a complex collection of two-terminal (or other multi-terminal) components and a derivation of the system characteristics attempted on the basis of this vast amount of detail, the number of equations resulting would be prohibitive for even the most simple control systems. The only practical procedure is first to derive a set of terminal characteristics for each packaged unit or subassembly, retaining only those terminals which are used to unite it with the remaining subassemblies of the system.

Having defined the broad objectives, one should next proceed with the details. The first question that must be answered is what constitutes a mathematical model of multi-terminal components. To answer this question, consider the four-terminal component shown in Fig. 5.19a. Let a pair of complementary variables v_i and f_i be identified with each pair of component terminals.

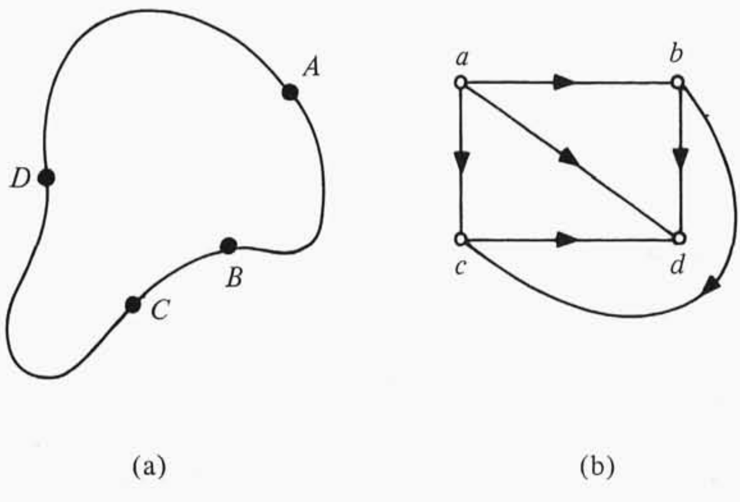

Fig. 5.19.

Generally, if a component has n terminals, then it is possible to identify a pair of complementary variables $v_i(t)$ and $f_i(t)$ with each possible pair of terminals on the component by a mapping that includes exactly one vertex for each component terminal and one edge for each pair of terminals. The edges of the mapping illustrated in Fig. 5.19b for a four-terminal component identify a pair of oriented complementary variables with every possible pair of terminals on the component.

However, not all these variables are required to model the characteristics of the component. They can be modelled by the complementary variables identified by a tree spanned on the n vertices. If the n vertices correspond to the terminals of an n-terminal component, then the spanning tree is called a **terminal graph** of the component.

Several terminal graphs for a four-terminal component are shown in Fig. 5.20.

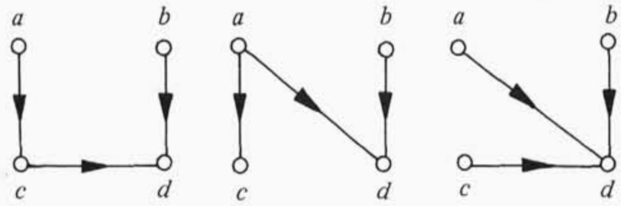

Fig. 5.20.

The terminal characteristics of an n-terminal component are completely specified by a set of $n-1$ equations in $n-1$ pairs of oriented complementary variables $v_i(t)$ and $f_i(t)$ identified by an arbitrarily chosen terminal graph. Such a set of equations called **terminal equations**, together with the terminal graph of the component, forms the **terminal representation** of the component.

The terminal equations are also known as **constitutive relationships** and the terminal representation is also called a **component model**.

The description of the component is not complete without both the terminal equation and the terminal graph.

Beyond selecting a terminal graph on the n terminals of a component to identify the $2(n-1)$ complementary terminal variables for the model, there remains the task of actually establishing the model. In general, one is required to select one set S_1 of $(n-1)$-terminal variables as independent variable functions of time and the remaining set S_2 of $(n-1)$-terminal variables as dependent variable functions of time. The only requirement on the sets S_1 and S_2 is that each contains $(n-1)$-terminal variables. The model consists of a set of $n-1$ relations or, more generally, a mapping showing the variable functions of time in S_2 as a function of time in set S_1. These relations can be given in the form of tables, curves, or mathematical functions; the latter form is, generally speaking, most suitable in all system studies.

Since we shall not apply multi-terminal components in our further considerations, their terminal representations will not be quoted here. The interested reader may find them in excellent monographs such as Shearer *et al.* (1967), Wellstead (1979).

5.3.3 A system graph

As long as the n-terminal component remains isolated from a system, exactly $n-1$ complementary terminal variables are taken as independent variables. When components are interconnected to form a system, these variables are no longer independent—they are constrained by the interconnections. The equation characterizing these constraints is derived from what is called the **system graph**. In the following we shall present a simple operational procedure for generating the system graph.

Consider the arbitrary system of interconnected components represented schematically in Fig. 5.21a. Let the terminal graph of each component be identified as indicated in Fig. 5.21b. The system graph is defined operationally as the collection of edges and vertices obtained by coalescing the vertices of the component terminal graphs in one-to-one

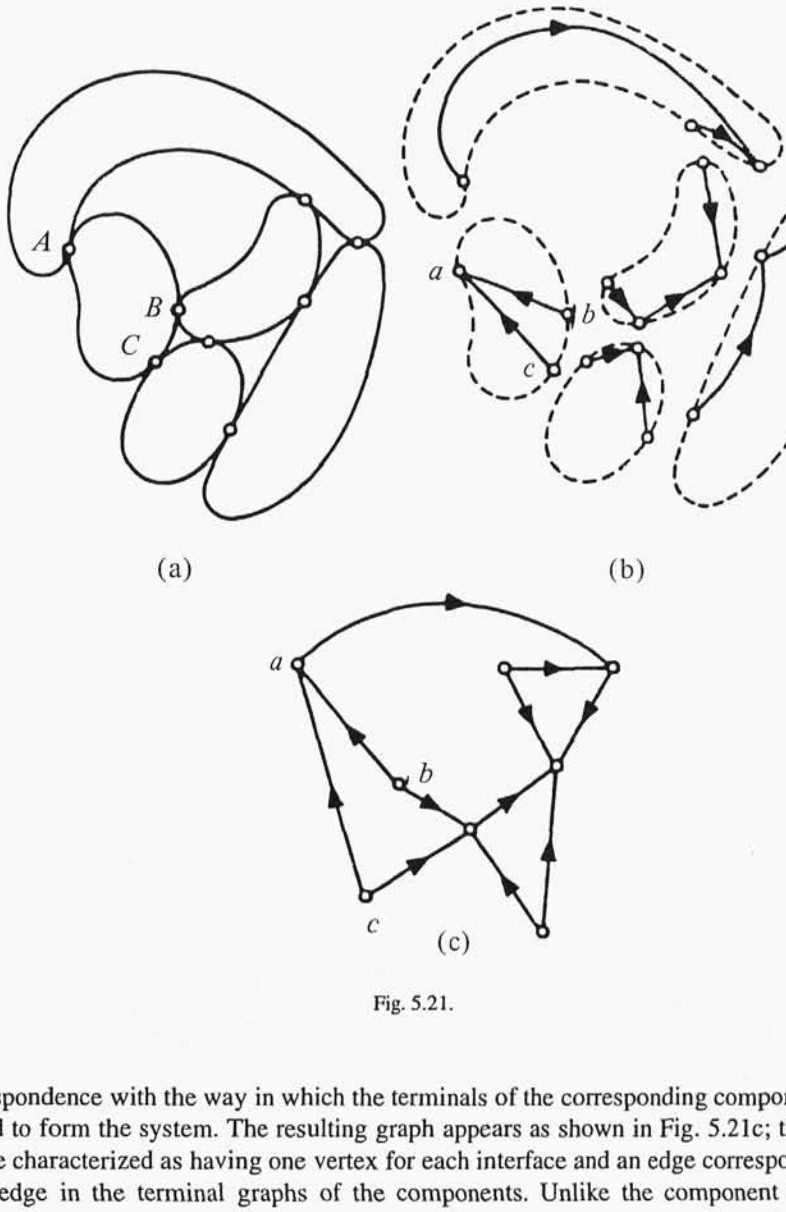

Fig. 5.21.

correspondence with the way in which the terminals of the corresponding components are united to form the system. The resulting graph appears as shown in Fig. 5.21c; the graph can be characterized as having one vertex for each interface and an edge corresponding to each edge in the terminal graphs of the components. Unlike the component terminal graphs, the edges in the system graph may form closed paths, called circuits. These circuits, as we shall see, form the basis of one set of equations which 'bind' the component models together to form a model of the system.

It should be clear that the system graph is unique for any given set of component models and any given interconnection pattern. However, since any one of several terminal graphs can be used as a basis for modelling the characteristics of multi-terminal components, the system graph is not unique until at least the terminal graph in each component model has been specified, i.e. the system graph depends upon the terminal graphs used in the component models.

5.3.4 Formulating techniques

Suppose now that both terminal equations of components and system graph are given. It remains to form the equations of the entire system i.e. a system model.

A physical base for through- and across-variable constraint equations provide conditions which must be satisfied when the elements are combined or connected together. These two conditions will be called **compatibility** and **continuity**. The compatibility requirement is established by the manner in which the elements are connected, and it results in a relation among the various across-variables. For example, when two elements are connected in parallel, compatibility requires equal voltages of velocities at the points where the elements are connected. In a mechanical system, the concept of compatibility means that the geometric constraints imposed on the motion of the elements are expressed. In an electrical circuit, the compatibility requirement is called *Kirchhoff's voltage law*.

Continuity implies that charge is conserved in an electrical circuit or that momentum is conserved in a mechanical system.

As we already know a real physical system consisting of multi-terminal components may be represented by a system graph. Suppose that the system graph consists of n vertices and e edges. In such a graph there are $m = e - n + 1$ fundamental circuits and $n - 1$ fundamental cut-sets.

Both compatibility and continuity requirements give rise to formulation of two following postulates:

Postulate 1 For any circuit row matrix $[\hat{\mathsf{B}}]_i$ and any arbitrary column matrix v of across-variables identified by the e edges of a system graph

$$[\hat{\mathsf{B}}]_i \mathsf{v} = 0, \qquad i = 1, \dots, b. \tag{5.10}$$

Equation (5.10) is called a **circuit equation** of the system graph.

Postulate 2 For any cut-set row matrix $[\hat{\mathsf{Q}}]_i$ and any arbitrary column matrix f of through-variables identified by the e edges of a system graph

$$[\hat{\mathsf{Q}}]_i \mathsf{f} = 0, \qquad i = 1, \dots, q. \tag{5.11}$$

Equation (5.11) is called a **cut-set equation** of the system graph.

Among b equations (5.10) only m are linearly independent, and among q equations (5.11) only $n - 1$ are linearly independent. These systems of linearly independent equations will be called B**-space base** and Q**-space base**, respectively. The B-space base form the equations corresponding to fundamental circuits, while the Q-space base form the equations, which correspond to fundamental cut-sets. Thus we can write

$$\mathsf{Bv} = 0 \tag{5.12}$$

$$\mathsf{Qf} = 0 \tag{5.13}$$

The equations (5.12) and (5.13), combined with the component terminal equations, are basic to any form of formulation techniques. Our objective here is to eliminate variables among these systems of equations in such a way that a system model, which involves solving simultaneously the smallest possible number of equations, can be obtained. We have three possible courses of action. The first, known as the **branch formulation**, involves substituting successively the terminal equations and, if necessary, fundamental circuit equations into the cut-set equations. The second method, which is a technique dual to the branch formulation, is the substitution of the terminal equations and, if necessary, the cut-set equations into the fundamental circuit equations, and is called the **chord formulation**. Lastly, the **branch–chord formulation** requires the substitution of both the cut-set and circuit equations into the terminal equations. In general, the branch–chord method is more difficult than either the branch or chord techniques, and is to be avoided if possible.

For details of the formulation techniques the interested reader is referred to excellent books on the subject such as Koenig and Blackwell (1961), Wellstead (1979) and Roe (1967). We shall simply present one of the techniques by an example.

Example. Let a mechanical system consisting of four bodies with masses m_2, m_5, m_8, m_{10}, five springs with stiffness k_1, k_3, k_4, k_6, k_9, and one damper with viscous damping coefficient b_7 be connected as in Fig. 5.22a. Suppose that the system is driven by a force f_{11}, and the system model is to be determined.

We shall begin the solution of a problem with the introduction of the terminal variables x_A, x_B, x_C, x_D, which mean displacements of bodies m_2, m_5, m_8, m_{10} from their initial positions.

The system graph is drawn by inspection from Fig. 5.22a. The graph is shown in Fig. 5.22b and its edges are oriented arbitrarily (the edge 11 representing f_{11} source being only non-arbitrary orientation). Let us denote the across- and through-variables of the respective elements by $x_1, x_2, \ldots, x_{10}$ and $f_1, f_2, \ldots, f_{11}$. In accordance with the definition of the terminal across-variable and the notation introduced in Fig. 5.22a, we have

$$
\begin{aligned}
&x_1 = x_A - x_G = x_A, && x_2 = x_A - x_G = x_A,\\
&x_3 = x_B - x_G = x_B, && x_4 = x_A - x_B,\\
&x_5 = x_B - x_G = x_B, && x_6 = x_B - x_D,\\
&x_7 = x_B - x_C, && x_8 = x_C - x_G = x_C,\\
&x_9 = x_C - x_D, && x_{10} = x_D - x_G = x_D.
\end{aligned}
\tag{5.14}
$$

The terminal equations of the elements are

$$
\begin{aligned}
&f_1 = k_1 x_A, && f_2 = m_2 \ddot{x}_A, && f_3 = k_3 x_B,\\
&f_4 = k_4 (x_A - x_B), && f_5 = m_5 \ddot{x}_B, && f_6 = k_6 (x_B - x_D),\\
&f_7 = b_7 (\dot{x}_A - \dot{x}_C), && f_8 = m_8 \ddot{x}_C, && f_9 = k_9 (x_C - x_D),\\
&f_{10} = m_{10} \ddot{x}_D, && f_{11} = f_{11}(t).
\end{aligned}
\tag{5.15}
$$

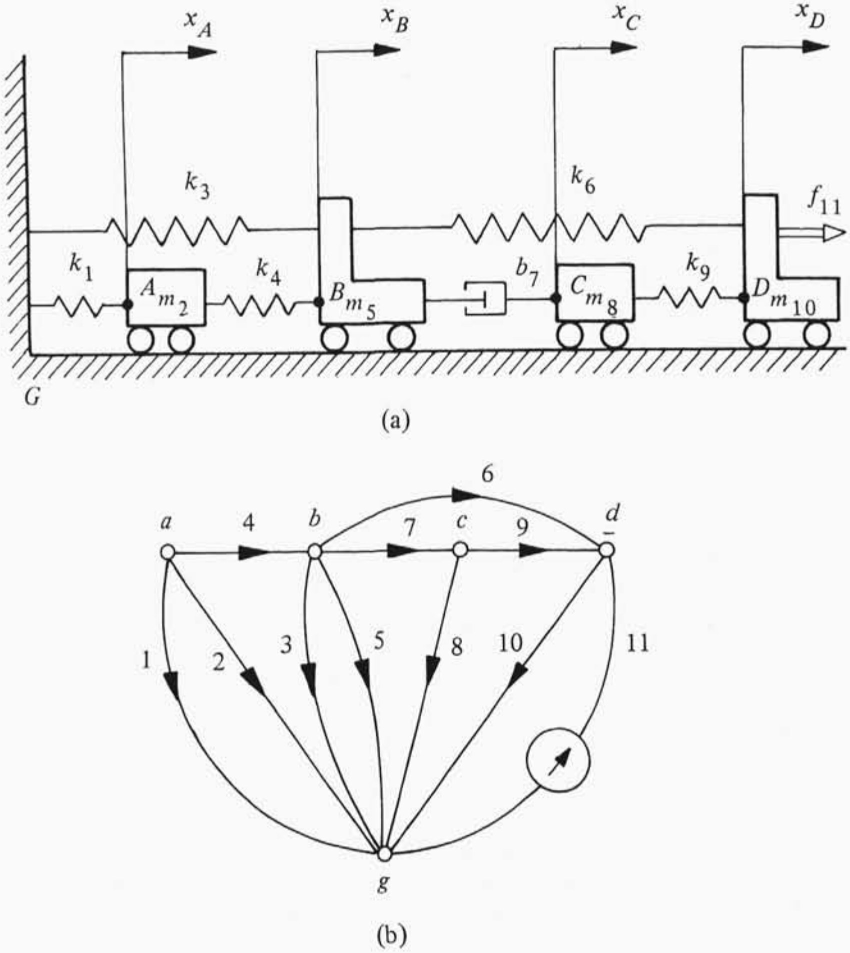

Fig. 5.22.

In the next stage we shall use a cut-set matrix. Suppose that as a formulating tree, $G_t = \{2, 5, 8, 10\}$ has been chosen. It is the so-called **Lagrange tree**, since all its branches have one common vertex. In Fig. 5.23 the fundamental cut-set and their positive orientations are shown. The cut-set matrix for these cut-sets is

$$Q = \begin{array}{c} \begin{array}{ccccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 1 & -1 \end{bmatrix} \end{array} . \quad \begin{array}{c} \\ \text{(i)} \\ \text{(ii)} \\ \text{(iii)} \\ \text{(iv)} \end{array} \qquad (5.16)$$

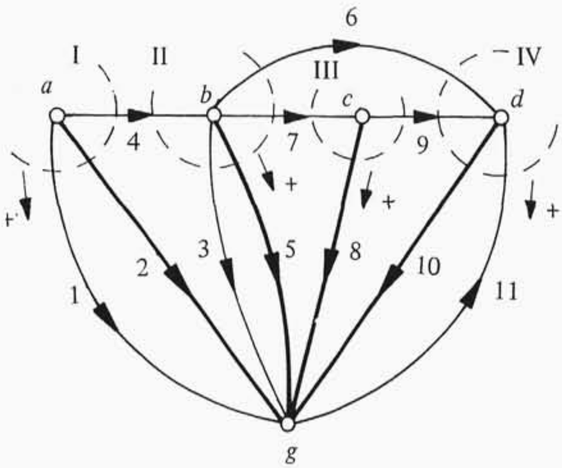

Fig. 5.23.

Performing a multiplication of the matrix Q and the column matrix $f = [f_1, f_2, \ldots, f_{11}]^{\mathrm{T}}$ gives

$$\begin{aligned} f_1 + f_2 + f_4 &= 0 \qquad \text{(i)} \\ f_3 - f_4 + f_5 + f_6 + f_7 &= 0 \qquad \text{(ii)} \\ -f_7 + f_8 + f_9 &= 0 \qquad \text{(iii)} \\ -f_6 - f_9 + f_{10} - f_{11} &= 0. \qquad \text{(iv)} \end{aligned} \tag{5.17}$$

Substituting the terminal equations (5.15) into (5.17) gives

$$\begin{aligned} m_2\ddot{x}_A + k_1 x_A + k_4(x_A - x_B) &= 0, \\ m_5\ddot{x}_B + b_7(\dot{x}_A - \dot{x}_C) + k_3 x_B - k_4(x_A - x_B) + k_6(x_B - x_D) &= 0, \\ m_8\ddot{x}_C - b_7(\dot{x}_A - \dot{x}_C) + k_9(x_C - x_D) &= 0, \\ m_{10}\ddot{x}_D - k_6(x_B - x_D) - k_9(x_C - x_D) - f_{11} &= 0. \end{aligned} \tag{5.18}$$

The system of equations (5.18) may be rewritten in the form

$$\begin{bmatrix} m_2 s^2 + k_1 + k_4 & -k_4 & 0 & 0 \\ -k_4 & m_5 s^2 + b_7 s + k_3 + k_4 + k_6 & -b_7 s & -k_6 \\ 0 & -b_7 s & m_8 s^2 + b_7 s + k_9 & -k_9 \\ 0 & -k_6 & -k_9 & m_{10} s^2 + k_6 + k_9 \end{bmatrix} \begin{bmatrix} x_A \\ x_B \\ x_C \\ x_D \end{bmatrix} =$$

$$= \begin{bmatrix} 0 \\ 0 \\ 0 \\ f_{11} \end{bmatrix} \tag{5.19}$$

where $s = \mathrm{d}/\mathrm{d}t$ and $s^2 = \mathrm{d}^2/\mathrm{d}t^2$.

Consider now the same example, but choosing another formulating tree, say the tree $\{1, 4, 7, 9\}$. In Fig. 5.24 the four fundamental cut-sets associated with this tree and their positive orientations are shown. The cut-set matrix for these cut-sets is as follows:

$$Q = \begin{array}{c} \begin{array}{ccccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 & -1 & 0 & 0 & -1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & -1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & -1 & 1 \end{bmatrix} \end{array} . \begin{array}{c} \\ (\text{v}) \\ (\text{vi}) \\ (\text{vii}) \\ (\text{viii}) \end{array} \tag{5.20}$$

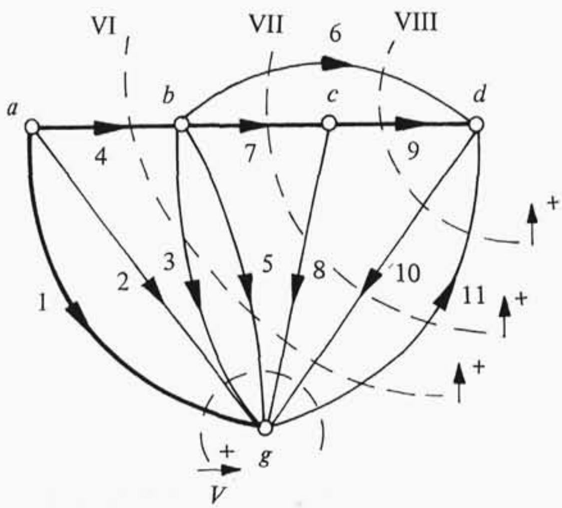

Fig. 5.24.

Performing a multiplication of the matrix Q (5.20) and the column matrix $f = [f_1, f_2, \ldots, f_{11}]^T$ according to equation (5.13) gives

$$\begin{aligned} f_1 + f_2 + f_3 + f_5 + f_8 + f_{10} - f_{11} &= 0 && (\text{v}) \\ -f_3 - f_4 - f_5 - f_8 - f_{10} + f_{11} &= 0 && (\text{vi}) \\ f_6 + f_7 - f_8 - f_{10} + f_{11} &= 0 && (\text{vii}) \\ f_6 + f_9 - f_{10} + f_{11} &= 0. && (\text{viii}) \end{aligned} \tag{5.21}$$

Comparing equations (5.17) with (5.21) we can observe that each equation from a system (5.21) is a linear combination of equations (5.17). Indeed we have

$$(\text{v}) = (\text{i}) + (\text{ii}) + (\text{iii}) + (\text{iv})$$
$$(\text{vi}) = -(\text{ii}) - (\text{iii}) - (\text{iv})$$
$$(\text{vii}) = -(\text{iii}) - (\text{iv})$$
$$(\text{viii}) = -(\text{iv}).$$

Thus, we may conclude that the equations obtained after substituting the terminal equations (5.15) into (5.21) and reordering them in a matrix form, i.e.

$$\begin{bmatrix} m_2 s^2 + k_1 & m_5 s^2 + k_3 & m_8 s^2 & m_{10} s^2 \\ k_4 & -m_5 s^2 - k_3 - k_4 & -m_8 s^2 & -m_{10} s^2 \\ 0 & b_7 s + k_6 & -m_8 s^2 - b_7 s & m_{10} s^2 - k_6 \\ 0 & -k_6 & -k_9 & -m_{10} s^2 - k_6 - k_9 \end{bmatrix} \begin{bmatrix} x_A \\ x_B \\ x_C \\ x_D \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ f_{11} \end{bmatrix}, \tag{5.22}$$

form a system model equivalent to the model (5.19).

We have presented one formulation technique using a relatively simple example—a system with four masses. This particular example, though, is probably too small to demonstrate the advantages of the linear graph modelling method. Still, even in this example we have obtained two models in a relatively simple, systematic manner. It should therefore be emphasized that the more complex a system is the greater are the advantages from application of the method. Besides this, when we need to model a system composed of subsystems of varied physical nature, such as mechanical and electrical, it is very convenient to use a unified method, common to both subsystems. The techniques involving linear graphs provide the appropriate tool for modelling such systems. For a deeper study the interested reader is referred to such books as those mentioned in section 5.3.2.

5.3.5 The limits of a method

We have presented one from three relatively popular graph-theoretical modelling methods. Although the foundations as well as the operational procedures of two other methods, i.e. *signal flow graph method* and *bond graph method*, are quite different, all of them require the terminal equations and terminal representations of system components. There is no problem in forming the terminal equations of relatively simple mechanical elements such as a rigid body with simple types of motion: translational movement along a fixed direction and rotational movement about a fixed axis, with two-terminal mechanical elements such as a linear spring and a damper if their terminals move along a fixed direction. Unfortunately mechanical elements appear in much more complicated situations and they cause various difficulties. To understand what causes the difficulties let us consider a certain fragment of a *kinematic chain* of rigid bodies as in Fig. 5.25.

Suppose the bodes B_i, B_j and their **direct carriers**, i.e. the bodies $B_{i\sim}$ and $B_{j\sim}$, can move in a rotational sense with angular coordinates φ_i, φ_j, $\varphi_{i\sim}$ and $\varphi_{j\sim}$, respectively. (We have here used a superscript '$\sim$' to denote the direct carrier. A detailed explanation of the meaning of this notation will be given in section 5.4.4.) Suppose some bodies of a system may additionally move in a translational sense relative to each other, and let $B_{i\sim}$ and B_j be two such bodies. Let this component of motion of interest be a simple linear translation and let the coordinates describing this degree of freedom be $x_{i\sim}$, x_j, respectively (see Fig. 5.25). Let there be a linear spring with the stiffness k_v between the bodies B_i and B_j.

The common problem when multi-body systems are considered is to generate the equations of motion in terms of the generalized coordinates and generalized velocities,

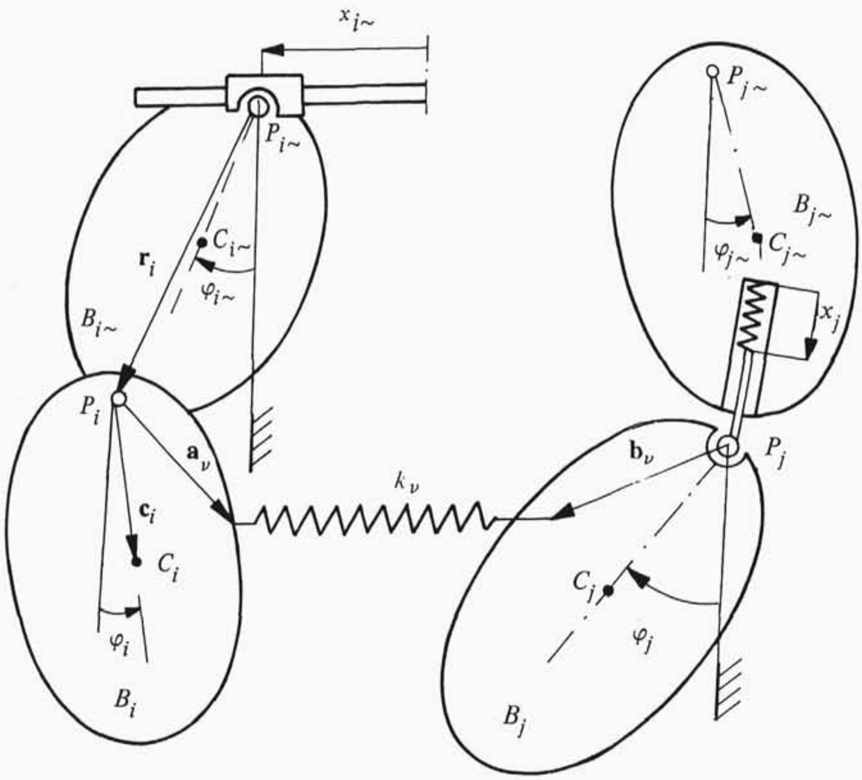

Fig. 5.25.

i.e. in terms of angular and linear displacements and velocities. This quite common requirement causes difficulties which make the application of the linear graph method useless. To see why, let us remember that within this method the mathematical model of each body is required, and that this model has to be independent of the manner in which the component is interconnected with other components to form the system. In the example considered, where we have a system of many rigid bodies, the motion of a body depends directly on the motion of its **carriers**, i.e. those bodies whose motions determine, via constraints, that of the given body. For example the total displacement as well as the absolute acceleration of the body B_i depends on the motion of the body $B_{i\sim}$ and in turn upon its carriers. This means that since the acceleration of the mass centre C_i depends, among other things, on the coordinates φ_i, $\varphi_{i\sim}$, $x_{i\sim}$, the inertia characteristics of a body B_i will appear in many equations of motion; specifically not only in the equation associated with the φ_i coordinate, but also in the equations associated with coordinate $\varphi_{i\sim}$, $x_{i\sim}$ and those which represent the motion of all carriers of the body B_i. Conversely, in the equation associated with coordinate φ_i, terms will appear depending on the coordinates associated with the carriers of the body B_i as well as those carried by the body B_i. Similar difficulties appear when we try to include elastic forces in the equations of motion. As is well known, the force in a linear spring with a stiffness k_ν is proportional to its extension.

The problem now relies upon the expression of the spring extension in terms of generalized coordinates. The positions where the springs are attached depend on many coordinates and it may require quite serious calculation to determine the current extension of a spring. Additionally the spring attachment point can assume any arbitrary position in the body. It means that the body may have an arbitrary number of terminals and only current data concerning the position of vectors $\mathbf{a}_\nu$ (or $\mathbf{b}_\nu$) with respect to body B_i (or B_j), establish an actual terminal position.

From what we have said above it follows that the mechanical elements appearing in a kinematic chain of bodies don't fulfil the fundamental axiom of system theory, on which the linear graph method hinges.

Additionally, three-dimensional rotation between two bodies cannot be represented simply by three independent numbers; the numerical values depend also on the sequence in which the three rotations occur.

Both features of multi-body systems described above affect the coupling of equations of motions and their complexity. Moreover, since the models of separate mechanical elements are useless when they are interconnected to form a system, the linear graph method is not adequate in modelling problems involving many rigid bodies.

In the next section we will present a method which provides a useful aid for a modelling of certain classes of rigid-body systems.

5.4 MODELLING OF RIGID-BODY SYSTEMS

5.4.1 Introductory remarks†

The problem of multi-rigid body system modelling was investigated, among others, by **J. Wittenburg**, and he solved it using the Newtonian approach combined with graph-theoretical aids. Although we shall present another method, most of the introductory definitions and comments made by J. Wittenburg in his excellent monograph (Wittenburg (1977)) will be useful for us. We shall therefore follow his development and his definitions in this section.

Mechanical systems investigated in most student textbooks consist of either a single rigid body or several rigid bodies in some particularly simple geometric configuration. The important role they play in classical mechanics is due to the fact that their equations of motion can be integrated in closed form. However, the engineer in his everyday practice is confronted with an endless variety of much more complex systems. To mention only a few examples, one may think of linkages in machines, of steering mechanisms in cars, of railway trains consisting of elastically connected cars, of walking machines and manipulators, etc. The assumption that the individual bodies of such systems are rigid is an idealization which may or may not be acceptable, depending largely on the kind of problem under investigation. Thus, in a crank-and-slider mechanism, the seemingly rigid connecting rod has to be treated, as an elastic member when its forced bending vibrations are of concern. At the other extreme, the human body, which is composed of

† A substantial part of the reasoning in this section, and the introduction of the mathematical description of the interconnection structure (section 5.4.4) is based on items from the book *Dynamics of Systems of Rigid Bodies* by Jeans Wittenburg. The authors make grateful acknowledgment to Teubner Verlag for permission to quote these items from the above-named book.