Because of regular labelling, P is upper triangular and because of the fact that the graph is regular directed, all nonzero entries of P have the same sign, in this case positive.

5.4.5 The kinetic energy

The kinetic energy for a point mass is defined as $T = \frac{1}{2} m\mathbf{v}^2$, where $\mathbf{v}$ is the absolute velocity of m, i.e. its velocity relative to an inertial reference base. For a rigid body, as for any deformable body, the kinetic energy is the integral

$$T = \frac{1}{2} \int_m \mathbf{v}^2 \, \mathrm{d}m, \tag{5.43}$$

where now $\mathbf{v}$ is the absolute velocity of mass particle $\mathrm{d}m$ of a body.

Let us now consider a rigid body in arbitrary motion (Fig. 5.36). The absolute velocity $\mathbf{v}$ of a mass particle $\mathrm{d}m$ is

$$\mathbf{v} = \mathbf{v}_P + \mathbf{\Omega} \times \mathbf{r}, \tag{5.44}$$

where $\mathbf{v}_P$ is the absolute velocity of the reference point P,

$\mathbf{\Omega}$ is the absolute angular velocity of the body, and

$\mathbf{r}$ is the radius vector from P to the mass particle.

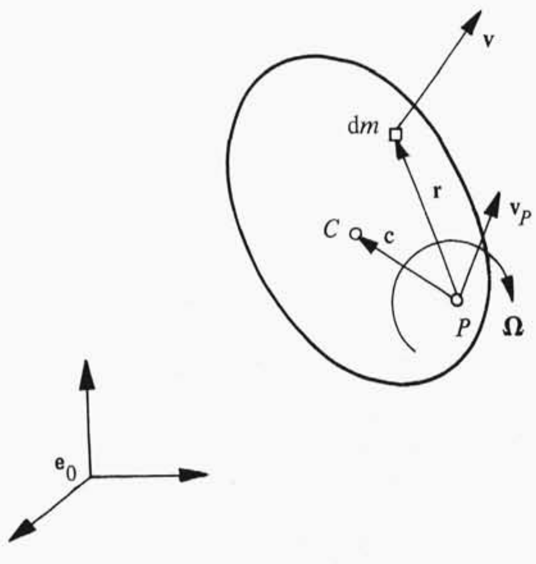

Fig. 5.36.

The radius vector $\mathbf{c} = \overrightarrow{PC}$ in Fig. 5.36 indicates the body mass centre C. Evaluation of the integral (5.43) yields

$$T = \frac{1}{2} m\mathbf{v}_P^2 + m\mathbf{v}_P\mathbf{u} + \frac{1}{2} \Omega^{\mathrm{T}} \mathsf{J}^P \Omega, \tag{5.45}$$

where Ω is the column matrix representing components of an absolute angular velocity vector $\boldsymbol{\Omega}$ of the body in the base $\mathbf{e}_0$;

$\mathbf{u} = \boldsymbol{\Omega} \times \mathbf{c}$ is the relative velocity of the body mass centre, C, in the rotation motion with respect to P;

m is the mass of the body, and

J^P is the inertia tensor of the body with respect to P.

The expression (5.45) becomes particularly simple if either the body-fixed point P is also fixed in inertial space or the mass centre C is used as reference point P. In the former case $\mathbf{v}_P = \mathbf{0}$ so that the first two terms equal zero. In the latter case $\mathbf{u} = \mathbf{0}$, so that the central term vanishes and the expression (5.45) takes the well-known form

$$T = \tfrac{1}{2} m \mathbf{v}_c^2 + \tfrac{1}{2} \Omega^{\mathrm{T}} \mathsf{J}^C \Omega = T_{\mathrm{trans}} + T_{\mathrm{rot}}, \tag{5.46}$$

where $\mathbf{v}_C$ is the absolute velocity of the mass centre C, and J^C is the inertia tensor of the body with respect to the mass centre C. However, for a reason which will be clear later, we shall use, in what follows, the formula (5.45) only.

For an arbitrary system of n rigid bodies, the kinetic energy is

$$T = \sum_{i=1}^{n} T_i = \tfrac{1}{2} \sum_{i=1}^{n} m \mathbf{v}_{P_i}^2 + \sum_{i=1}^{n} m \mathbf{v}_{P_i} \mathbf{u}_i + \tfrac{1}{2} \sum_{i=1}^{n} \Omega_i^{\mathrm{T}} \mathsf{J}^{P_i} \Omega_i \tag{5.47}$$

In order to simplify the notation, we shall adopt the following convention

$$\mathbf{v}_i \equiv \mathbf{v}_{P_i}, \quad \mathsf{J}_i \equiv \mathsf{J}^{P_i}, \quad T_i^{\nabla} \equiv \tfrac{1}{2} m_i \mathbf{v}_i^2, \quad T_i^{*} \equiv m_i \mathbf{v}_i \mathbf{u}_i, \quad T_i^{0} \equiv \tfrac{1}{2} \Omega_i^{\mathrm{T}} \mathsf{J}_i \Omega_i$$

Thus

$$T = T^{\nabla} + T^{*} + T^{0}, \tag{5.48}$$

where

$$T^{\nabla} = \sum_{i=1}^{n} T_i^{\nabla}, \qquad T^{*} = \sum_{i=1}^{n} T_i^{*}, \qquad T_i^{0} = \sum_{i=1}^{n} T_i^{0}. \tag{5.49}$$

It is obvious that the kinetic energy of an arbitrary system of rigid bodies depends on the number of bodies, their inertial characteristics m_i, J_i and a group of velocities $\mathbf{v}_i$, $\mathbf{u}_i$, and $\boldsymbol{\Omega}_i$.

Consider now a system of interconnected rigid bodies forming an open kinematic chain, such as in Fig. 5.37, and let P_i be chosen as the joint on the ith body. Suppose that the bodies of a system may move in both the translational and rotational senses. For simplicity, we shall assume that: (1) each rotational body motion can be expressed in terms of angular coordinates measured from the same line in the same direction, and (2) the external base is immovable. Now the vectors $\mathbf{v}_i$, $\mathbf{u}_i$, $\boldsymbol{\Omega}_i$ depend, in general, on the motion of the direct carrier. In order to establish relationships between the velocity $\mathbf{v}_i$ and the characteristics of the motion of a direct carrier, let us consider Fig. 5.38.

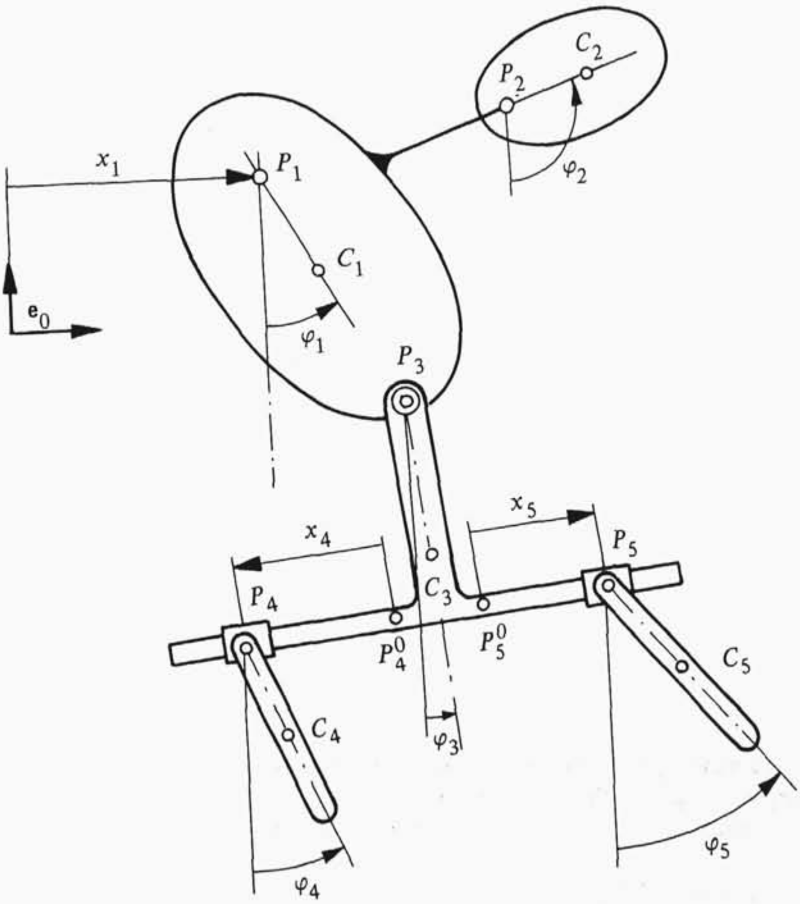

Fig. 5.37.

This figure shows two adjacent bodies B_i and B_j. The body B_j slides along curved slide-way with a relative velocity $\boldsymbol{\rho}_j'$. The problem is to determine the absolute velocity $\mathbf{v}_j$ of the point P_j using the velocity characteristics of a direct carrier. We have

$$\mathbf{v}_j = \mathbf{v}_i + \boldsymbol{\Omega}_i \times (\mathbf{r}_j + \boldsymbol{\rho}_j) + \boldsymbol{\rho}_j'. \tag{5.50}$$

Since the body B_i is a direct carrier of the body B_j the index $i = j^{\sim}$. Introducing the notation

$$\mathbf{a}_j = \boldsymbol{\Omega}_{j\sim} \times \mathbf{r}_j, \qquad \mathbf{b}_j = \boldsymbol{\Omega}_{j\sim} \times \boldsymbol{\rho}_j, \tag{5.51}$$

$$\mathbf{w}_j = \mathbf{a}_j + \mathbf{b}_j + \boldsymbol{\rho}_j', \tag{5.52}$$

the relation (5.50) may be expressed in the form

$$\mathbf{v}_j = \mathbf{v}_{j\sim} + \mathbf{w}_j. \tag{5.53}$$

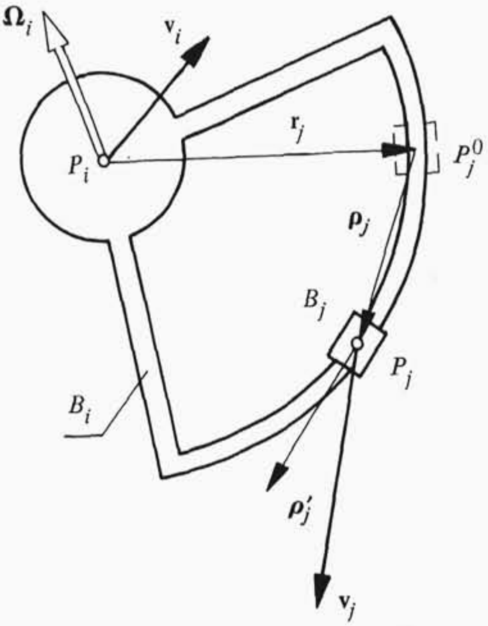

Fig. 5.38.

There is a simple physical interpretation of the velocity $\mathbf{w}_j$. It is the velocity of the joint point P_j with respect to the base constrained with P_i.

Since relation (5.53) holds for any arbitrary j, we have

$$\mathbf{v}_j = \mathbf{v}_0 + \sum_{\nu=1}^{j} \mathbf{w}_\nu, \tag{5.54}$$

where index ν change its value according to the labels of the arcs in the path between vertices s_j and s_0. This can be expressed as

$$\mathbf{v}_j = \left\lfloor \mathsf{P}^{\mathrm{T}} \right\rfloor_j \mathbf{w} \tag{5.55}$$

where $\left\lfloor \mathsf{P}^{\mathrm{T}} \right\rfloor_j$ is the jth row of P^{T} and $\mathbf{w} = [\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n]^{\mathrm{T}}$. Thus the column matrix $\mathbf{v} = [\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n]^{\mathrm{T}}$ can be expressed as

$$\mathbf{v} = \mathsf{P}^{\mathrm{T}}\mathbf{w}. \tag{5.56}$$

Two components of (5.48), namely T^{∇} and T^{*}, can be calculated by use of (5.56). We have

$$T^{\nabla} = \tfrac{1}{2}\sum_{i=1}^{n} m\mathbf{v}_i^2 = \tfrac{1}{2}\mathbf{v}^{\mathrm{T}}\mathsf{m}\mathbf{v} \tag{5.57}$$

where $\mathsf{m} = \mathrm{diag}\,(m_1, m_2, \ldots, m_n)$. Hence

$$T^{\nabla} = \tfrac{1}{2}\,\mathbf{w}^{\mathrm{T}}\mathsf{PmP}^{\mathrm{T}}\mathbf{w}, \tag{5.58}$$

or, more briefly,

$$T^{\nabla} = \tfrac{1}{2}\,\mathbf{w}^{\mathrm{T}}\mathsf{M}\mathbf{w}, \tag{5.59}$$

where $\mathsf{M} = \mathsf{PmP}^{\mathrm{T}}$ is an $n \times n$ constant matrix. There is a simple physical interpretation of the elements M_{ij} of the matrix M. To present it briefly (for details see Arczewski (1987a)), let us introduce the following designation: $M_i = m_i$ + *mass of those bodies whose carrier is the ith body*. The masses M_i, $i = 1, \ldots, n$, can be regarded as the elements of a diagonal matrix

$$\overset{*}{\mathsf{M}} = \mathrm{diag}\,(M_1, \ldots, M_n). \tag{5.60}$$

In the matrix M the diagonal elements $M_{ii} = M_i$, and the off-diagonal elements are

$$M_{ij} = \begin{cases} 0, & \text{if in the graph } G \text{ there is not a path directed to the vertex } s_0, \\ & \text{and consisting simultaneously of both vertices } s_i \text{ and } s_j, \\ M_l, & \text{where } l = \max\,(i, j), \text{ if in the graph } G \text{ there is a path directed} \\ & \text{to the vertex } s_0 \text{ and consisting simultaneously of both} \\ & \text{vertices } s_i \text{ and } s_j. \end{cases}$$

Now we proceed to the determination of the second component of the sum (5.48), i.e.

$$T^{*} = \sum_{i=1}^{n} m_i \mathbf{v}_i \mathbf{u}_i = \mathbf{v}^{\mathrm{T}}\mathsf{m}\mathbf{u}, \tag{5.61}$$

where

$$\mathbf{u} = [\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n]^{\mathrm{T}}, \quad \text{and} \qquad \mathbf{u}_l = \boldsymbol{\Omega}_i \times \mathbf{c}_i. \tag{5.62}$$

Substituting (5.56) into (5.61) we obtain

$$T^{*} = \mathbf{w}^{\mathrm{T}}\mathsf{Pm}\mathbf{u}, \tag{5.63}$$

According to the assumption about the manner in which the body rotational motion is measured, the angular velocities $\boldsymbol{\Omega}_i$ do not depend on each other. Therefore the third component of the kinetic energy (5.48) is the simplest one, i.e.

$$T^{0} = \tfrac{1}{2}\sum_{i=1}^{n} \Omega_i^{\mathrm{T}} \mathsf{J}_i \Omega_i = \tfrac{1}{2}\,\Omega^{\mathrm{T}}\mathsf{J}\Omega, \tag{5.64}$$

where Ω is a column matrix $3n \times 1$ formed from column matrices Ω_i representing angular velocities vectors $\boldsymbol{\Omega}_i$, and J is a block diagonal matrix with dimension $3n$ by $3n$ composed from matrices J_i.

Finally, the kinetic energy T can be expressed as

$$T = \frac{1}{2}\mathbf{w}^{\mathrm{T}}\mathsf{M}\mathbf{w} + w^{\mathrm{T}}\mathsf{Pm}\mathbf{u} + \frac{1}{2}\Omega^{\mathrm{T}}\mathsf{J}\Omega. \tag{5.65}$$

This expression (5.65) provides only a general frame for the determination of the kinetic energy of a particular class of systems in terms of generalized coordinates and generalized velocities. In order to use this, we should further specify a system, i.e. we should introduce generalized coordinates that take into account all constraints, then specify the location of joints, mass centres and all inertia characteristics. However, since our main objective is not to determine the kinetic energy by means of formula (5.65) but to use it as a starting point for the mathematical model determination, we shall not present any example of kinetic energy determination, and the interested reader is referred to Arczewski (1987a).

5.4.6 The potential energy of gravity forces

Consider a system of n rigid bodies situated in gravity field (Fig. 5.39). The potential energy of gravity forces of the system with respect to a certain level e.g. that of P_0 may be expressed as

$$V^{\mathrm{G}} = -\sum_{i=1}^{n} m_i \mathbf{g}\mathbf{r}_{c_i}, \tag{5.66}$$

where m is the mass of the body, $\mathbf{g}$ is the vector gravitational acceleration, $\mathbf{r}_{c_i}$ is the position vector of the mass centre of the ith body.

The potential energy (5.66) may be written as

$$V^{\mathrm{G}} = -\mathbf{g}\tilde{\mathsf{m}}^{\mathrm{T}}\mathbf{r}_c \tag{5.67}$$

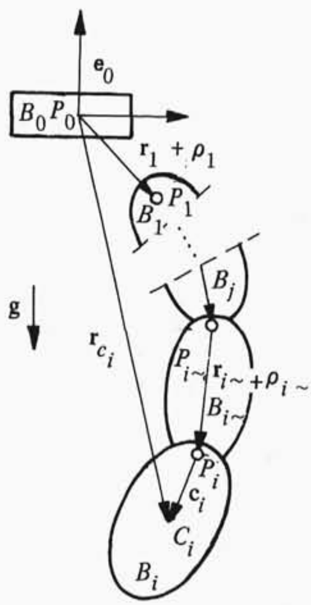

Fig. 5.39.

where

$$\tilde{\mathsf{m}} = \mathsf{m}\mathsf{I}_n = [m_1 . m_2, \ldots, m_n]^{\mathrm{T}}, \qquad \mathbf{r}_c = \left[\mathbf{r}_{c_1}, \mathbf{r}_{c_2}, \ldots, \mathbf{r}_{c_n}\right]^{\mathrm{T}}. \tag{5.68}$$

For a system of interconnected rigid bodies, the position vector $\mathbf{r}_{c_i}$ depends on the configurations of all those bodies which are carriers of the ith body, so we have

$$\mathbf{r}_{c_i} = \mathbf{r}_i + \boldsymbol{\rho}_1 + \ldots + \mathbf{r}_1 + \boldsymbol{\rho}_i + \mathbf{c}_i = \sum_{\nu=1}^{i} (\mathbf{r}_\nu + \boldsymbol{\rho}_\nu) + \mathbf{c}_i, \tag{5.69}$$

where

$$\mathbf{r}_\nu = \overrightarrow{P_{\nu\sim} P_\nu^0}, \quad \boldsymbol{\rho}_\nu = \overrightarrow{P_\nu^0 P_\nu}, \quad \mathbf{c}_i = \overrightarrow{P_i C_i},$$

and the current summation index ν assumes values from 1 to i, but only from the set of labels of carriers of the ith body. The relation (5.69) is easily expressible in terms of the path matrix, P, as follows:

$$\mathbf{r}_{c_i} = \varepsilon_i^{\mathrm{T}} \mathsf{P}^{\mathrm{T}} (\tilde{\mathbf{r}} + \tilde{\boldsymbol{\rho}}) + \mathbf{c}_i, \tag{5.70}$$

where ε_i is an isolating $(n \times 1)$ matrix, so $\varepsilon_i^{\mathrm{T}} \mathsf{P}^{\mathrm{T}}$ is the ith row of P^{T}, and $\tilde{\mathbf{r}} = [\mathbf{r}_2, \mathbf{r}_2, \ldots, \mathbf{r}_n]^{\mathrm{T}}$, $\tilde{\boldsymbol{\rho}} = [\rho_1, \rho_2, \ldots, \rho_n]^{\mathrm{T}}$. Consequently, the column matrix $\mathbf{r}_c$ can be expressed as

$$\mathbf{r}_c = \mathsf{P}^{\mathrm{T}} (\tilde{\mathbf{r}} + \tilde{\boldsymbol{\rho}}) + \tilde{\mathbf{c}}, \tag{5.71}$$

where $\tilde{\mathbf{c}} = [\mathbf{c}_{1,} \mathbf{c}_2, \ldots, \mathbf{c}_n]^{\mathrm{T}}$.

Introducing (5.71) into (5.67), we have

$$V^{\mathrm{G}} = -\mathrm{g}\tilde{\mathsf{m}}^{\mathrm{T}} \mathsf{P}^{\mathrm{T}} (\tilde{\mathbf{r}} + \tilde{\boldsymbol{\rho}}) - \mathrm{g}\tilde{\mathsf{m}}^{\mathrm{T}} \tilde{\mathbf{c}}. \tag{5.72}$$

Comparing (5.72) with (5.67), we note that the initial form, i.e. (5.67), is simpler than the final form (5.72). However, the latter form (5.72) has an advantage, which will become very clear when it is applied to an interconnected system of bodies: namely, each component of the column matrices $\tilde{\mathbf{r}}$, $\tilde{\boldsymbol{\rho}}$, and $\tilde{\mathbf{c}}$ is relatively easily expressed in terms of the generalized coordinates usually introduced to describe the configuration of the bodies. At the same time, the components of the column matrix $\mathbf{r}_c$ are complex functions of these coordinates. Therefore a differentiation of the expression (5.72) with respect to generalized coordinates is many times easier than differentiation of (5.67). The particular expressions for vectors $\mathbf{r}_i$, $\boldsymbol{\rho}_i$, $\mathbf{c}_i$ or $\mathbf{r}_{c_i}$ depend, of course, on the particular choice of coordinate system. Now we will not specify any particular system of coordinates, but we shall assume two features which the system, when considered further, needs to possess. It is well known that the position of a rigid body is completely described when the position of any one reference point and its angular orientation with respect to a non-rotating frame are known. In connection with this, we shall assume that a system of coordinates describing the motion of the ith body consists of (1) relative linear translations of the reference point P_i referred to a frame affixed to a direct carrier of the ith body, i.e. to the $i^{\sim}$th body; (2), rotational (angular) displacements referred to a basis whose axis remains parallel to the external basis, $\mathbf{e}_0$. In a system of coordinates possessing both of the above features, the vector $\mathbf{c}_i$ depends only on the angular coordinate of the ith body, while the vectors $\mathbf{r}_i$, $\boldsymbol{\rho}_i$ depend on the translational coordinates of the ith body and the angular coordinates of its direct carrier, i.e. the $i^{\sim}$th body.

For purposes of economy, we will not deal with the determination of the potential energy of springs. This problem, however, has been solved and details of the solution can be found in Arczewski (1987b).

5.4.7 The equations of motion

The main task of this section is the determination of equations of motion for open kinematic rigid-body chains. However, solving a quite general problem, even within the considered class of systems, would be too difficult, since three-dimensional rotations of a body involve complex mathematics. An important group of mechanisms are those whose motion is constrained to a plane. Therefore our further considerations will be limited to that subclass of systems. Thus we shall assume that:

(1) all bodes can move in a rotational sense, with angular coordinates $\varphi_1, \varphi_2, \ldots, \varphi_n$, all measured in the same sense (compare Fig. 5.37);
(2) some or all bodies may additionally move in a translational sense relative to each other. Let this second component of motion be a simple linear translational and let the coordinates describing these degrees of freedom be $x_1, x_2, \ldots, x_n$.

For a better comprehension of the following we now explain the main steps leading from the expressions for the kinetic and potential energies given by formulae (5.65) and (5.72) to the final mathematical model of a considered system of rigid bodies.

Bearing in mind the application of Lagrange's equations of the second kind, the expressions (5.65) and (5.72) ought to be brought to a form convenient for further differentiation with respect to the generalized coordinates q_σ, the generalized velocities $\dot{q}_\sigma$, and then with respect to time t. In the first step, the general formulae (5.65) and (5.72) have to be adapted to that of a special case of planar motion. In particular, the vectors appearing in all the formulae should be decomposed as far as possible and then expressed in such a manner as to disclose their dependence on the generalized coordinates q_σ and the generalized velocities $\dot{q}_\sigma$.

In the second step, the kinetic and potential energies must be expressed as a product of certain matrices, i.e. in a form enabling subsequent, relatively easy, differentiations. Finally, the third step, the required differentiations of the expressions for kinetic and potential energies, are performed and the expressions reduced if possible.

Let us now consider two adjacent bodies of a certain kinematic chain (Fig. 5.40). Let the body $B_{i\sim}$ be a direct carrier of the body B_i, let $P_{i\sim}$, P_i, $C_{i\sim}$, C_i be reference points and mass centres of these bodies as shown in Fig. 5.40, and let us denote the absolute angular velocities of the bodies by $\boldsymbol{\Omega}_{i\sim}$, $\boldsymbol{\Omega}_i$, the velocity of P_i with respect to the base constrained with $P_{i\sim}$ by $\mathbf{w}_i$, and the relative velocity of the ith body mass centre C_i in the rotational motion with respect to P_i by $\mathbf{u}_i$. For the considered case of planar motion, we have

$$\mathbf{u}_i = \boldsymbol{\Omega}_i \times \mathbf{c}_i = c_i \dot{\varphi}_i \hat{\mathbf{u}}_i, \qquad \mathbf{w}_i = \mathbf{a}_i + \mathbf{b}_i + \mathbf{x}'_i, \tag{5.73}$$

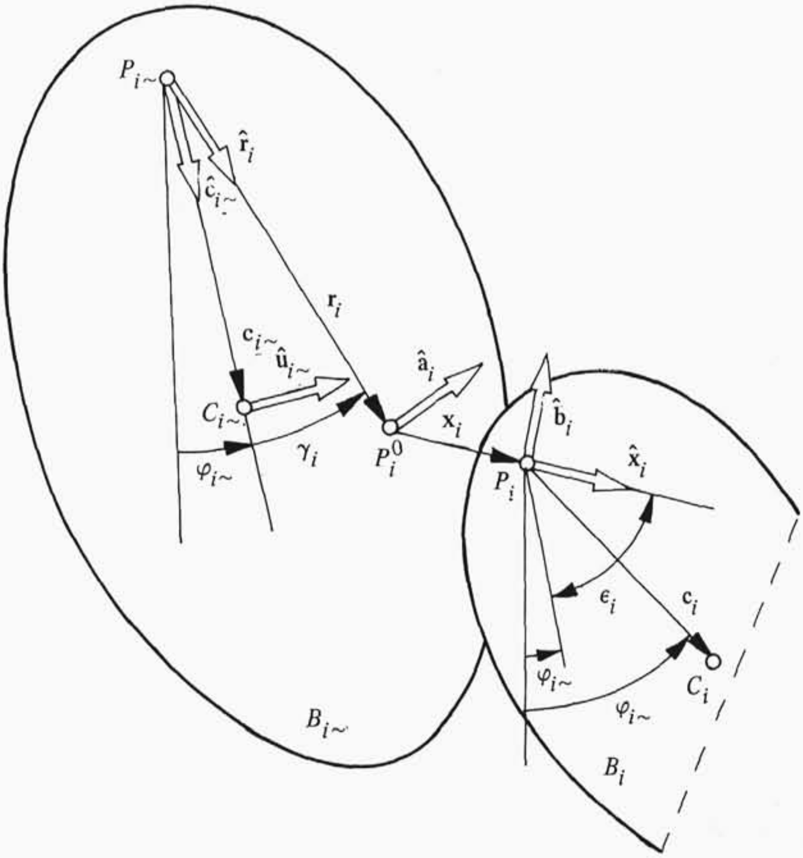

Fig. 5.40.

where

$$\mathbf{a}_i = \mathbf{\Omega}_{i\sim} \times \mathbf{r}_i = r_i \dot{\varphi}_{i\sim} \hat{\mathbf{a}}_i, \qquad \mathbf{b}_i = \mathbf{\Omega}_{i\sim} \times \mathbf{x}_i = x_i \dot{\varphi}_{i\sim} \hat{\mathbf{b}}_i, \qquad \mathbf{x}'_i = \dot{x}_i \hat{\mathbf{x}}_i. \tag{5.74}$$

The system of unit vectors within rigid bodies as well as vectors $\mathbf{c}_i$, $\mathbf{r}_i$, $\mathbf{x}_i$ are shown in Fig. 5.40 and the angular positions of the unit vectors are shown in Fig. 5.41.

In order to develop and then to prove the equations of motion, let us introduce:

— the diagonal matrices of unit vectors:

$$\begin{aligned} &\hat{\mathbf{a}} = \operatorname{diag} \hat{\mathbf{a}}_i, \qquad \hat{\mathbf{b}} = \operatorname{diag} \hat{\mathbf{b}}_i, \qquad \hat{\mathbf{c}} = \operatorname{diag} \hat{\mathbf{c}}_i, \\ &\hat{\mathbf{r}} = \operatorname{diag} \hat{\mathbf{r}}_i, \qquad \hat{\mathbf{u}} = \operatorname{diag} \hat{\mathbf{u}}_i, \qquad \hat{\mathbf{x}} = \operatorname{diag} \hat{\mathbf{x}}_i; \end{aligned} \tag{5.75}$$

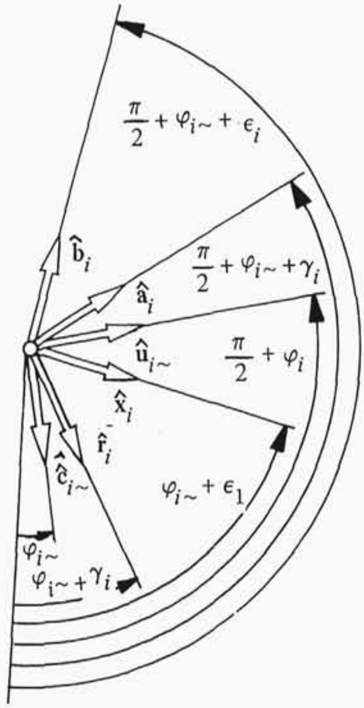

Fig. 5.41.

— the diagonal matrices related to different velocities and position vectors

$$\begin{aligned} &\mathbf{a} = \operatorname{diag} \mathbf{a}_i, \quad &&\mathbf{b} = \operatorname{diag} \mathbf{b}_i, \quad &&\mathbf{c} = \operatorname{diag} \mathbf{c}_i, \\ &\mathbf{r} = \operatorname{diag} \mathbf{r}_i, &&\mathbf{u} = \operatorname{diag} \mathbf{u}_i, &&\mathbf{x} = \operatorname{diag} \mathbf{x}_i, \\ &\dot{\mathbf{x}} = \operatorname{diag} \dot{\mathbf{x}}_i, &&\dot{\varphi} = \operatorname{diag} \dot{\varphi}_i, &&i = 1,\ldots,n; \end{aligned} \tag{5.76}$$

— the column matrices associated with diagonal matrices (5.76):

$$\begin{aligned} &\tilde{\mathbf{a}} = \{\mathbf{a}\} = [\mathbf{a}_1,\ldots,\mathbf{a}_n]^{\mathrm{T}}, \quad &&\tilde{\mathbf{b}} = \{\mathbf{b}\} = [\mathbf{b}_1,\ldots,\mathbf{b}_n]^{\mathrm{T}}, \\ &\tilde{\mathbf{c}} = \{\mathbf{c}\} = [\mathbf{c}_1,\ldots,\mathbf{c}_n]^{\mathrm{T}}, &&\tilde{\mathbf{r}} = \{\mathbf{r}\} = [\mathbf{r}_1,\ldots,\mathbf{r}_n]^{\mathrm{T}}, \\ &\tilde{\mathbf{u}} = \{\mathbf{u}\} = [\mathbf{u}_1,\ldots,\mathbf{u}_n]^{\mathrm{T}}, &&\tilde{\mathbf{x}} = \{\mathbf{x}\} = [\mathbf{x}_1,\ldots,\mathbf{x}_n]^{\mathrm{T}}, \\ &\dot{\mathbf{x}} = \{\dot{\mathbf{x}}\} = [\dot{x}_1,\ldots,\dot{x}_n]^{\mathrm{T}}, &&\dot{\varphi} = \{\dot{\varphi}\} = [\dot{\varphi}_1,\ldots,\dot{\varphi}_n]^{\mathrm{T}}; \end{aligned} \tag{5.77}$$

— and finally the column matrix of the reference point relative velocity vectors $\mathbf{x}'_i$

$$\mathbf{x}' = \{\mathbf{x}'\} = [\mathbf{x}'_1,\ldots,\mathbf{x}'_n]^{\mathrm{T}}. \tag{5.78}$$

The notation of column matrices introduced above preserves the rules established by Conventions 5.3 and 5.4 (see section 5.4.3).

Using notations (5.77) and (5.78), the three components T^{∇}, T^{*}, T^{0} of the kinetic energy T given by expressions (5.59), (5.63) and (5.64), respectively, take the following form now:

$$T^{\nabla} = \tfrac{1}{2}\left(\tilde{\mathsf{a}} + \tilde{\mathbf{b}} + \tilde{\mathbf{x}}'\right)^{\mathrm{T}} \mathsf{M}\left(\tilde{\mathsf{a}} + \tilde{\mathbf{b}} + \tilde{\mathbf{x}}'\right), \tag{5.79}$$

$$T^{*} = \tilde{\mathbf{u}}^{\mathrm{T}} \mathsf{m} \mathsf{P}^{\mathrm{T}}\left(\tilde{\mathsf{a}} + \tilde{\mathbf{b}} + \tilde{\mathbf{x}}'\right), \tag{5.80}$$

$$T^{0} = \tfrac{1}{2}\dot{\varphi}^{\mathrm{T}} \mathsf{J} \dot{\varphi}, \tag{5.81}$$

where the matrices M, P, m, and J have the same meaning as defined previously, i.e. in sections 5.4.4 and 5.4.5. Similarly, we may adopt the expression (5.72) for the considered class of systems. The potential energy of gravity forces, V^{G}, takes the following form now:

$$V^{\mathrm{G}} = -\mathsf{g}\tilde{\mathsf{m}}^{\mathrm{T}} \mathsf{P}^{\mathrm{T}}\left(\tilde{\mathbf{r}} + \tilde{\mathbf{x}}\right) - \mathsf{g}\tilde{\mathsf{m}}^{\mathrm{T}} \tilde{\mathbf{c}}. \tag{5.82}$$

In order to carry out all the differentiations of the expressions (5.79)–(5.82) for use in the Lagrange equations, the real dependence of the column matrices (5.77) on φ_σ, $\dot{\varphi}_\sigma$, x_σ, $\dot{x}_\sigma$, and t has to be determined. For this reason, the following decomposition of the column matrices (5.77) and (5.78) is very useful:

$$\begin{aligned} &\{\mathbf{c}\} = \hat{\mathbf{c}}\{c\}, \qquad \{\mathbf{r}\} = \hat{\mathbf{r}}\{r\}, \qquad \{\mathbf{x}\} = \hat{\mathbf{x}}\{x\}, \qquad \{\mathbf{x}'\} = \hat{\mathbf{x}}\{\dot{x}\}, \\ &\{\mathbf{a}\} = \mathsf{r}\hat{\mathbf{a}}\mathsf{S}_{-}^{\mathrm{T}}\{\dot{\varphi}\}, \qquad \{\mathbf{b}\} = \mathsf{x}\hat{\mathbf{b}}\mathsf{S}_{-}^{\mathrm{T}}\{\dot{\varphi}\}, \qquad \{\mathbf{u}\} = \mathsf{c}\hat{\mathbf{u}}\{\dot{\varphi}\}, \end{aligned} \tag{5.83}$$

where S_{-} is the $(n \times n)$ matrix obtained from the incidence $(n \times n)$ matrix S by replacing the 1 entries by zeros and by changing the sign of all the -1 entries.

Using this latter form of the column matrices the formulae (5.78), (5.79) and (5.81) may be now expressed as follows:

$$T^{\nabla} = \sum_{i=1}^{6} T_i^{\nabla}, \tag{5.84}$$

where

$$\begin{aligned} T_1^{\nabla} &= \tfrac{1}{2}\{\mathbf{a}\}^{\mathrm{T}} \mathsf{M}\{\mathbf{a}\} = \tfrac{1}{2}\{\dot{\varphi}\}^{\mathrm{T}} \mathsf{S}_{-}\mathsf{r}\hat{\mathbf{a}}\mathsf{M}\hat{\mathbf{a}}\mathsf{r}\mathsf{S}_{-}^{\mathrm{T}}\{\dot{\varphi}\}, \\ T_2^{\nabla} &= \tfrac{1}{2}\{\mathbf{b}\}^{\mathrm{T}} \mathsf{M}\{\mathbf{b}\} = \tfrac{1}{2}\{\dot{\varphi}\}^{\mathrm{T}} \mathsf{S}_{-}\hat{\mathbf{b}}\mathsf{M}\hat{\mathbf{b}}\mathsf{x}\mathsf{S}_{-}^{\mathrm{T}}\{\dot{\varphi}\}, \\ T_3^{\nabla} &= \tfrac{1}{2}\{\mathbf{x}'\}^{\mathrm{T}} \mathsf{M}\{\mathbf{x}'\} = \tfrac{1}{2}\{\dot{\mathsf{x}}\}^{\mathrm{T}} \hat{\mathbf{x}}\mathsf{M}\hat{\mathbf{x}}\{\dot{x}\}, \\ T_4^{\nabla} &= \{\mathbf{a}\}^{\mathrm{T}} \mathsf{M}\{\mathbf{b}\} = \{\dot{\varphi}\}^{\mathrm{T}} \mathsf{S}_{-}\mathsf{r}\hat{\mathbf{a}}\mathsf{M}\hat{\mathbf{b}}\mathsf{x}\mathsf{S}_{-}^{\mathrm{T}}\{\dot{\varphi}\}, \\ T_5^{\nabla} &= \{\mathbf{a}\}^{\mathrm{T}} \mathsf{M}\{\mathbf{x}'\} = \{\dot{\varphi}\}^{\mathrm{T}} \mathsf{S}_{-}\mathsf{r}\hat{\mathbf{a}}\mathsf{M}\hat{\mathbf{x}}\{\dot{x}\}, \\ T_6^{\nabla} &= \{\mathbf{b}\}^{\mathrm{T}} \mathsf{M}\{\mathbf{x}'\} = \{\dot{\varphi}\}^{\mathrm{T}} \mathsf{S}_{-}\mathsf{x}\hat{\mathbf{b}}\mathsf{M}\hat{\mathbf{x}}\{\dot{\mathsf{x}}\}, \end{aligned}$$

then

$$T^{*} = \sum_{i=1}^{3} T_i^{*}, \tag{5.85}$$

where

$$T_1^* = \{\mathbf{u}\}^{\mathrm{T}}\mathrm{mP}^{\mathrm{T}}\{\mathbf{a}\} = \{\dot{\varphi}\}^{\mathrm{T}}\mathrm{c\hat{u}mP^T\hat{a}rS_-^T}\{\dot{\varphi}\},$$
$$\mathrm{T}_2^* = \{\mathbf{u}\}^{\mathrm{T}}\mathrm{mP}^{\mathrm{T}}\{\mathbf{b}\} = \{\dot{\varphi}\}^{\mathrm{T}}\mathrm{c\hat{u}mP^T\hat{b}xS_-^T}\{\dot{\varphi}\},$$
$$\mathrm{T}_3^* = \{\mathbf{u}\}^{\mathrm{T}}\mathrm{mP}^{\mathrm{T}}\{\mathbf{x}'\} = \{\dot{\varphi}\}^{\mathrm{T}}\mathrm{c\hat{u}mP^T\hat{x}}\{\dot{\mathrm{x}}\},$$

and

$$V = -\mathrm{g\tilde{m}^TP^T}\left(\hat{\mathbf{r}}\{\mathrm{r}\} + \hat{\mathbf{x}}\{\mathrm{x}\}\right) - \mathrm{g\tilde{m}^T\hat{c}}\{\mathrm{c}\}. \tag{5.86}$$

Performing all differentiation due to the Lagrange's method and substituting the results in the Lagrange equations (for details see Arczewski (1990)) gives

— the equation of motion with respect to angular coordinate φ_σ:

$$\begin{aligned}
J_\sigma\ddot{\varphi}_\sigma + \epsilon_\sigma^{\mathrm{T}}\mathrm{S}_-\Big((\mathrm{r\hat{r}} + \mathrm{x\hat{x}})\mathrm{M}(\mathrm{r\hat{r}} + \mathrm{x\hat{x}})\{\ddot{\varphi}_-\} + (\mathrm{x\hat{b}} + \mathrm{r\hat{a}})\mathrm{M\hat{x}}\{\ddot{\mathrm{x}}\}\\
- (\mathrm{r\hat{a}} + \mathrm{x\hat{b}})\mathrm{M}(\mathrm{r\hat{r}} + \mathrm{x\hat{x}})\{\dot{\varphi}_-^2\} + 2(\mathrm{x\hat{b}} + \mathrm{r\hat{a}})\mathrm{M\hat{b}\dot{x}}\{\dot{\varphi}_-\}\\
+ (\mathrm{r\hat{r}} + \mathrm{x\hat{x}})\mathrm{Pmc\hat{c}}\{\ddot{\varphi}\} - (\mathrm{r\hat{a}} + \mathrm{x\hat{b}})\mathrm{Pmc\hat{c}}\{\dot{\varphi}^2\}\Big)\\
+c_\sigma\epsilon_\sigma^{\mathrm{T}}\mathrm{\hat{u}mP^T}\Big((\mathrm{r\hat{a}} + \mathrm{x\hat{b}})\{\ddot{\varphi}_-\} - (\mathrm{r\hat{r}} + \mathrm{x\hat{x}})\{\dot{\varphi}_-^2\} + \hat{\mathbf{x}}\{\dot{\mathrm{x}}\} + 2\hat{\mathbf{b}}\dot{\mathrm{x}}\{\dot{\varphi}_-\}\Big)\\
= \mathrm{g}\epsilon_\sigma^{\mathrm{T}}\mathrm{S}_-\overset{*}{M}(\hat{\mathbf{a}}\{\mathrm{r}\} + \hat{\mathbf{b}}\{\mathrm{x}\}) + \mathrm{g}m_\sigma \mathrm{c}_\sigma\hat{\mathbf{u}}_\sigma,
\end{aligned} \tag{5.87}$$

— and the equation of motion with respect to translational coordinate x_σ:

$$\begin{aligned}
\epsilon_\sigma^{\mathrm{T}}\Big((\mathrm{\hat{x}M\hat{x}}\{\ddot{\mathrm{x}}\} + \mathrm{\hat{x}M}(\mathrm{r\hat{a}} + \mathrm{x\hat{b}})\{\ddot{\varphi}_-\} - \mathrm{\hat{x}M}(\mathrm{r\hat{r}} + \mathrm{x\hat{x}})\{\dot{\varphi}_-^2\}\\
+ 2\mathrm{\hat{x}M\hat{b}\dot{x}}\{\dot{\varphi}_-\} + \mathrm{\hat{x}Pm\hat{u}c}\{\ddot{\varphi}\} - \mathrm{\hat{x}Pm\hat{c}c}\{\dot{\varphi}^2\}\Big) = \mathrm{g}M_\sigma\hat{\mathrm{x}}_\sigma,
\end{aligned} \tag{5.88}$$

where

$$\{\dot{\varphi}_-\} = \mathrm{S}_-^{\mathrm{T}}\{\dot{\varphi}\} = [\dot{\varphi}_{1\sim},\ldots,\dot{\varphi}_{n\sim}]^{\mathrm{T}}, \qquad \{\dot{\varphi}_-^2\} = [\dot{\varphi}_{1\sim}^2,\ldots,\dot{\varphi}_{n\sim}^2]^{\mathrm{T}},$$
$$\{\ddot{\varphi}_-\} = \frac{\mathrm{d}}{\mathrm{d}t}\{\dot{\varphi}_-\}, \qquad \{\ddot{\varphi}\} = \frac{\mathrm{d}}{\mathrm{d}t}\{\dot{\varphi}\}, \qquad \{\ddot{x}\} = \frac{\mathrm{d}^2}{\mathrm{d}t^2}\{x\}.$$

Equations (5.87), (5.88) form a mathematical model for the whole class of rigid body systems, namely, for the open kinematic chains where the motion is constrained to a plane.

Despite the rather complicated form of (5.87) and (5.88) their use is simple. This will be shown in the following section.

5.4.8 Example

A system which we are going to consider now may be thought as a physical model of four-link robot (see Fig. 5.42). We will assume that the bodies' motion is constrained to a

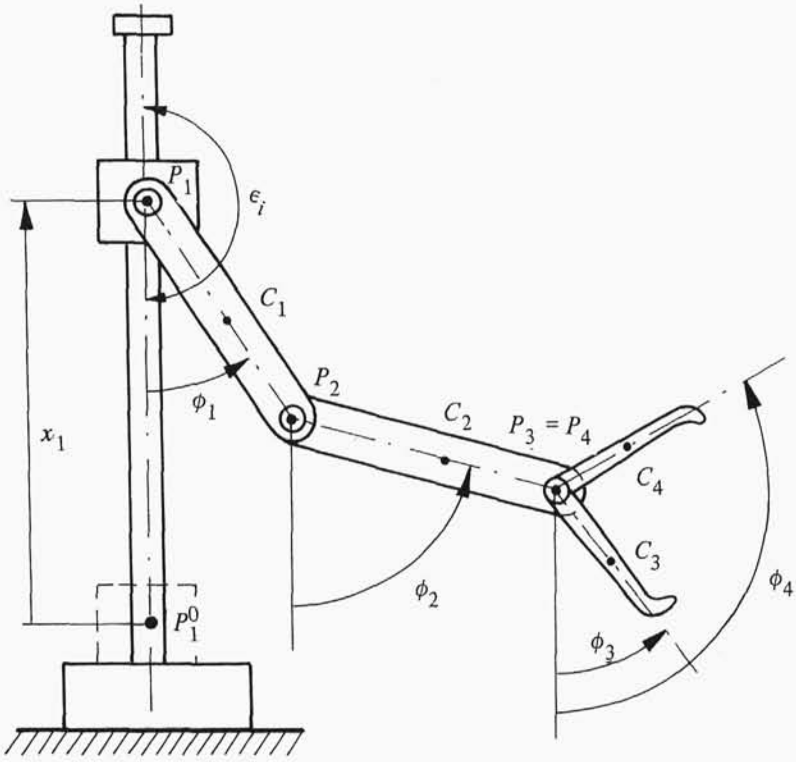

Fig. 5.42.

plane (i.e. that of the paper). In addition to the rotational degrees of freedom described by angular coordinates $\varphi_1, \varphi_2, \varphi_3, \varphi_4$, the first body has a translational degree of freedom described by coordinate x_1. The following data are given: masses of bodies m_1, m_2, m_3, m_4, moments of intertia J_1, J_2, J_3, J_4 with respect to joint points P_1, P_2, P_3, P_4, respectively, location of mass centres: $|\overrightarrow{P_1C_1}| = c_1$, $|\overrightarrow{P_2C_2}| = c_2$, $|\overrightarrow{P_3C_3}| = c_3$, $|\overrightarrow{P_4C_4}| = c_4$; location of joints within a system: $|\overrightarrow{P_1C_2}| = r_2$, $|\overrightarrow{P_2C_3}| = r_3$, $|\overrightarrow{P_2C_4}| = r_4$, and the orientation of the x_1-axis: $\varepsilon_1 = \pi/2$.

The equations of motion are to be determined.

The problem will be solved in three stages.

Stage 1 Determination of topological characteristics of the system, i.e. matrices S_{-} and P.

The regular graph associated with a system is shown in Fig. 5.43, hence the matrices S, S_{-}, and P are as follows:

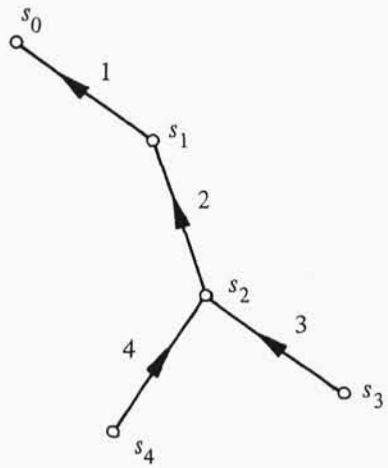

Fig. 5.43.

$$\mathsf{S}=\begin{bmatrix}1 & -1 & 0 & 0\\ 0 & 1 & -1 & -1\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix},\quad \text{thus}\quad \mathsf{S}_-=\begin{bmatrix}0 & 1 & 0 & 0\\ 0 & 0 & 1 & 1\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix},\quad \text{and}\quad \mathsf{P}=\begin{bmatrix}1 & 1 & 1 & 1\\ 0 & 1 & 1 & 1\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}.$$

Stage 2 Determination of all quantities appearing in (5.87) and (5.88) which do not depend on a particular coordinate, i.e. those quantities which are necessary for the determination of any equation of a system and at the same time are calculated only once.

We have

$$\mathsf{m}=\operatorname{diag}(m_1,m_2,m_3,m_4),\qquad \mathsf{J}=\operatorname{diag}(J_1,J_2,J_3,J_4),$$

$$\mathsf{mP}^{\mathsf{T}}=\begin{bmatrix}m_1 & 0 & 0 & 0\\ m_2 & m_2 & 0 & 0\\ m_3 & m_3 & m_3 & 0\\ m_4 & m_4 & 0 & m_4\end{bmatrix},\qquad \mathsf{M}=\mathsf{PmP}^{\mathsf{T}}=\begin{bmatrix}M_1 & M_2 & M_3 & M_4\\ M_2 & M_2 & 0 & M_4\\ M_3 & 0 & M_3 & 0\\ M_4 & M_4 & 0 & M_4\end{bmatrix},$$

where

$$M_1=m_1+m_2+m_3+m_4,\quad M_2=m_2+m_3+m_4,\quad M_3=m_3,\quad M_4=m_4.$$

$$\overset{*}{M}=\operatorname{diag}(M_1,M_2,M_3,M_4),$$

$$\hat{\mathbf{a}}=\operatorname{diag}(\mathbf{0},\hat{\mathbf{a}}_2,\hat{\mathbf{a}}_3,\hat{\mathbf{a}}_4),\qquad \hat{\mathbf{b}}=\operatorname{diag}(\hat{\mathbf{b}}_1,\mathbf{0},\mathbf{0},\mathbf{0}),\qquad \hat{\mathbf{c}}=\operatorname{diag}(\hat{\mathbf{c}}_1,\hat{\mathbf{c}}_2,\hat{\mathbf{c}}_3,\hat{\mathbf{c}}_4),$$

$$\hat{\mathbf{r}}=\operatorname{diag}(\mathbf{0},\hat{\mathbf{r}}_2,\hat{\mathbf{r}}_3,\hat{\mathbf{r}}_4),\qquad \hat{\mathbf{u}}=\operatorname{diag}(\hat{\mathbf{u}}_1,\hat{\mathbf{u}}_2,\hat{\mathbf{u}}_3,\hat{\mathbf{u}}_4),\qquad \hat{\mathbf{x}}=\operatorname{diag}(\hat{\mathbf{x}}_1,\mathbf{0},\mathbf{0},\mathbf{0}),$$

$$\mathsf{c}=\operatorname{diag}(c_1,c_2,c_3,c_4),\qquad \mathsf{r}=\operatorname{diag}(0,r_2,r_3,r_4),\qquad \mathsf{x}=\operatorname{diag}(x_1,0,0,0),$$

$$\dot{\mathsf{x}}=\operatorname{diag}(\dot{x}_1,0,0,0),\qquad \dot{\varphi}=\operatorname{diag}(\dot{\varphi}_1,\dot{\varphi}_2,\dot{\varphi}_3,\dot{\varphi}_4)$$

$$\{\dot{\varphi}\} = [\dot{\varphi}_1, \dot{\varphi}_2, \dot{\varphi}_3, \dot{\varphi}_4]^{\mathrm{T}}, \quad \{\ddot{\varphi}\} = [\ddot{\varphi}_1, \ddot{\varphi}_2, \ddot{\varphi}_3, \ddot{\varphi}_4]^{\mathrm{T}},$$
$$\{\dot{\varphi}^2\} = [\dot{\varphi}_1^2, \dot{\varphi}_2^2, \dot{\varphi}_3^2, \dot{\varphi}_4^2]^{\mathrm{T}},$$
$$\{\dot{\varphi}_{\sim}\} = \mathsf{S}_{-}^{\mathrm{T}}\{\dot{\varphi}\} = [0, \dot{\varphi}_1, \dot{\varphi}_2, \dot{\varphi}_2]^{\mathrm{T}},$$
$$\{\ddot{\varphi}_{\sim}\} = [0, \ddot{\varphi}_1, \ddot{\varphi}_2, \ddot{\varphi}_2]^{\mathrm{T}}, \qquad \{\dot{\varphi}_{\sim}^2\} = [0, \dot{\varphi}_1^2, \dot{\varphi}_2^2, \dot{\varphi}_2^2]^{\mathrm{T}},$$
$$\{c\} = [c_1, c_2, c_3, c_4]^{\mathrm{T}}, \qquad \{r\} = [0, r_2, r_3, r_4]^{\mathrm{T}}, \qquad \{x\} = [x_1, 0, 0, 0]^{\mathrm{T}},$$
$$\{\dot{x}\} = [\dot{x}_1, 0, 0, 0]^{\mathrm{T}}, \qquad \{\ddot{x}\} = [\ddot{x}_1, 0, 0, 0]^{\mathrm{T}}, \qquad \dot{\mathsf{x}}\{\dot{\varphi}_{\sim}\} = [0, 0, 0, 0]^{\mathrm{T}}.$$

Additionally the solar products of the unit vectors appearing in (5.87) and (5.88) are:

$$\begin{aligned}
&\hat{\mathbf{r}}_i\hat{\mathbf{r}}_j = \cos(\varphi_{i\sim} - \varphi_{j\sim} + \gamma_i - \gamma_j),\\
&\hat{\mathbf{r}}_i\hat{\mathbf{x}}_j = \hat{\mathbf{a}}_i\hat{\mathbf{b}}_j = \cos(\varphi_{i\sim} - \varphi_{j\sim} + \gamma_i - \varepsilon_j),\\
&\hat{\mathbf{x}}_i\hat{\mathbf{x}}_j = \hat{\mathbf{b}}_i\hat{\mathbf{b}}_j = \cos(\varphi_{i\sim} - \varphi_{j\sim} + \varepsilon_i - \varepsilon_j),\\
&\hat{\mathbf{b}}_i\hat{\mathbf{x}}_j = -\sin(\varphi_{i\sim} - \varphi_{j\sim} + \varepsilon_i - \varepsilon_j),\\
&\hat{\mathbf{a}}_i\hat{\mathbf{x}}_j = -\sin(\varphi_{i\sim} - \varphi_{j\sim} + \gamma_i - \varepsilon_j),\\
&\hat{\mathbf{a}}_i\hat{\mathbf{r}}_j = -\sin(\varphi_{i\sim} - \varphi_{j\sim} + \gamma_i - \gamma_j),\\
&\hat{\mathbf{b}}_i\hat{\mathbf{r}}_j = -\sin(\varphi_{i\sim} - \varphi_{j\sim} + \varepsilon_i - \gamma_j),\\
&\hat{\mathbf{r}}_i\hat{\mathbf{c}}_j = \cos(\varphi_{i\sim} - \varphi_j + \gamma_i), \qquad \hat{\mathbf{x}}_i\hat{\mathbf{c}}_j = \cos(\varphi_{i\sim} - \varphi_j + \varepsilon_i),\\
&\hat{\mathbf{a}}_i\hat{\mathbf{c}}_j = -\sin(\varphi_{i\sim} - \varphi_j + \gamma_i), \qquad \hat{\mathbf{b}}_i\hat{\mathbf{c}}_j = -\sin(\varphi_{i\sim} - \varphi_j + \varepsilon_i),\\
&\hat{\mathbf{u}}_i\hat{\mathbf{a}}_j = \cos(\varphi_i - \varphi_{j\sim} + \gamma_j), \qquad \hat{\mathbf{u}}_i\hat{\mathbf{b}}_j = \cos(\varphi_i - \varphi_{j\sim} + \varepsilon_j),\\
&\hat{\mathbf{u}}_i\hat{\mathbf{r}}_j = -\sin(\varphi_i - \varphi_{j\sim} + \gamma_j), \qquad \hat{\mathbf{u}}_i\hat{\mathbf{x}}_j = -\sin(\varphi_i - \varphi_{j\sim} + \varepsilon_j).
\end{aligned} \tag{5.89}$$

Stage 3 Performing calculations due to formulae (5.87) or (5.88) for the subsequent generalized coordinates.

First we shall determine the equations corresponding to the angular coordinates. Thus we shall use (5.87). Substituting in it $\sigma = 1$, we get

$$J_\sigma\ddot{\varphi}_\sigma = J_1\ddot{\varphi}_1,$$
$$\varepsilon_1^{\mathrm{T}}\mathsf{S}_{-} = [0 \quad 1 \quad 0 \quad 0],$$
$$\varepsilon_1^{\mathrm{T}}\mathsf{S}_{-}\mathsf{r}\hat{\mathsf{r}}\mathsf{M}\mathsf{r}\hat{\mathsf{r}}\{\ddot{\varphi}_{\sim}\} = M_2 r_2^2\ddot{\varphi}_1 + (M_3 r_2 r_3\hat{\mathbf{r}}_2\hat{\mathbf{r}}_3 + M_4 r_2 r_4\hat{\mathbf{r}}_2\hat{\mathbf{r}}_4)\ddot{\varphi}_2,$$
$$\varepsilon_1^{\mathrm{T}}\mathsf{S}_{-}\mathsf{r}\hat{\mathsf{a}}\mathsf{M}\hat{\mathsf{x}}\{\ddot{\mathsf{x}}\} = M_2 r_2\hat{\mathbf{a}}_2\hat{\mathbf{x}}_1\ddot{x}_1,$$
$$-\varepsilon_1^{\mathrm{T}}\mathsf{S}_{-}\mathsf{r}\hat{\mathsf{a}}\mathsf{M}\mathsf{r}\hat{\mathsf{r}}\{\dot{\varphi}_{\sim}^2\} = -(M_3 r_2 r_3\hat{\mathbf{a}}_2\hat{\mathbf{r}}_3 + M_4 r_2 r_4\hat{\mathbf{a}}_2\hat{\mathbf{r}}_4)\dot{\varphi}_2^2,$$
$$\varepsilon_1^{\mathrm{T}}\mathsf{S}_{-}\mathsf{r}\hat{\mathsf{r}}\mathsf{P}\mathsf{m}\mathsf{c}\hat{\mathsf{c}}\{\ddot{\varphi}\} = m_2 r_2 c_2\hat{\mathbf{r}}_2\hat{\mathbf{c}}_2\ddot{\varphi}_2 + m_3 r_2 c_3\hat{\mathbf{r}}_2\hat{\mathbf{c}}_3\ddot{\varphi}_3 + m_4 r_2 c_4\hat{\mathbf{r}}_2\hat{\mathbf{c}}_4\ddot{\varphi}_4,$$
$$-\varepsilon_1^{\mathrm{T}}\mathsf{S}_{-}\mathsf{r}\hat{\mathsf{a}}\mathsf{P}\mathsf{m}\mathsf{c}\hat{\mathsf{c}}\{\dot{\varphi}^2\} = -m_2 r_2 c_2\hat{\mathbf{a}}_2\hat{\mathbf{c}}_2\dot{\varphi}_2^2 - m_3 r_2 c_3\hat{\mathbf{a}}_2\hat{\mathbf{c}}_3\dot{\varphi}_3^2 - m_4 r_2 c_4\hat{\mathbf{a}}_2\hat{\mathbf{c}}_4\dot{\varphi}_4^2,$$

$$c_1\varepsilon_1^{\mathrm{T}}\hat{\mathbf{u}}\mathrm{m}\mathrm{P}^{\mathrm{T}}\mathrm{r}\hat{\mathbf{a}}\{\ddot{\varphi}_-\} = m_2c_1r_2\hat{\mathbf{a}}_2\hat{\mathbf{u}}_2\ddot{\varphi}_1,$$

$$-c_1\varepsilon_1^{\mathrm{T}}\hat{\mathbf{u}}\mathrm{m}\mathrm{P}^{\mathrm{T}}\mathrm{r}\hat{\mathbf{r}}\{\dot{\varphi}_-^2\} = -m_2c_1r_2\hat{\mathbf{u}}_2\hat{\mathbf{r}}_2\dot{\varphi}_1^2,$$

$$c_1\varepsilon_1^{\mathrm{T}}\hat{\mathbf{u}}\mathrm{m}\mathrm{P}^{\mathrm{T}}\hat{\mathbf{x}}\{\ddot{\mathrm{x}}\} = m_2c_1\hat{\mathbf{u}}_2\hat{\mathbf{x}}_1\ddot{x}_1,$$

$$\mathbf{g}\varepsilon_1^{\mathrm{T}}\mathrm{S}_-\dot{\mathrm{M}}(\hat{\mathbf{a}}\{\mathrm{r}\}+\hat{\mathbf{b}}\{x\}) = M_2r_2\mathbf{g}\hat{\mathbf{a}}_2,$$

$$\mathbf{g}m_\sigma c_\sigma\hat{\mathbf{u}}_\sigma = m_1c_1\mathbf{g}\hat{\mathbf{u}}_1.$$

Remaining terms equal zero. After reducing and ordering of the above computed terms, we finally obtain the equation corresponding to generalized coordinate φ_1:

$$\begin{aligned}
&(J_1 + M_2r_2^2 + m_2c_1r_2\hat{\mathbf{a}}_2\hat{\mathbf{u}}_2)\ddot{\varphi}_1 + (M_3r_3\hat{\mathbf{r}}_3 + M_4r_4\hat{\mathbf{r}}_4 + m_2c_2\hat{\mathbf{c}}_2)r_2\hat{\mathbf{r}}_2\ddot{\varphi}_2\\
&+m_3r_2c_3\hat{\mathbf{r}}_2\hat{\mathbf{c}}_3\ddot{\varphi}_3 + m_4r_2c_4\hat{\mathbf{r}}_2\hat{\mathbf{c}}_4\ddot{\varphi}_4 + (M_2r_2\hat{\mathbf{a}}_2\hat{\mathbf{x}}_1 + m_2c_1\hat{\mathbf{u}}_2\hat{\mathbf{x}}_1)\ddot{x}_1\\
&-m_2c_1r_2\hat{\mathbf{u}}_2\hat{\mathbf{r}}_2\dot{\varphi}_1^2 - (M_3r_3\hat{\mathbf{r}}_3 + M_4r_4\hat{\mathbf{r}}_4 + m_2c_2\hat{\mathbf{c}}_2)r_2\hat{\mathbf{a}}_2\dot{\varphi}_2^2 - m_3r_2c_3\hat{\mathbf{a}}_2\hat{\mathbf{c}}_3\dot{\varphi}_3^2\\
&-m_4r_2c_4\hat{\mathbf{a}}_2\hat{\mathbf{c}}_4\dot{\varphi}_4^2 = M_2r_2\mathbf{g}\hat{\mathbf{a}}_2 + m_1c_1\mathbf{g}\hat{\mathbf{u}}_1.
\end{aligned} \tag{5.90}$$

Now we proceed to the calculation of terms appearing in the equation corresponding to the angular coordinate φ_2. Thus we use (5.87) substituting in it $\sigma = 2$. We have

$$\begin{aligned}
J_\sigma\ddot{\varphi}_\sigma &= J_2\ddot{\varphi}_2,\\
\varepsilon_2^{\mathrm{T}}\mathrm{S}_- &= [0 \quad 0 \quad 1 \quad 1],\\
\varepsilon_2^{\mathrm{T}}\mathrm{S}_-\mathrm{r}\hat{\mathbf{r}}\mathrm{M}\mathrm{r}\hat{\mathbf{r}}\{\ddot{\varphi}_-\} &= (M_3r_3\hat{\mathbf{r}}_3 + M_4r_4\hat{\mathbf{r}}_4)r_2\hat{\mathbf{r}}_2\ddot{\varphi}_1 + (M_3r_3^2 + M_4r_4^2)\ddot{\varphi}_2,\\
\varepsilon_2^{\mathrm{T}}\mathrm{S}_-\mathrm{r}\hat{\mathbf{a}}\mathrm{M}\hat{\mathbf{x}}\{\ddot{\mathrm{x}}\} &= M_3r_3\hat{\mathbf{a}}_3\hat{\mathbf{x}}_1\ddot{\mathrm{x}}_1,\\
-\varepsilon_2^{\mathrm{T}}\mathrm{S}_-\mathrm{r}\hat{\mathbf{a}}\mathrm{M}\mathrm{r}\hat{\mathbf{r}}\{\dot{\varphi}_-^2\} &= -(M_3r_3\hat{\mathbf{a}}_3 + M_4r_4\hat{\mathbf{a}}_4)r_2\hat{\mathbf{r}}_2\dot{\varphi}_1^2 - (M_3r_3^2\hat{\mathbf{a}}_3\hat{\mathbf{r}}_3 + M_4r_4^2\hat{\mathbf{a}}_4\hat{\mathbf{r}}_4)\dot{\varphi}_2^2\\
\varepsilon_2^{\mathrm{T}}\mathrm{S}_-\mathrm{r}\hat{\mathbf{r}}\mathrm{P}\mathrm{m}\mathrm{c}\hat{\mathbf{c}}\{\ddot{\varphi}\} &= m_3r_3c_3\hat{\mathbf{r}}_3\hat{\mathbf{c}}_3\ddot{\varphi}_3 + m_4r_4c_4\hat{\mathbf{r}}_4\hat{\mathbf{c}}_4\ddot{\varphi}_4,\\
-\varepsilon_2^{\mathrm{T}}\mathrm{S}_-\mathrm{r}\hat{\mathbf{a}}\mathrm{P}\mathrm{m}\mathrm{c}\hat{\mathbf{c}}\{\dot{\varphi}^2\} &= -m_3r_3c_3\hat{\mathbf{a}}_3\hat{\mathbf{c}}_3\dot{\varphi}_3^2 - m_4r_4c_4\hat{\mathbf{a}}_4\hat{\mathbf{c}}_4\dot{\varphi}_4^2,\\
c_2\varepsilon_2^{\mathrm{T}}\hat{\mathbf{u}}\mathrm{m}\mathrm{P}^{\mathrm{T}}\mathrm{r}\hat{\mathbf{a}}\{\ddot{\varphi}_-\} &= (m_3\hat{\mathbf{u}}_3 + m_4\hat{\mathbf{u}}_4)c_2r_2\hat{\mathbf{a}}_2\ddot{\varphi}_1 + (m_3r_3\hat{\mathbf{u}}_3\hat{\mathbf{a}}_3 + m_4r_4\hat{\mathbf{u}}_4\hat{\mathbf{a}}_4)c_2\ddot{\varphi}_2\\
-c_2\varepsilon_2^{\mathrm{T}}\hat{\mathbf{u}}\mathrm{m}\mathrm{P}^{\mathrm{T}}\mathrm{r}\hat{\mathbf{r}}\{\dot{\varphi}_-^2\} &= -(m_3\hat{\mathbf{u}}_3 + m_4\hat{\mathbf{u}}_4)c_2r_2\hat{\mathbf{r}}_2\dot{\varphi}_1^2 - (m_3r_3\hat{\mathbf{u}}_3\hat{\mathbf{r}}_3 + m_4r_4\hat{\mathbf{u}}_4\hat{\mathbf{r}}_4)c_2\dot{\varphi}_2^2\\
c_2\varepsilon_2^{\mathrm{T}}\hat{\mathbf{u}}\mathrm{m}\mathrm{P}^{\mathrm{T}}\hat{\mathbf{x}}\{\ddot{\mathrm{x}}\} &= (m_3\hat{\mathbf{u}}_3 + m_4\hat{\mathbf{u}}_4)c_2\hat{\mathbf{x}}_1\ddot{\mathrm{x}}_1,\\
\mathbf{g}\varepsilon_2^{\mathrm{T}}\mathrm{S}_-\dot{\mathrm{M}}(\hat{\mathbf{a}}\{\mathrm{r}\} + \hat{\mathbf{b}}\{\mathrm{x}\}) &= M_3r_3\mathbf{g}\hat{\mathbf{a}}_3 + M_4r_4\mathbf{g}\hat{\mathbf{a}}_4,\\
\mathbf{g}m_\sigma c_\sigma\hat{\mathbf{u}}_\sigma &= m_2c_2\hat{\mathbf{g}}_2.
\end{aligned}$$

Again after reducing and ordering of the above computed terms, we finally obtain the equation corresponding to coordinate φ_2:

$$\begin{aligned}
&\left[(M_3 r_3 \hat{\mathbf{r}}_3 + M_4 r_4 \hat{\mathbf{r}}_4) r_2 \hat{\mathbf{r}}_2 + (m_3 \hat{\mathbf{u}}_3 + m_4 \hat{\mathbf{u}}_4) c_2 r_2 \hat{\mathbf{a}}_2\right] \ddot{\varphi}_1 \\
&+\left[(J_2 + M_3 r_3^2 + M_4 r_4^2) + (m_3 r_3 \hat{\mathbf{u}}_3 \hat{\mathbf{a}}_3 + m_4 r_4 \hat{\mathbf{u}}_4 \hat{\mathbf{a}}_4) c_2\right] \ddot{\varphi}_2 \\
&+\left[M_3 r_3 \hat{\mathbf{a}}_3 + (m_3 \hat{\mathbf{u}}_3 + m_4 \hat{\mathbf{u}}_4) c_2\right] \hat{\mathbf{x}}_1 \ddot{x}_1 \\
&-\left[(M_3 r_3 \hat{\mathbf{a}}_3 + M_4 r_4 \hat{\mathbf{a}}_4) + (m_3 \hat{\mathbf{u}}_3 + m_4 \hat{\mathbf{u}}_4) c_2\right] r_2 \hat{\mathbf{r}}_2 \dot{\varphi}_1^2 \\
&-\left[(M_3 r_3^2 \hat{\mathbf{a}}_3 \hat{\mathbf{r}}_3 + M_4 r_4^2 \hat{\mathbf{a}}_4 \hat{\mathbf{r}}_4) + (m_3 r_3 \hat{\mathbf{u}}_3 \hat{\mathbf{r}}_3 + m_4 r_4 \hat{\mathbf{u}}_4 \hat{\mathbf{r}}_4) c_2\right] \dot{\varphi}_2^2 \\
&-m_3 r_3 c_3 \hat{\mathbf{a}}_3 \hat{\mathbf{c}}_3 \dot{\varphi}_3^2 - m_4 r_4 c_4 \hat{\mathbf{a}}_4 \hat{\mathbf{c}}_4 \dot{\varphi}_4^2 = M_3 r_3 \mathbf{g} \hat{\mathbf{a}}_3 + M_4 r_4 \mathbf{g} \hat{\mathbf{a}}_4 + m_2 c_2 \mathbf{g} \hat{\mathbf{u}}_2 .
\end{aligned} \tag{5.91}$$

Similarly for φ_3, substituting in (5.87) $\sigma = 3$, we have

$$\begin{aligned}
J_\sigma \ddot{\varphi}_\sigma &= J_3 \ddot{\varphi}_3, \\
c_1 \varepsilon_1^{\mathrm{T}} \hat{\mathbf{u}} \mathsf{m} \mathsf{P}^{\mathrm{T}} \mathsf{r} \hat{\mathbf{a}} \{\ddot{\varphi}_\sim\} &= m_3 c_3 r_2 \hat{\mathbf{u}}_3 \hat{\mathbf{a}}_2 \ddot{\varphi}_1 + m_3 c_3 r_3 \hat{\mathbf{u}}_3 \hat{\mathbf{a}}_3 \ddot{\varphi}_2, \\
-c_1 \varepsilon_1^{\mathrm{T}} \hat{\mathbf{u}} \mathsf{m} \mathsf{P}^{\mathrm{T}} \mathsf{r} \hat{\mathbf{r}} \{\dot{\varphi}_\sim^2\} &= -m_3 c_3 r_2 \hat{\mathbf{u}}_3 \hat{\mathbf{r}}_2 \dot{\varphi}_1^2 - m_3 c_3 r_2 \hat{\mathbf{u}}_3 \hat{\mathbf{r}}_3 \dot{\varphi}_2^2, \\
c_1 \varepsilon_1^{\mathrm{T}} \hat{\mathbf{u}} \mathsf{m} \mathsf{P}^{\mathrm{T}} \hat{\mathbf{x}} \{\ddot{\mathrm{x}}\} &= m_3 c_3 \hat{\mathbf{u}}_3 \hat{\mathbf{x}}_1 \ddot{\mathrm{x}}_1, \\
\mathbf{g} m_3 c_3 \hat{\mathbf{u}}_3 &= m_3 c_3 \mathbf{g} \hat{\mathbf{u}}_3 .
\end{aligned}$$

All the other terms equal zero. The ordering of the above computed terms, yields the equation corresponding to coordinate φ_3:

$$\begin{aligned}
m_3 c_3 r_2 \hat{\mathbf{u}}_3 \hat{\mathbf{a}}_2 \ddot{\varphi}_1 + m_3 c_3 r_3 \hat{\mathbf{u}}_3 \hat{\mathbf{a}}_3 \ddot{\varphi}_2 + J_3 \ddot{\varphi}_3 + m_3 c_3 \hat{\mathbf{u}}_3 \hat{\mathbf{x}}_1 \ddot{\mathrm{x}}_1 - m_3 c_3 r_2 \hat{\mathbf{u}}_3 \hat{\mathbf{r}}_2 \dot{\varphi}_1^2 \\
- m_3 c_3 r_2 \hat{\mathbf{u}}_3 \hat{\mathbf{r}}_3 \dot{\varphi}_2^2 = m_3 c_3 \mathbf{g} \hat{\mathbf{u}}_3 .
\end{aligned} \tag{5.92}$$

and for φ_4:

$$\begin{aligned}
m_4 c_4 r_2 \hat{\mathbf{u}}_4 \hat{\mathbf{a}}_2 \ddot{\varphi}_1 + m_4 c_4 r_4 \hat{\mathbf{u}}_4 \hat{\mathbf{a}}_4 \ddot{\varphi}_2 + J_4 \ddot{\varphi}_4 + m_4 c_4 \hat{\mathbf{u}}_4 \hat{\mathbf{x}}_1 \ddot{\mathrm{x}}_1 - m_4 c_4 r_2 \hat{\mathbf{u}}_4 \hat{\mathbf{r}}_2 \dot{\varphi}_1^2 \\
- m_4 c_4 r_2 \hat{\mathbf{u}}_4 \hat{\mathbf{r}}_4 \dot{\varphi}_2^2 = m_4 c_4 \mathbf{g} \hat{\mathbf{u}}_4 .
\end{aligned} \tag{5.93}$$

Now we determine the equation of motion corresponding to the translational coordinate x_1. We will use the equation (5.88) assuming $\sigma = 1$. Thus $\varepsilon_1^{\mathrm{T}} = [1\ \ 0\ \ 0\ \ 0]$ and successive components of the equation are of the form

$$\begin{aligned}
\varepsilon_1^{\mathrm{T}} \hat{\mathbf{x}} \mathsf{M} \hat{\mathbf{x}} \{\ddot{\mathrm{x}}\} &= M_1 \ddot{x}_1, \\
\varepsilon_1^{\mathrm{T}} \hat{\mathbf{x}} \mathsf{M} \mathsf{r} \hat{\mathbf{a}} \{\ddot{\varphi}_\sim\} &= M_2 r_2 \hat{\mathbf{x}}_1 \hat{\mathbf{a}}_2 \ddot{\varphi}_1 + M_3 r_3 \hat{\mathbf{x}}_1 \hat{\mathbf{a}}_3 \ddot{\varphi}_2 + M_4 r_4 \hat{\mathbf{x}}_1 \hat{\mathbf{a}}_4 \ddot{\varphi}_2, \\
\varepsilon_1^{\mathrm{T}} \hat{\mathbf{x}} \mathsf{P} \mathsf{m} \hat{\mathbf{u}} \mathsf{c} \{\ddot{\varphi}\} &= m_1 c_1 \hat{\mathbf{x}}_1 \hat{\mathbf{u}}_1 \ddot{\varphi}_1 + m_2 c_2 \hat{\mathbf{x}}_1 \hat{\mathbf{u}}_2 \ddot{\varphi}_2 + m_3 c_3 \hat{\mathbf{x}}_1 \hat{\mathbf{u}}_3 \ddot{\varphi}_3 + m_4 c_4 \hat{\mathbf{x}}_1 \hat{\mathbf{u}}_4 \ddot{\varphi}_4, \\
-\varepsilon_2^{\mathrm{T}} \hat{\mathbf{x}} \mathsf{P} \mathsf{m} \hat{\mathbf{c}} \mathsf{c} \{\dot{\varphi}^2\} &= -m_1 c_1 \hat{\mathbf{x}}_1 \hat{\mathbf{c}}_1 \dot{\varphi}_1^2 - m_2 c_2 \hat{\mathbf{x}}_1 \hat{\mathbf{c}}_2 \dot{\varphi}_2^2 - m_3 c_3 \hat{\mathbf{x}}_1 \hat{\mathbf{c}}_3 \dot{\varphi}_3^2 - m_4 c_4 \hat{\mathbf{x}}_1 \hat{\mathbf{c}}_4 \dot{\varphi}_4^2, \\
\mathbf{g} M_\sigma \hat{\mathbf{x}}_\sigma &= M_1 \mathbf{g} \hat{\mathbf{x}}_1 .
\end{aligned}$$

All the other terms equal zero. The ordering of the above computed terms gives the equation corresponding to coordinate x_1:

$$M_1\ddot{x}_1+(M_2r_2\hat{\mathbf{a}}_2+m_1c_1\hat{\mathbf{u}}_1)\hat{\mathbf{x}}_1\ddot{\varphi}_1+(M_3r_3\hat{\mathbf{a}}_3+M_4r_4\hat{\mathbf{a}}_4+m_2c_2\hat{\mathbf{u}}_2)\hat{\mathbf{x}}_1\ddot{\varphi}_2$$
$$+m_3c_3\hat{\mathbf{x}}_1\hat{\mathbf{u}}_3\ddot{\varphi}_3+m_4c_4\hat{\mathbf{x}}_1\hat{\mathbf{u}}_4\ddot{\varphi}_4-m_1c_1\hat{\mathbf{x}}_1\hat{\mathbf{c}}_1\dot{\varphi}_1^2-m_2c_2\hat{\mathbf{x}}_1\hat{\mathbf{c}}_2\dot{\varphi}_2^2$$
$$-m_3c_3\hat{\mathbf{x}}_1\hat{\mathbf{c}}_3\dot{\varphi}_3^2-m_4c_4\hat{\mathbf{x}}_1\hat{\mathbf{c}}_4\dot{\varphi}_4^2=M_1\mathbf{g}\hat{\mathbf{x}}_1. \tag{5.94}$$

If we want to eliminate scalar products from the equations (5.90)–(5.94) we can use the functions (5.89).

5.4.9 Concluding remarks

We have shown that the amalgamation of topological information for a system of rigid bodies and essential system data (i.e. inertial characteristics, location of certain points within the bodies, orientations of certain axes) provides a relatively simple and systematic means of mathematical model determination. The apparent advantage of the method presented in this section is that the formulae for kinetic energy (5.65), for potential energy (5.72) as well as the equations of motion (5.87) and (5.88) can be used for many different systems without performing all the calculations required for the Lagrange equations. The only requirement is to insert specific problem data into the general formulae; a great deal of preparatory work is thus avoided. Moreover, each equation of motion, even each term of the equation, may be calculated independently of all others, which can lead to substantial savings of computer storage. We hope that the example considered sufficiently proves the efficiency and great labour saving of the proposed method, in comparison with the classical approach.

The method presented in this section can be further creatively developed. There are many questions which have remained unsolved until now. The quite general formulae for kinetic energy, (4.65), was used to generate the equation of motion for planar systems only. We are sure that the problem of complex mechanical system modelling is far from the final solution. It seems that there is a lot to do in the field. A good understanding of the key idea of the method presented, together with the use of graph theory as a means of topological structure representation may result in further development of mechanical system modelling methods.