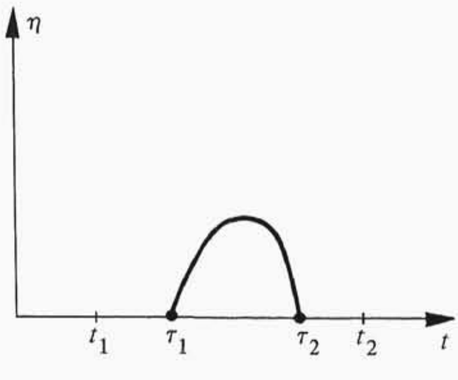


Fig. 4.12.

$$\frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial f}{\partial y'} = 0 \tag{4.103}$$

must hold. This is the famous **Euler–Lagrange equation** and its solution is called the **extremal**.

All our results till now may be generalized in a natural way for the multidimensional case—that is, the case of a multidimensional functional space of functions that still depend upon just one variable, $x$. If we then denote by $\{y\}$ the set of functions $y_1,\dots,y_n$, we will be analysing the extremum of the functional $I[\{y\}]$. After an adequate generalization of the notions of proximity, variation of a functional etc., we can obtain the Euler–Lagrange equations in the form

$$\frac{\partial f}{\partial y_i} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial f}{\partial y_i'} = 0, \qquad i = 1,\dots,n. \tag{4.104}$$

### 4.2.3 Differential variational principles

#### *4.2.3.1 The common property of differential principles*

Before we pass over to consideration of selected differential principles we would like to turn attention to some questions which, in our opinion, are essential. First of all we recall the remark from section 4.2.1 that variations should be understood in a broader sense and must not necessarily mean extremalization. That is why there is no objection to including differential non-extremal principles to variational ones. It is only essential that admissible variations of certain functions appear; in classical mechanics they may be those of the positions of mass particles (the d'Alembert principle), and in thermodynamics they are the variations of the so-called local dissipative potentials (the Onsager principle).

The fact that in section 4.2.2.1 only virtual displacements were considered does not imply that they are the 'construction material' of all the differential principles. True, virtual displacements are the central concept and a difficult one, and that is why they were taken up at the beginning of this section. This should not hinder the proper perception of the fact that other quantities could be equally 'good', for instance virtual velocity or virtual acceleration. This is a suitable place for emphasizing that the principle of virtual work in a form elaborated by Lagrange contained the very notion of 'virtual velocity'! That is why we want to emphasize that all the differential principles known in theoretical mechanics have a common property, namely that each of them contains one element from the set of elements $\mathbf{r}_\nu$, $\mathbf{v}_\nu = \dot{\mathbf{r}}_\nu$, $\mathbf{a}_\nu = \ddot{\mathbf{r}}_\nu$, and eventually higher differentials $\mathbf{r}_\nu^{(p)}$ of the position vector.

Let us try , in the light of the above, to grasp the essence of differential principles in classical mechanics. There is, it seems, no need to prove that actual motion fulfils Newton's second law:

$$m_\nu \ddot{\mathbf{r}}_\nu = \mathbf{F}_\nu + \mathbf{R}_\nu, \tag{4.105}$$

in which $\mathbf{F}_\nu$ are active forces, and $\mathbf{R}_\nu$ are reactions of constraints. In this connection, it is suggested that the actual states taken for comparison with the imagined ones (see 4.1.2) are called 'Newtonian states'. Thus, we can write down the general form of the differential variational principle

$$\sum_{\nu=1}^{n} (\mathbf{F}_\nu - m_\nu \ddot{\mathbf{r}}_\nu)\delta \mathbf{r}_\nu^{(p)} = 0. \tag{4.106}$$

We assume in this that $\delta t = 0$ (synchronous variation), $\delta \mathbf{r}_\nu = 0, \ldots, \delta \mathbf{r}_\nu^{(p-1)} = 0$, and $\delta \mathbf{r}_\nu^{(p)} \neq 0$. Particular forms of principles known in mechanics can be obtained from equation (4.106). Thus, for instance, for $p = 2$ we would have the principle of Gauss (see section 4.2.3.6).

#### *4.2.3.2 The principle of virtual work*

In 1717 Johann Bernoulli proposed the principle of virtual work, which is essentially a definition of equilibrium for the mechanical system. Obviously, every engineer in mechanics knows that the necessary and sufficient conditions for a rigid body to be in equilibrium are that the resultant force and the resultant couple be zero vectors everywhere. The question therefore arises: why are we recalling such an old-fashioned device? The answer is that in the study of statics we have followed the procedure of isolating a body to expose certain unknown forces and then writing equations of equilibrium that include all the forces acting on the body. Such a method of establishing the conditions of equilibrium is less useful for constrained multi-body systems. This is even more obvious when applied to the study of the equilibrium of deformable bodies. An invaluable tool is the principle of virtual work, which reflects in a simple manner the dependence of the internal strain state upon the external load, but we shall be writing about this in Volume 2. Here we present only the principles valid for mechanical systems composed of a finite number of particles.

Assume, then, that a system of $n$ particles is tied with holonomic scleronomic bilateral and perfect constraints. The necessary and sufficient condition for the equilibrium of this system is that equation

$$\sum_{\nu=1}^{n} \mathbf{F}_\nu \cdot \delta \mathbf{r}_\nu = 0 \tag{4.107}$$

hold. Note that equation (4.107) results from (4.106) if we accept the assumption that $\ddot{\mathbf{r}}_\nu = 0$.

Since $\delta\mathbf{r}_\nu$ is a displacement, the whole expression on the left-hand side of equation (4.107) has the dimensions of work. That is why the quantity from the definition is called **virtual work** and is denoted by $\delta W$:

$$\delta W = \sum_{\nu=1}^{n} \mathbf{F}_\nu \cdot \delta\mathbf{r}_\nu. \tag{4.108}$$

Beware of the misunderstanding that may arise here: the virtual work $\delta W$ is not, in general, the variation of the work $W$ and, to avoid confusion, it must be remembered that $\delta W$ is merely a shorthand notation for the quantity

$$\sum_{\nu=1}^{n} \mathbf{F}_\nu \cdot \delta\mathbf{r}_\nu.$$

As we remember from section 2.2.3 it is best to present the description of constrained systems with the aid of generalized coordinates. Let us see how they help in the formulation of equation (4.107), and for this purpose we use relation (4.60). Taking the latter into account virtual work (4.108) can be represented as

$$\delta W = \sum_{\nu=1}^{n} \mathbf{F}_\nu \cdot \sum_{\sigma=1}^{s} \frac{\partial\mathbf{r}_\nu}{\partial q_\sigma}\delta q_\sigma = \sum_{\sigma=1}^{s}\left(\sum_{\nu=1}^{n} \mathbf{F}_\nu \cdot \frac{\partial\mathbf{r}_\nu}{\partial q_\sigma}\right)\delta q_\sigma. \tag{4.109}$$

The quantities

$$Q_\sigma = \sum_{\nu=1}^{n} \mathbf{F}_\nu \cdot \frac{\partial\mathbf{r}_\nu}{\partial q_\sigma}, \qquad \sigma = 1,\dots,s \tag{4.110}$$

will be called **generalized forces** and expression (4.110) takes the form of

$$\delta W = \sum_{\sigma=1}^{s} Q_\sigma \delta q_\sigma \tag{4.111}$$

and expresses virtual work in generalized coordinates. If $q_\sigma$ ($\sigma = 1,\dots, s$) are generalized independent coordinates, then variations $\delta q_\sigma$ are independent and, on the basis of (4.107) and (4.111) we have

$$Q_\sigma = 0, \qquad \sigma = 1,\dots,s, \tag{4.112}$$

which means that the vanishing of the generalized force $Q_\sigma$ for each independent generalized coordinate $q_\sigma$ is a concise way of writing down the conditions of equilibrium of a system of particles subject to constraints.

Hence, we see that we can obtain as many conditions of equilibrium as there are independent variations $\delta q_\sigma$ ($\sigma = 1,\dots, s$) which can be realized in the system. In other words this means that the number of conditions of equilibrium which can be obtained for the system is equal to the number of degrees of freedom $s$. Thus, the principle of virtual work makes it possible to obtain all the conditions of equilibrium of a system of particles.

From the point of view of modelling, though, something else is most interesting. We shall see, in many instances, that we have to determine non-potential generalized forces. It turns out that it is far better to use for this purpose not the formula defining generalized forces, i.e. (4.110), but the principle of virtual work, (4.111), in which these forces appear, and this will be demonstrated in section 4.3.1.

One final remark; since the principle of virtual work concerns the state of equilibrium, the constraints applied are stationary. Then the virtual displacements are identical with admissible displacements. That is why the principle is also known as the principle of admissible displacements. Sometimes, especially in the theory of elasticity, the name of Lagrangian principle is also used.

#### *4.2.3.3 D'Alembert's principle*

In his *Traité de dynamique* in 1743, **Jean le Roland d'Alembert** (1717–1783) proposed a principle of which it is often said that it reduces a problem of dynamics to one of statics, and, in a sense this statement is true. It is commonly held that the history of analytical mechanics starts with this principle. **D'Alembert's principle** occupies as a crucial position in dynamics as the Lagrangian principle in statics. Initially, studies were connected with the work conducted by **Jacob Bernoulli** (1654–1705), who noticed that the actual motion of a pendulum is composed of, in a sense, 'hidden' motions, one caused by the force of gravity and the second by the reaction of the string. Lagrange saw in this observation a seed of the future principle of d'Alembert.

Before we discuss d'Alembert's principle, we will formulate it, as usual, in the form of a postulate. Thus, consider a system consisting of $n$ particles, whose motion is subject to $a$ holonomic and $b$ nonholonomic bilateral constraints. We shall further assume that these constraints are ideal, and that they are given by the following equations:

$$f_\alpha(\mathbf{r}_1,\mathbf{r}_2,\ldots,\mathbf{r}_n,t)=0, \qquad \alpha=1,\ldots,a, \tag{4.113}$$

$$\varphi_\beta(\mathbf{r}_1,\ldots,\mathbf{r}_n;\dot{\mathbf{r}}_1,\ldots,\dot{\mathbf{r}}_n,t)=0, \qquad \beta=1,\ldots,b. \tag{4.114}$$

The d'Alembert principle states that the motion of such a system takes place in such a way that equation

$$\sum_{\nu=1}^{n}(\mathbf{F}_\nu-m_\nu\ddot{\mathbf{r}}_\nu)\cdot\delta\mathbf{r}_\nu=0 \tag{4.115}$$

holds, in which $\delta\mathbf{r}_\nu$ are virtual displacements subject to the constraints (4.113) and (4.114).

First we note that equation (4.115) can be obtained from equation (4.106) for $p=0$. But this time something else is worth particular emphasis. We have already remarked that the reduction of the problem of dynamics to the one of statics requires explanation. We should explain, therefore, that in handbooks of mechanics the postulate of reaction for dynamic systems, i.e. the formula

$$\mathbf{F}_\nu+\mathbf{R}_\nu+\mathbf{B}_\nu=0 \tag{4.116}$$

where

$$\mathbf{B}_\nu = -m_\nu \ddot{\mathbf{r}}_\nu \tag{4.117}$$

is sometimes called the principle of d'Alembert. The postulate mentioned should be understood as follows: if the external forces $\mathbf{F}_\nu$ acting upon the points of a constrained system are complemented with forces of inertia, $\mathbf{B}_\nu$, then these forces are equilibrated by the reaction of constraints, $\mathbf{R}_\nu$. As noted by Hamel (see Hamel (1949), p. 220), such a view is an insult to d'Alembert, for it is an intolerable trivialization of the principle. A nice illustration is provided here by *Rosenberg* (see Rosenberg (1977), p. 124), which, in view of its value, we will quote in full. Thus, authors who do this proceed as follows: they rewrite Newton's second law for a single particle

$$m\ddot{\mathbf{r}} = \mathbf{F} \tag{4.118}$$

in the form

$$\mathbf{F} - m\ddot{\mathbf{r}} = 0, \tag{4.119}$$

where $\mathbf{F}$ is the resultant of all forces acting on the particle, and they call (4.118) Newton's principle, and (4.119) d'Alembert's principle. They argue that, if $\mathbf{F}$ in (4.119) is a force it may be added to $(-m\ddot{\mathbf{r}})$, and then it follows from homogeneity requirements that $(-m\ddot{\mathbf{r}})$ is also a force (usually called the *reversed effective force*, while $m\ddot{\mathbf{r}}$ is called the *inertia force*). Thus, (4.119) states that the sum of two forces vanishes. This is the statement of a static problem; hence, the dynamic problem (4.118) has been reduced to the static problem (4.119).

Now, it is evident that (4.118) and (4.119) are the same equations, their only difference being that in (4.119) all the nonzero terms have been transferred to the same side of the equal sign. Certainly, (4.119) does not involve any new 'principle' not contained in (4.118), and thus the sharp judgement of Hamel's, for the essence of the principle of d'Alembert is equation (4.115), in which variations $\delta\mathbf{r}_\nu$ are interdependent due to holonomic and nonholonomic constraints. It is therefore not permissible to conclude from equation (4.115) that $\mathbf{F}_\nu = m\ddot{\mathbf{r}}_\nu = 0$ for $\nu = 1, \ldots, n$, as suggested by formulation (4.119).

One further comment is worth making. It is usual that in place of kinematic constraints (4.114), constraints which are applied to velocities are of the form

$$\sum_{\nu=1}^{n} \mathbf{B}_\nu^{(\beta)} \cdot \mathbf{v}_\nu + D_\beta = 0, \qquad \beta = 1, \ldots, b. \tag{4.120}$$

This is quite understandable, for only with such constraints can one proceed. The question is that of 'plucking out' the vectors of virtual displacements from the general form (4.114), and this is not possible.

We shall give now the modified form of d'Alembert's principle, which will be used to construct models of motion of holonomic and—above all—nonholonomic systems. The modification was introduced by Lagrange already in 1780 and presently is known as the **fundamental equation**.

#### *4.2.3.4 The fundamental equation*

Using the virtual work expression (4.111), d'Alembert's principle (4.115) takes the form

$$\sum_{\nu=1}^{n} m_\nu \ddot{\mathbf{r}}_\nu \cdot \delta \mathbf{r}_\nu = \sum_{\sigma=1}^{s} Q_\sigma \delta q_\sigma. \tag{4.121}$$

It now remains to convert the right-hand side to achieve the double objective of introducing generalized coordinates and replacing scalar for vectorial kinematical properties. To do this, we make use of relation (4.60), and equation (4.121) can be then expressed in the form

$$\sum_{\sigma=1}^{s} Q_\sigma \delta q_\sigma = \sum_{\sigma=1}^{s} \left( \sum_{\nu=1}^{n} m_\nu \ddot{\mathbf{r}}_\nu \cdot \frac{\partial \mathbf{r}_\nu}{\partial q_\sigma} \right) \delta q_\sigma. \tag{4.122}$$

Since

$$\frac{\mathrm{d}}{\mathrm{d}t}\left( m\dot{\mathbf{r}} \cdot \frac{\partial \mathbf{r}}{\partial q} \right) = m\ddot{\mathbf{r}} \cdot \frac{\partial r}{\partial q} + m\mathbf{r} \cdot \frac{\mathrm{d}}{\mathrm{d}t}\left( \frac{\partial \mathbf{r}}{\partial q} \right), \tag{4.123}$$

the right-hand side of equation (4.122) takes now the form

$$\sum_{\sigma=1}^{s} \left[ \sum_{\nu=1}^{n} \frac{\mathrm{d}}{\mathrm{d}t}\left( m_\nu \dot{\mathbf{r}}_\nu \cdot \frac{\partial \mathbf{r}_\nu}{\partial q_\sigma} \right) - \sum_{\nu=1}^{n} m_\nu \dot{\mathbf{r}}_\nu \cdot \frac{\mathrm{d}}{\mathrm{d}t}\left( \frac{\partial \mathbf{r}_\nu}{\partial q_\sigma} \right) \right] \delta q_\sigma. \tag{4.124}$$

If relation $\partial \dot{\mathbf{r}}_\nu / \partial \dot{q}_\nu = \partial \mathbf{r}_\nu / \partial q_\nu$ is used in the first term, (4.124) becomes

$$\sum_{\sigma=1}^{s} \left[ \sum_{\nu=1}^{n} \frac{\mathrm{d}}{\mathrm{d}t}\left( m_\nu \dot{\mathbf{r}}_\nu \cdot \frac{\partial \dot{\mathbf{r}}_\nu}{\partial \dot{q}_\sigma} \right) - \sum_{\nu=1}^{n} m_\nu \dot{\mathbf{r}}_\nu \cdot \frac{\mathrm{d}}{\mathrm{d}t}\left( \frac{\partial \mathbf{r}_\nu}{\partial q_\sigma} \right) \right] \delta q_\sigma. \tag{4.125}$$

It can be shown, simply by performing the indicated operations, that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left( \frac{\partial \mathbf{r}_\nu}{\partial q_\sigma} \right) = \frac{\partial}{\partial q_\sigma}\left( \frac{\mathrm{d}\mathbf{r}_\nu}{\mathrm{d}t} \right), \tag{4.126}$$

and employing this result in the second term of the above leads to the complete equation

$$\sum_{\sigma=1}^{s} \left[ \sum_{\nu=1}^{n} \frac{\mathrm{d}}{\mathrm{d}t}\left( m_\nu \dot{\mathbf{r}}_\nu \cdot \frac{\partial \dot{\mathbf{r}}_\nu}{\partial \dot{q}_\sigma} \right) - \sum_{\nu=1}^{n} m_\nu \dot{\mathbf{r}}_\nu \cdot \frac{\partial \dot{\mathbf{r}}_\nu}{\partial q_\sigma} \right] \delta q_\sigma = \sum_{\sigma=1}^{s} Q_\sigma \delta q_\sigma. \tag{4.127}$$

Recalling the identities

$$m\dot{\mathbf{r}}_\nu \cdot \frac{\partial \dot{\mathbf{r}}}{\partial \dot{q}} = \frac{\partial}{\partial \dot{q}}\left( \tfrac{1}{2} m\dot{\mathbf{r}}^2 \right) \qquad \text{and} \qquad m\dot{\mathbf{r}} \cdot \frac{\partial \dot{\mathbf{r}}}{\partial q} = \frac{\partial}{\partial q}\left( \tfrac{1}{2} m\dot{\mathbf{r}}^2 \right) \tag{4.128}$$

and introducing them into (4.127) leads to

$$\sum_{\sigma=1}^{s}\left[\sum_{\nu=1}^{n}\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial\dot{q}_\sigma}\left(\tfrac{1}{2}m_\nu\dot{\mathbf{r}}_\nu^2\right)-\sum_{\nu=1}^{n}\frac{\partial}{\partial q_\sigma}\left(\tfrac{1}{2}m_\nu\dot{\mathbf{r}}_\nu^2\right)\right]\delta q_\sigma=\sum_{\sigma=1}^{s}Q_\sigma\delta q_\sigma. \tag{4.129}$$

However, the expressions inside the parentheses are merely the kinetic energies associated with the particles in the system; denoting the sum of these kinetic energies by $T$, the final equation of motion, expressed in terms of scalar quantities, reads

$$\sum_{\sigma=1}^{s}\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial\dot{q}_\sigma}-\frac{\partial T}{\partial q_\sigma}-Q_\sigma\right)\delta q_\sigma=0. \tag{4.130}$$

This is the modified form of d'Alembert's principle in generalized coordinates—also called the **fundamental equation**. This holds both for holonomic and nonholonomic systems.

#### *4.2.3.5 The modified fundamental equation*

In order to obtain the form taking account of the commutability conditions, we develop the equation (4.130) as follows:

(1) differentiate with respect to time the term $(\partial T/\partial\dot{q}_\sigma)\delta q_\sigma$, obtaining

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial\dot{q}_\sigma}\delta q_\sigma\right)=\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial\dot{q}_\sigma}\right)\delta q_\sigma+\frac{\partial T}{\partial\dot{q}_\sigma}\frac{\mathrm{d}}{\mathrm{d}t}(\delta q_\sigma); \tag{4.131}$$

(2) in the identity (4.131) add and subtract expression $(\partial T/\partial\dot{q}_\sigma)\delta\dot{q}_\sigma$ to create the term

$$(\partial T/\partial\dot{q}_\sigma)\left(\frac{\mathrm{d}}{\mathrm{d}t}\delta q_\sigma-\delta\dot{q}_\sigma\right);$$

(3) reintroduce the definition (4.111) of virtual work, and hence equation (4.130) can be presented in the form

$$\sum_{\sigma=1}^{s}\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial\dot{q}_\sigma}\delta q_\sigma\right)\right]=\sum_{\sigma=1}^{s}\left[\frac{\partial T}{\partial\dot{q}_\sigma}\left(\frac{\mathrm{d}}{\mathrm{d}t}\delta q_\sigma-\delta\dot{q}_\sigma\right)+\frac{\partial T}{\partial\dot{q}_\sigma}\delta\dot{q}_\sigma+\frac{\partial T}{\partial q_\sigma}\delta q_\sigma\right]+\delta W. \tag{4.132}$$

Now integrating equation (4.132) over the values from initial position at time $t_1$ to final position at time $t_2$ we obtain

$$\sum_{\sigma=1}^{s}\left(\frac{\partial T}{\partial\dot{q}_\sigma}\delta q_\sigma\right)\Bigg|_{t_1}^{t_2}=\int_{x_1}^{x_2}\left[\sum_{\sigma=1}^{s}\frac{\partial T}{\partial\dot{q}_\sigma}\left(\frac{\mathrm{d}}{\mathrm{d}t}\delta q_\sigma-\delta\dot{q}_\sigma\right)+\delta T+\delta W\right]\mathrm{d}t. \tag{4.133}$$

Then applying zero variations at the ends of the integration interval gives

$$\int_{x_1}^{x_2}\left[\delta T+\delta W+\sum_{\sigma=1}^{s}\frac{\partial T}{\partial\dot{q}_\sigma}\left(\frac{\mathrm{d}}{\mathrm{d}t}\delta q_\sigma-\delta\dot{q}_\sigma\right)\right]\mathrm{d}t=0. \tag{4.134}$$

This is the most general form of the d'Alembert principle for both holonomic and nonholonomic systems, and is also called the **modified fundamental equation**.

The final form of the variational principle for nonholonomic systems depends upon the definitions of operations $dq_\sigma$ and $\delta q_\sigma$ that are finally selected. In fact, the value of the commutator

$$\left(\frac{d}{dt}\delta q_\sigma - \delta \dot{q}_\sigma\right)$$

in equation (4.133) depends on this definition.

At this point considerable progress has been made starting with d'Alembert's principle and generalized coordinates. The form of (4.134) bears little or no resemblance to equation (4.115); if we now consider yet another step, namely to assume commutability of operators (4.78), then we get

$$\int_{t_1}^{t_2} (\delta T + \delta W)\, dt = 0. \tag{4.135}$$

#### *4.2.3.6 Gauss' principle of least constraint*

If it is true that the modified fundamental equation belongs to the most general principles of mechanics—and at least that is what we believe—one could ask why we present yet another differential principle? Doubts may be reinforced by the fact that in Rosenberg's excellent book (Rosenberg (1977)), **Gauss's principle** is not discussed, although admissible accelerations, i.e. those elements which can be used to construct it, are considered. We decided, however, to present Gauss's principle because of its pervasive relevance to the various branches of integrated mechanics, and its usefulness in the synthesis of systems with program constraints, and therefore to modelling of systems with unilateral constraints. Gauss's principle has been applied to the description of random systems, to which other principles cannot be applied.

In general, Gauss's principle takes a special place among the differential principles of mechanics. While these and other principles have evolved over long periods of time, being the work of many people, Gauss's principle has not been the subject of much development. **Carl Friedrich Gauss** (1777–1855) formulated it in 1829, and he himself admitted it was a by-product of his studies concerning the method of least squares. The principle is sometimes presented under the name of the *principle of the least curvature of Gauss and Hertz*. **Hertz** was the best-known proponent of the work of Gauss.

Consider a system of $n$ mass particles subject to the action of holonomic and nonholonomic constraints, which, as in section 4.2.3.3, are given by equations (4.113) and (4.114). The principle of Gauss says that in the actual motion of a system

$$\sum_{\nu=1}^{n} (\mathbf{F}_\nu - m_\nu \ddot{\mathbf{r}}_\nu) \cdot \delta\ddot{\mathbf{r}}_\nu = 0 \tag{4.136}$$

must be satisfied, in which $\delta\ddot{\mathbf{r}}_\nu$ are virtual accelerations that conform with all constraints. It is assumed that accelerations are subject to variations only when position and velocity variations are zero. Such variations are sometimes called variations of Gauss, and we can easily see that equation (4.136) is a special case of equation (4.106) for $p = 2$.

Gauss's principle is the only one of the differential variational principles which has a lucid physical sense. In order to demonstrate this it is necessary to remember that the active forces $\mathbf{F}_\nu$ and masses $m_\nu$ are given and are not subject to variation. Then, virtual acceleration can be expressed in the form of

$$\delta w_\nu = -\delta\left(\frac{\mathbf{F}_\nu}{m_\nu} - \mathbf{w}_\nu\right), \tag{4.137}$$

where $\mathbf{w}_\nu$ is the acceleration of $\nu$th particle.

Having introduced (4.137) to (4.136) and put $m_\nu$ before the brackets we get

$$\sum_{\nu=1}^{n} m_\nu \left(\frac{\mathbf{F}_\nu}{m_\nu} - \mathbf{w}_\nu\right)\delta\left(\frac{\mathbf{F}_\nu}{n_\nu} - \mathbf{w}_\nu\right) = 0, \tag{4.138}$$

which can be presented as

$$\delta\left\{\tfrac{1}{2}\sum_{\nu=1}^{n} m_\nu \left(\frac{\mathbf{F}_\nu}{m_\nu} - \mathbf{w}_\nu\right)^2\right\} = 0. \tag{4.139}$$

The latter form is extremely interesting for two reasons. First, if we define the quantity called **constraint** (in our notation we use letter $Z$, the first letter of the corresponding German word '*der Zwang*') to be

$$Z = \tfrac{1}{2}\sum_{\nu=1}^{n} m_\nu \left(\frac{\mathbf{F}_\nu}{m_\nu} - \mathbf{w}_\nu\right)^2, \tag{4.140}$$

then equation (4.139), the transformed principle of Gauss, can be expressed by means of the formula

$$\delta Z = 0, \tag{4.141}$$

which means that constraint attains its extremum in actual motion!

Secondly the 'contents' of all the parentheses under the sum in definition (4.140) can be interpreted in the following manner: the quantity $\mathbf{F}_\nu / m_\nu = \mathbf{a}_\nu$ represents the acceleration that, under the action of an active force $\mathbf{F}_\nu$, a free particle (i.e. not constrained one) would attain. In reality, however, constraints exist and so the particle has acceleration $\mathbf{w}_\nu$, and hence the difference $(\mathbf{F}_\nu / m_\nu) - \mathbf{w}_\nu$ is the measure of the limitations imposed on the freedom of motion by the constraints of the system or more briefly a measure of constraint set upon a particle by constraints.

It can be demonstrated that the extremum obtained by the constraint in actual motion is a minimum. We will not prove this, as, in our opinion, it has no influence upon modelling. We shall, however, return to this in considering integral principles (see section 4.2.4.1). The fact that the extremum appearing in Gauss's principle is a minimum justifies the complete name of this principle.

### 4.2.4 Integral variational principles

#### *4.2.4.1 Contemporary formulation of integral principles*

Very probably, every reader who has looked for more information on the variational principles used in mechanics has been surprised by the lack of generally accepted terminology. There are various names given to one and the same principle, e.g. *Hamilton's principle* is also called the *principle of the least action*, the *conventional form of Hamilton's principle*, the *modified form of Hamilton's principle*, the *elementary form of Hamilton's principle*.

We do not intend to concentrate on unifying various views or on establishing common terminology, which could be of importance in the framework of a course in analytical mechanics. We will rather focus on applying these principles to modelling. Thus, we maintain that only the vanishing of the first variation of a functional is important, and consequently the occurrence of the characteristic equation for the definite functional (for example equation (4.103) for functional (4.79)). These equations are all that one needs for modelling with the variational–integral method. This approach is characteristic of the contemporary formulation of integral variational principles.

For someone who feels the above insufficient, the following information is provided:

(1) the necessary condition for the extremum of a functional, defined by the vanishing of the first variation of (4.79), should be complemented by the condition on the second variation of this functional, whence we can obtain

$$\frac{\partial^2 f}{\partial y'^2} \geq 0, \tag{4.142}$$

known as the *Legendre condition*; we emphasize that this is only a necessary, and not a sufficient, condition for the minimum of functional (4.79) (for details see Gelfand and Fomin (1963);

(2) the definition of sufficient conditions is a complicated matter; this is because firstly, the *theorem of Hilbert* should be applied, and secondly, the so-called *Weierstrass function* should be introduced; one can then formulate the appropriate sufficient conditions (usually only the so-called *condition of Jacobi*);

(3) the procedure for proving conditions sufficient for the existence of an extremum of a functional is so complicated that it is best left to specialized mathematicians; in this situation we propose not to analyse whether the real motion corresponds to a maximum or a minimum of a functional; a model is any case subject to an even more rigorous verification, namely an experiment.

#### *4.2.4.2 The Hamilton–Ostrogradski principle*

Consider a holonomic system with independent generalized coordinates $q_\sigma$ $(\sigma = 1, \ldots, s)$ and the Lagrangian function $L(t, q_\sigma, \dot{q}_\sigma) = T - V$. The integral

$$S = \int_{t_1}^{t_2} L \, \mathrm{d}t \tag{4.143}$$

bears the name of **action in the Hamiltonian** sense over the time interval $[t_1, t_2]$. Note that action is a functional depending upon the motion of a system, since in order to calculate action one must have the function $q_\sigma = q_\sigma(t)$ defined in the time interval. It should be emphasized that in this version of the principle we establish the initial time point $t_1$ and the terminal time point $t_2$.

Suppose that among the trajectories considered here there exists a natural Newtonian trajectory, i.e. the one over which a system can move in a given force field. All the other trajectories are called variational or comparative.

The **Hamilton–Ostrogradski** principle states that the motion of a system is given by the stationary value of the scalar integral (4.143). The expression can be mathematically formulated as the equation

$$\delta S = 0. \tag{4.144}$$

The principle considered is contained in the works of **William Rowan Hamilton** (1805–1865) published over the years 1834–1836. Hamilton assumed, in this context, that the system is subject to scleronomic constraints. For a more general case—that is, for rheonomic constraints—this principle was formulated and proved by **Mikhail Vasilievich Ostrogradski** (1801–1861) in 1848. In connection with this and in order to emphasize the mathematical character of the principle we refer to it as the *Hamilton–Ostrogradski principle*, a title used as a rule by Russian authors. For brevity, we will refer to it as the *classical Hamilton principle* or simply *Hamilton's principle*.

Physicists consider that Hamilton's principle plays a very prominent role in mechanics, whereas engineers are somewhat less enthusiastic. The difference results from divergent tasks and expectations. First, let us quote a statement by the famous physicist **Richard Feynman** (1918–1988) who, in his special lecture on the principle of least action, said (see Chapter 19, Vol. 2 of Feynman *et al.* (1965)):

> I have been saying we get Newton's law. That is not quite true, because Newton's law includes nonconservative forces like friction. Newton said that $m\mathbf{a}$ is equal to any $\mathbf{F}$. But the principle of least action only works for conservative systems—where all forces can be gotten from a potential function. You know, however, that on the microscopic level—on the deepest level of physics—there are no nonconservative forces. Nonconservative forces, like friction, appear only because we neglect microscopic complications—there are just too many particles to analyze. But the fundamental laws can be put in the form of a principle of least action.

This is undoubtedly very beautiful, but an engineer in mechanics is not as much interested in fundamental laws of the micro-world as in the possibility of describing the phenomena of the macro-world. For example, every day an aeronautical engineer deals uniquely with nonconservative forces. What should be do in such a situation? In fact he performs a generalization of Hamilton's principle for the case in which he is interested, and this will be dealt in the next section.

#### *4.2.4.3 The 'engineering' formulation of Hamilton's principle*

For physical systems that contain nonconservative forces, we have postulated that, for the same assumptions as for the classical Hamilton's principle, the actual motion of the system results in

$$\delta\int_{t_1}^{t_2} L\,\mathrm{d}t + \int_{t_1}^{t_2} \delta W\,\mathrm{d}t = 0, \tag{4.145}$$

in which $\delta W$ represents the virtual work done on the system by the nonconservative forces. For discrete systems, $\delta W$ is given by the formula (4.111), and for a continuous system we have

$$\delta W = \int_S \mathbf{F}\cdot\delta\mathbf{r}\,\mathrm{d}S + \int_\Omega \mathbf{R}\cdot\delta\mathbf{r}\,\mathrm{d}\Omega, \tag{4.146}$$

where the surface and body forces denoted by $\mathbf{F}$ and $\mathbf{R}$ are those forces in the system not derivable from a potential function; they are known functions of the time $t$, and of the position $\mathbf{r}$.

Equation (4.145) is often called the **extended Hamilton principle**, especially by aeronautical engineers. Students of theoretical mechanics may protest against such a title, since between (4.143) and (4.145) there is an essential difference; in the classical form of Hamilton's principle (4.143) the subject is the function $S$ and the search is for the necessary conditions of stationarity of this functional. The problem of mechanics thus reduces to a question of variational calculus. In contrast, equation (4.145) is only a statement that the quantity

$$\delta\tilde{S} = \delta S + \int_{t_1}^{t_2} \delta W\,\mathrm{d}t = \int_{t_1}^{t_2} (\delta L + \delta W)\,\mathrm{d}t \tag{4.147}$$

vanishes. However, the functional does not exist, for there is no magnitude whose variation would be equal $\delta\tilde{S}$! This would mean, for instance, that the following statement would not be admissible: 'out of all possible forms of motion the one that will be realized will follow a trajectory such that the functional

$$\bar{S} = \int_{t_1}^{t_2} (L + W)\,\mathrm{d}t \tag{4.148}$$

attains its minimum, and this is expressed by the requirement that $\delta\tilde{S} = 0$'. In reality, however, expressions of this type are often encountered in modelling of complex continuous systems with variational methods.

It appears that it is possible to establish a commonality between the two statements if the deviation from orthodoxy is not too great, on one hand, and words are more carefully chosen, on the other. In order to demonstrate this, let us refer to an example. A good one is provided by the lateral oscillations of a beam loaded by aerodynamic forces; these forces undoubtedly belong to the large category of nonconservative forces.

Let $y = y(x, t)$ denote a small deflection of the neutral axis of the beam with respect to the initial state. From elementary strength of materials, we know that strain energy, $V$, of a beam is

$$V = \tfrac{1}{2}\int_0^l EI(x)\left(\frac{\partial^2 y}{\partial x^2}\right)^2 \, dx, \tag{4.149}$$

where $EI$ is the flexural stiffness of the beam and $l$ is its span.

The kinetic energy, $T$, of the beam, neglecting rotation of the elements of the beam with respect to the axis perpendicular to the axis of the beam and the plane of vibrations, has the form

$$T = \tfrac{1}{2}\int_0^l \mu(x)\left(\frac{\partial y}{\partial t}\right)^2 \, dx, \tag{4.150}$$

where $\mu$ denotes the mass per unit length of the beam.

Assuming that vibrations take place in one plane and that they are small, the virtual work done by the nonconservative (aerodynamic) forces is

$$\delta W = \int_0^l p(x,t)\,\delta y \, dx, \tag{4.151}$$

where $p$ denotes the aerodynamic force per unit length.

It can therefore be seen from all the forms of the principle that we will need the variations $\delta T$ and $\delta V$. On the basis of (4.149) and (4.150) we obtain

$$\begin{aligned} \delta T &= \int_0^l \mu(x)\frac{\partial y}{\partial t}\,\delta\!\left(\frac{\partial y}{\partial t}\right) dx, \\ \delta V &= \int_0^l EI(x)\frac{\partial^2 y}{\partial x^2}\,\delta\!\left(\frac{\partial^2 y}{\partial x^2}\right) dx. \end{aligned} \tag{4.152}$$

Using (4.152) and (4.151) in equation (4.147) results in

$$\int_{t_1}^{t_2}\int_0^l\left[\mu(x)\frac{\partial y}{\partial t}\,\delta\!\left(\frac{\partial y}{\partial t}\right) - EI(x)\frac{\partial^2 y}{\partial x^2}\,\delta\!\left(\frac{\partial^2 y}{\partial x^2}\right) + p(x,t)\delta y\right] dx\, dt = 0. \tag{4.153}$$

Now we must extract the variations $\delta y$ from expressions $\delta(\partial y/\partial x)$ and $\delta(\partial^2 y/\partial x^2)$. This performed conventionally through integration by parts. In doing this we use the property (4.78). Hence

$$\begin{aligned} \int_{t_1}^{t_2} \mu(x)\frac{\partial y}{\partial t}\,\delta\!\left(\frac{\partial y}{\partial t}\right) dt &= \int_{t_1}^{t_2} \mu(x)\frac{\partial y}{\partial t}\frac{\partial}{\partial t}(\delta y)\, dt \\ &= -\mu(x)\frac{\partial y}{\partial t}\,\delta y\Big|_{t_1}^{t_2} - \int_{t_1}^{t_2}\frac{\partial}{\partial t}\left(\mu(x)\frac{\partial y}{\partial t}\right)\delta y\, dt \\ &= -\int_{t_1}^{t_2} m(x)\frac{\partial^2 y}{\partial t^2}\,\delta y\, dt, \end{aligned} \tag{4.154}$$

because $\delta y$ vanishes at $t = t_1$ and $t = t_2$ as the initial and terminal configurations are specified (classical Hamilton's principle). Integration over the spatial variable in a similar fashion yields

$$\int_0^l EI(x)\frac{\partial^2 y}{\partial x^2}\delta\left(\frac{\partial^2 y}{\partial x^2}\right)\mathrm{d}x = \int_0^l \frac{\partial^2}{\partial x^2}\left(EI(x)\frac{\partial^2 y}{\partial x^2}\right)\delta y\,\mathrm{d}x - \left[\frac{\partial}{\partial x}\left(EI(x)\frac{\partial^2 y}{\partial x^2}\right)\delta y\right]_0^l + \left[EI(x)\frac{\partial^2 y}{\partial x^2}\delta\left(\frac{\partial y}{\partial x}\right)\right]_0^l. \tag{4.155}$$

Note an important detail here, namely that the second term does not vanish this time, unless we only consider specific boundary conditions, i.e. supports at both terminals of a beam.

Using (4.154) and (4.155) in equation (4.153) results in

$$\int_{t_1}^{t_2}\int_0^l \left\{\mu(x)\frac{\partial^2 y}{\partial t^2} + \frac{\partial^2}{\partial x^2}\left(EI(x)\frac{\partial^2 y}{\partial x^2}\right) - p\right\}\delta y\,\mathrm{d}x\,\mathrm{d}t = 0. \tag{4.156}$$

On the basis of an adequate lemma of variational calculus, analogous to the Du Bois–Raymond lemma presented in section 4.2.2.5, we obtain the field equation

$$\mu(x)\frac{\partial^2 y}{\partial t^2} + \frac{\partial^2}{\partial x^2}\left(EI(x)\frac{\partial^2 y}{\partial x^2}\right) = p(x,t). \tag{4.157}$$

We have thus obtained the correct equation and this is most important for a modeller. From this point of view we may see that the controversies mentioned are not irreconcilable. First of all, it appears, according to the section 4.2.4.1, that the requirement of minimality is not necessary at all; it is quite sufficient to have stationarity—that is vanishing of $\delta\tilde{S}$. This means, further, that it is not important whether equation (4.147) constitutes the principle, and whether $\tilde{S}$ is a functional.

#### *4.2.4.4 Postulating versus derivation once more*

Now let us divide the generalized forces appearing in formula (4.111) into two parts, i.e. potential forces, $Q_\sigma^{\mathrm{pot}}$, and nonconservative ones, $Q_\sigma^{\mathrm{nc}}$, and then represent them in the following form:

$$Q_\sigma = Q_\sigma^{\mathrm{pot}} + Q_\sigma^{\mathrm{nc}} \tag{4.158}$$

and we make use of the definition of potential energy $V = V(t, q_\sigma)$:

$$Q_\sigma^{\mathrm{pot}} = -\frac{\partial V}{\partial q_\sigma}; \tag{4.159}$$

then

$$\delta W^{\mathrm{pot}} = -\sum_{\sigma=1}^{s}\frac{\partial V}{\partial q_\sigma}\delta q_\sigma = -\delta V. \tag{4.160}$$

Equation (4.135) now becomes

$$\int_{t_1}^{t_2} \delta(T-V)\,\mathrm{d}t + \int_{t_1}^{t_2} \delta W^{\mathrm{nc}}\,\mathrm{d}t = 0 \tag{4.161}$$

or, if use is made of the definition of the Lagrangian function $L = T - V$ and of the property (4.98),

$$\delta\int_{t_1}^{t_2} L\,\mathrm{d}t + \int_{t_1}^{t_2} \delta W^{\mathrm{nc}}\,\mathrm{d}t = 0. \tag{4.162}$$

Thus, it has the same form as the engineering form of Hamilton's principle (4.145), since $\delta W$ now denotes the virtual work of only the nonconservative forces. One could therefore assign to (4.162) the label 'D', as it arises by derivation of the engineering form of Hamilton's principle from d'Alembert's principle, and to (4.145) the label 'P', indicating that it is the result of postulation.

Hamilton's principle could be considered not as a consequence of Newton's second law (although, of course, it is in agreement with the latter) but as an equivalent postulate of mechanics. There is nothing new in the idea that it is very useful in cases in which direct application of Newton's second law is cumbersome. We maintain, namely an engineer may find himself in a practical situation in which application of the method of Newton is not only cumbersome but impossible. This usually occurs when we are dealing not with a purely mechanical system but with coupled systems. Recalling section 4.2.4.3, and through the example, we would like to encourage beginners in modelling to be bold, leave the footprints of the past and go where the familiar track—that is, the relation between the Newton's second law and Hamilton's principle—is no longer visible. There, only the 'compass of analogy' can be used if for some reasons the variational principle is chosen. More simply, a well-founded tool in classical mechanics is a general instrument for developing the equations in integrated mechanics.

## 4.3 MODELLING OF HOLONOMIC SYSTEMS

Quite numerous complex mechanical systems encountered in engineering practice may be presented in the form of models with holonomic constraints. The equations of motion of such systems can be obtained with the help of *Lagrange's equations of the second kind*. These equations constitute the most important instrument for modelling complex holonomic systems and that is why the present action begins with various methods of derivation of Lagrange's equations. However, in some specific cases, it may be more suitable or even necessary to use other types of equations. That is why the *Boltzmann–Hamel* and *Lagrange–Maxwell* equations are presented in sections 4.3.2 and 4.3.3.

### 4.3.1 Lagrange equations of the second kind

We shall show two methods of derivation: one originating from d'Alembert's principle in the Lagrange form, which is also called the fundamental equation, and the second from Hamilton's principle in the standard form. The fundamental equation in generalized coordinates (4.130) has been found to have the form