$$i = \frac{\mathrm{d}e}{\mathrm{d}t}, \tag{2.37}$$

with e being an electric charge. To avoid confusion we do not use the popular notation q, because q is traditionally used for the mechanical generalized coordinate.

If a source of electromotive force $E(t)$ is yet introduced into the circuit considered then, on the basis of the *second Kirchhoff law* we obtain

$$Ri = L\frac{\mathrm{d}i}{\mathrm{d}t} + \frac{1}{C}\int i \,\mathrm{d}t = E(t), \tag{2.38}$$

from which, taking into account (2.37), we get

$$L\ddot{e} + R\dot{e} + \frac{1}{C}e = E(t). \tag{2.39}$$

Let us refer not to equation (1.25) of damped oscillator with excitation, i.e.

$$m\ddot{x} + b\dot{x} + kx = F(t). \tag{2.40}$$

We now see the correspondence of parameters, and more importantly that the electric charge e corresponds to coordinate x, which, in this case, is a generalized coordinate. This means that an electric charge can also be treated analogously as a generalized coordinate.

2.2.4 The number of degrees of freedom

According to the traditional definition (see, e.g. Thompson (1961)) *the number of degrees of freedom of a body corresponds to the minimum number of independent coordinates required to define its position.* Resulting from this definition, the number of degrees of freedom is given by the formula (2.20). Some comments on nonholonomic systems will now follow. Considerations on continuous system will be contained in section 2.3.4.

The notion of the number of degrees of freedom was introduced on the basis of independent generalized coordinates, which are determined with the holonomic constraint equations, assuming there are no nonholonomic constraints. When there are nonholonomic constraints present along with the holonomic ones, the number of degrees of freedom, l, of a system, is defined as the difference between the minimum number of independent coordinates, s (i.e. the number of degrees of freedom of the holonomic system), and the number b of equations of nonholonomic constraints, that is

$$l = s - b. \tag{2.41}$$

Thus, the nonholonomic system of Fig. 2.2 has $5 - 2 = 3$ degrees of freedom, and not 5, as would be suggested by the standard definition, quoted before. This has serious consequences for modelling, since a proper model is obtained when the number of equations of motion is equal to the number of degrees of freedom. It is worth mentioning at this point that the various texts on analytical mechanics often contain the definition of the number of degrees of freedom for nonholonomic systems, which involves the notion of virtual displacement, namely *the number of degrees of freedom of a mechanical system is the number of virtual displacements of this system* (see, e.g. Neimark & Fufaev (1972)). This will be discussed in further detail in section 4.2.2.1, after the notion of virtual displacement is commented upon in more detail.

The view of equivalence of the number of degrees of freedom has yet one more quite interesting aspect. Thus, if we accept the definition of Arnold (see Arnold (1978), chapter 2, §4), saying that the system with one degree of freedom is the system described by one differential equation

$$\ddot{x} = f(x), \tag{2.42}$$

then we have, consequently, to accept the notion of the semi-degree of freedom related to the system described by the equation

$$\dot{x} = \varphi(x). \tag{2.43}$$

This situation can be encountered when we are performing a simplification of a model already obtained. Let us consider the motion of a thin plate of mass m, immersed in a liquid and attached to the end of a helical spring with a stiffness k. Assume that the plate was allowed to oscillate vertically in the fluid. Referring to Fig. 2.7, and applying Newton's second law of motion, we get

$$m\ddot{x} = mg - b\dot{x} - k(d + x), \tag{2.44}$$

where b denotes viscous damping coefficient, d is the static extension of the spring. Since $mg = kd$, we obtain

$$m\ddot{x} + b\dot{x} + kx = 0, \tag{2.45}$$

which corresponds to equation (2.42).

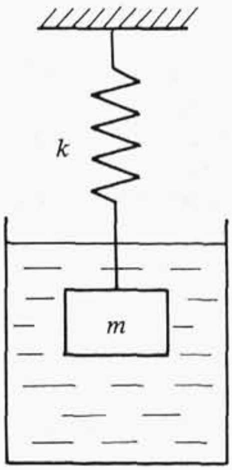

Fig. 2.7.

If we assume then that the mass of the plate can be neglected, the inertial component disappears from the model:

$$b\dot{x} + kx = 0, \tag{2.46}$$

which, in turn, corresponds to equation (2.43). Thus, in such a natural way, the mathematical model with half a degree of freedom has been obtained.

Such a situation occurs where one performs far more substantial idealization, consisting in this case in neglecting the mass of an element. Thus, for instance, the system composed of an elastic but massless rod, with certain mass concentrated at the end of the rod, and with an attached plate placed in a liquid (see Fig. 2.8), will have one and a half degree of freedom.

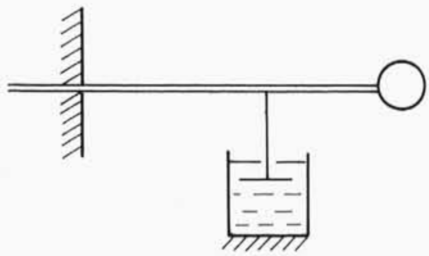

Fig. 2.8.

Studies of models with non-integer numbers of degrees of freedom started somewhat intuitively in the monograph (Andronov *et al.* (1966)) and have since resulted in unexpected developments within the framework of optimal control theory. One should perhaps mention that control theory refers also to the notion of the number of degrees of freedom, which is taken to mean the number of magnitudes which can be used to control a system.

2.2.5 Representations of the motion in space

On the basis of abundant literature one can say that the *science of dynamics is the science of motion*. Motion can be represented in different ways, depending upon the nature of the problem. In particular, the aim of proceeding may consist in obtaining a simple form of equations of motion of a mechanical system, or in 'seeing' their solutions. Various descriptions may be put together on the basis of variously defined spaces, being generalizations of the usual physical space. The purpose of generalization always remains, however, the same, namely: transformation of the motion from the usual space in which the motion is representable by a point, or by a curve (called a *trajectory*) traced by a point.

In classical mechanics one usually refers to four kinds of spaces: the **configuration space**, the **space of events**, the **phase space**, and the **state space** (compare Synge (1960)). For the convenience of the reader we shall give here Table 2.1 in which the representative spaces are listed in order of increasing dimensionality.

Of all the representation spaces, Q is the simplest. If the system consists of a single particle moving in ordinary space, then Q is ordinary space; and if the particle is constrained to move on a curve or surface (e.g. described by equation (2.7)), then Q is that curve or surface. However, the picture of the totality of trajectories is somewhat complicated since a trajectory is not determined by a point in Q and a direction in Q. It is easier to visualize the totality of trajectories in QT, in which a trajectory is determined by a point and a direction.

Table 2.1.

Name	Symbol	Coordinates	Dimensionality
Configuration space	Q	q_σ	s
Space of events	QT	q_σ, t	$s+1$
Phase space	QP	q_σ, p_σ	$2s$
State space	QPT	q_σ, p_σ, t	$2s+1$

To illustrate these considerations let us find the configuration space for a double pendulum (Fig. 2.9a). We choose angles ϕ and ψ of rotation of both pendula about their suspension axes as the generalized coordinates. Thus, any position of such a system is determined by two angles ϕ, ψ ($0 \le \phi$, $\psi \le 2\pi$). Therefore, the position of the system can be represented by a point of a square with sides equal to 2π on the plane (ϕ, ψ). Then, however, the correspondence between the positions of the double pendulum and points of the square will not be continuous, although it will be unique. In order to make it continuous we first roll the square to form a cylinder, by joining appropriate sides, and then we join the bases of the cylinder to form a torus, see Fig. 2.9b. Thus, the configuration space for the double pendulum is a torus.

The two spaces Q and QT are sufficient for analysing phenomena using the methods of Lagrangian mechanics. In Hamiltonian mechanics, however, we choose to replace all

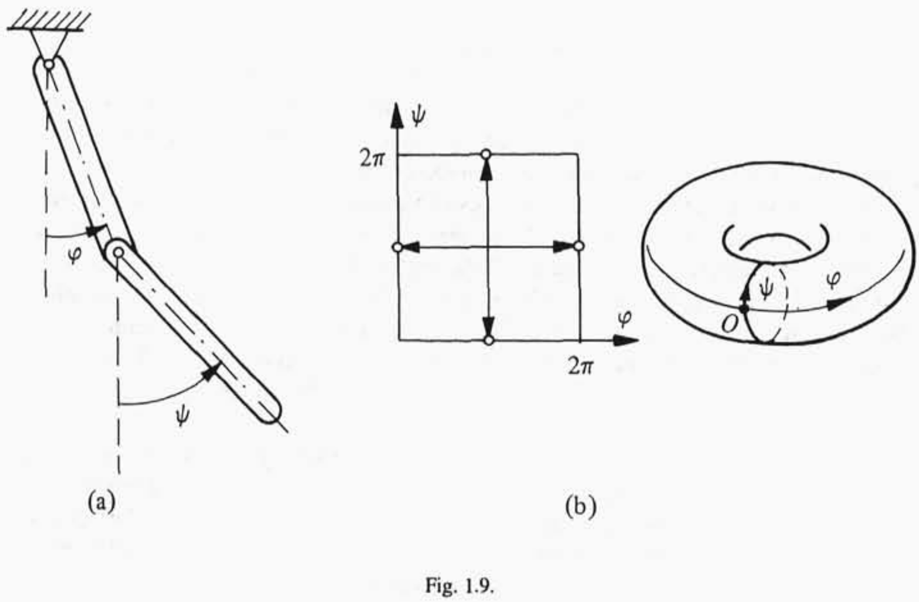

Fig. 1.9.

the generalized velocities $\dot{q}_\sigma$ by independent coordinates p_σ, called the generalized momenta, being defined as

$$p_\sigma = \frac{\partial L}{\partial \dot{q}_\sigma}, \tag{2.47}$$

where

$$L = T - V \tag{2.48}$$

is the **Lagrangian** of a system, where T is the kinetic energy and V the potential energy of the system.

The space QP is, thanks largely to the work of **Josiah W. Gibbs** (1839–1903) on statistical mechanics, probably the best known of spaces listed above. If the system is *conservative*, then the totality of trajectories appears as a congruence of curves in QP, one curve passing through each point. This is a satisfying simple picture, but it is complicated in the non-conservative case, for then there is a single infinity of trajectories passing through each point.

In the space QPT the time is treated on a part with the coordinates q_σ and the momenta p_σ. The picture of the trajectories is simpler than in QP for a non-conservative system, for now we have a congruence of curves, one through each point. QPT differs from QP in having an odd dimensionality—an important difference from the mathematical standpoint.

The simple pendulum will serve to illustrate the concept of phase space. We may write, for small angular deflections,

$$L = T - V = \tfrac{1}{2} ml^2 \dot{\phi}^2 - \tfrac{1}{2} mgl\,\phi^2$$

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}_\sigma} = ml^2 \dot{\phi}. \tag{2.49}$$

Hence $\dot{\phi} = p_\phi / ml^2$, and the mechanical energy of the pendulum,

$$E = T + V = \frac{p_\phi^2}{2ml^2} + \frac{mgl}{2}\phi^2, \tag{2.50}$$

is constant, since the system is conservative. Here p_ϕ and ϕ are the coordinates in phase space. Figure 2.10 illustrates the simple two-dimensional space in this instance. For a given E, fixed by the initial circumstances of the motion, the *representative point P* is restricted to move on an ellipse $(\phi^2/A^2) + (p_\phi^2/B^2) = 1$, $A = \sqrt{(2E/mgl)}$, and $B = \sqrt{(2Eml^2)}$, as is shown in the figure.

If the phase space is multidimensional it is convenient to use partial phase space diagrams, which assist in visualizing the behaviour of a complex system. In many applications when the mass remains constant, the phase space (or phase plane in two dimensions) is adequately described by the coordinates q (displacement) and $\dot{q}$ (velocity).

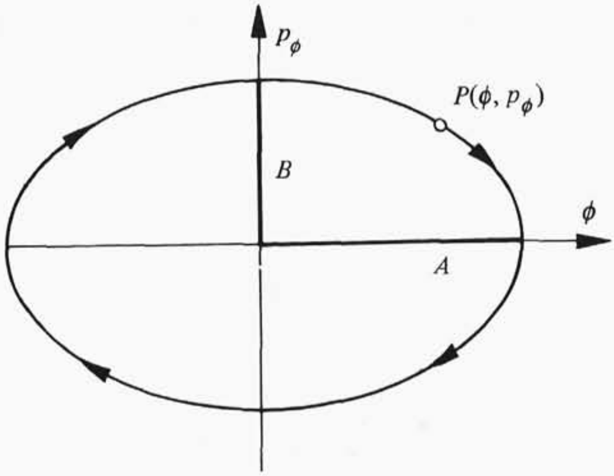

Fig. 2.10.

2.2.6 Quasi-coordinates

As demonstrated in section 2.2.3, due to introduction of generalized coordinates the equations of holonomic constraints were trivially satisfied, which is quite an important advantage. Still, there are yet the equations of nonholonomic constraints, which were represented in the form (2.24). The question therefore arises as to whether there is a way of doing away also with these constraint equations. Luckily, the answer is positive: the method sought is based upon the introduction of so-called *quasi-coordinates* (sometimes also called *pseudo-coordinates*). The notion of quasi-coordinates was considered by **Vito Volterra** (1860–1940) in 1898.

Notice first that the above equations, whose number is b, may be used to determine b generalized velocities, e.g. $\dot{q}_{l+1},\ldots,\dot{q}_s$, through other velocities $\dot{q}_1,\ldots,\dot{q}_l$, (where $l = s - b$ is the number of degrees of freedom of the nonholonomic system, see formula (2.41)). Velocities $\dot{q}_\lambda$ $(\lambda = 1,\ldots,l)$ can be given any values, thereby determining the values of other velocities. We shall, however, take another more general way, in the sense that we shall not adopt l generalized velocities as independent magnitudes, but rather l independent linear combinations of these velocities:

$$\overset{\circ}{\vartheta}_\lambda = \sum_{\sigma=1}^{s} f_{\sigma\lambda}(t, q_\sigma)\dot{q}_\sigma + f_\lambda(q_\sigma), \qquad \lambda = 1,\ldots,l. \tag{2.51}$$

Note that the notation $\overset{\circ}{\vartheta}_\lambda$ was used in expression (2.51) to emphasize the fact that notation $\dot{\vartheta}_\lambda$ may not make sense at all, since the right-hand side of (2.51) may not necessarily represent a total derivative. Were it so, variables ϑ_λ would represent the usual generalized coordinates, although differing from the original ones, i.e. q_λ. It ought also to be stressed that linear forms (2.51) should be subject to just one condition, namely that these l linear expressions form together with b linear expressions

$$\sum_{\sigma=1}^{s} B_{\beta\sigma}(t,q_\sigma)\dot{q}_\sigma + B_\beta(t,q_\sigma) = 0, \qquad \beta = 1,\ldots,b, \tag{2.52}$$

a complete system of $s = l + b$ linearly independent forms, that is, the determinant formed out of the coefficients of these s expressions does not vanish. Then, $\overset{\circ}{\vartheta}_\lambda\,(\lambda = 1,\ldots,l)$ may assume any values, and consequently it is possible to compute corresponding $\dot{q}_\sigma$ $(\sigma = 1,\ldots,s)$ after the system of linear equations (2.51), (2.52) has been solved. This leads to

$$\dot{q}_\sigma = \sum_{\lambda=1}^{l} h_{\sigma\lambda}\,\overset{\circ}{\vartheta}_\lambda + h_\sigma, \qquad \sigma = 1,\ldots,s, \tag{2.53}$$

where $h_{\sigma\lambda}$ and h_σ are functions of t and $q_1,\ldots,q_s$.

Magnitudes $\overset{\circ}{\vartheta}_\lambda$, given by (2.51), being linear combinations of generalized velocities, are called **quasi-velocities**, and ϑ_λ **quasi-coordinates** $(\lambda = 1,\ldots,l)$. In particular, $\overset{\circ}{\vartheta}_\lambda$ may be identical with some generalized velocities. In the general case, however, $s + l$ magnitudes $\dot{q}_\sigma$ and $\overset{\circ}{\vartheta}_\lambda$ are interrelated via (2.51) and (2.53).

It should be noted that the notion of quasi-coordinates may alternatively be introduced in a somewhat different manner, namely through the assumption that the number of equations relating generalized velocities $\dot{q}_\sigma$ is s, and not l. Equation (2.51) then leads to another equation

$$\overset{\circ}{\vartheta}_\rho = \sum_{\sigma=1}^{s} a_{\rho\sigma}\dot{q}_\sigma, \qquad \rho = 1,\ldots,s. \tag{2.54}$$

It seems to us that the latter method corresponds better to the purpose of elimination of nonholonomic constraints, to be shown later on. Furthermore, this method also encompasses the so-called *kinematic characteristics* or *parameters* (see section 4.4.3), which is introduced for nonholonomic systems because of its greater generality.

A popular example of quasi-coordinates in the sense of (2.54) is provided by projections of the angular velocity vector of a rigid body on the main axes of inertia of this body (ξ, η, ζ) (Chorlton (1983)):

$$\begin{aligned} \omega_\xi &= p = (\sin\theta\sin\phi)\,\dot{\psi} + (\cos\phi)\dot{\theta}, \\ \omega_\eta &= q = (\sin\theta\cos\phi)\,\dot{\psi} - (\sin\phi)\dot{\theta}, \\ \omega_\zeta &= r = \dot{\phi} + (\cos\theta)\,\dot{\psi}. \end{aligned} \tag{2.55}$$

Magnitudes p, q and r differ from the generalized velocities $\dot{\phi}, \dot{\eta}, \dot{\psi}$ by the fact that they are not the total derivatives with regard to time of any of the generalized coordinates. This is an important difference! In order to better illustrate this fact let us represent relations (2.54) in the form

$$\mathrm{d}\vartheta_\rho = \sum_{\sigma=1}^{s} a_{\rho\sigma}\,\mathrm{d}q_\sigma, \qquad \rho = 1,\ldots,s, \tag{2.56}$$

which, by the way, sometimes constitutes the initial form for the introduction of quasi-coordinates, and now let us return, in a more formalized manner, to the question of the symbolic meaning of the magnitude $\overset{\circ}{\vartheta}_\lambda$, already considered before. Thus, there is a condition, known from mathematical literature, for every differential form (2.56) to be integrable; this condition has the form of

$$\frac{\partial a_{\rho\sigma}}{\partial q_\tau} - \frac{\partial a_{\rho\tau}}{\partial q_\sigma} = 0, \qquad \rho,\sigma,\tau = 1,\ldots,s. \tag{2.57}$$

If condition (2.57) is satisfied, then variables ϑ_σ simply represent the usual generalized coordinates. When, however, this condition is not satisfied (i.e. at least some differential forms (2.56) are not total differentials), then ϑ_σ cannot be the usual generalized coordinates. The consequence of the notion of generalized coordinates is that there are such magnitudes which uniquely determine the position of the mechanical system (see section 2.2.3). Simultaneously, it can be demonstrated, for instance, that the counterpart of a closed loop in the space of generalized coordinates q_σ is not a closed loop in the space of quasi-coordinates ϑ_σ. It is just this gap in the correspondence between the position of a mechanical system and the values of quasi-coordinates that constitute this important feature which differentiates the latter from the usual generalized coordinates. In such a situation a question arises as to whether such magnitudes can be at all useful in modelling? It turns out that they can, since in spite of the fact that quasi-coordinates are not uniquely related to position of the system, the motion along a curve in the space of quasi-coordinates uniquely represents, due to (2.55), the motion of the mechanical system.

In order to illustrate these considerations with a concrete example let us look at the motion called *regular precession*:

$$\phi = at + \phi_0, \quad \psi = bt + \psi_0, \qquad \theta = \theta_0, \tag{2.58}$$

where a, b, ϕ_0, ψ_0, and θ_0 are given constants. In the Euler angle space (ϕ, ψ, θ) this motion is represented by a straight line in the plane parallel ($\theta = \theta_0$) to the plane (ϕ, ψ). On the other hand in the space of quasi-coordinates, denoted (P, Q, R), this motion is represented by a spiral line lying on the surface of a circular cylinder; parametric equations of this line have the form:

$$\begin{aligned} P &= \int_0^t p\,\mathrm{d}t = \frac{b}{a}\sin\theta_0[\cos\phi_0 - \cos(at+\phi_0)], \\ Q &= \int_0^t q\,\mathrm{d}t = \frac{b}{a}\sin\theta_0[-\sin\phi_0 + \sin(at+\phi_0)], \\ R &= \int_0^t r\,\mathrm{d}t = (a + b\cos\theta_0)t. \end{aligned} \tag{2.59}$$

Now calculate the velocity of translation of a point along this curve:

$$v = \sqrt{(\dot P^2 + \dot Q^2 + \dot R^2)} = \sqrt{(a^2 + 2ab\cos\theta_0 + b^2)}. \tag{2.60}$$

It is then an angular velocity of rotation of a rigid body with a fixed point. If, therefore, we are interested first of all in changes over time of the angular velocity vector, then representation of the body's motion in the space of quasi-coordinates turns out even more convenient!

In a way, however, this is just a by-product of the properties of quasi-coordinates, useful in analysis. The notion of quasi-coordinates becomes especially useful in modelling, when a mechanical system is subject to nonholonomic constraints besides the holonomic ones. We are able then, notwithstanding the nonintegrable nature of nonholonomic constraints, to define a set of quasi-coordinates in such a way as to make them satisfy these very equations as identities. In other words, if nonholonomic constraints take the analytic form (2.24), then the introduction of these linear combinations as quasi-velocities results, on the basis of (2.51), in

$$\overset{\circ}{\vartheta}_\beta = \sum_{\sigma=1}^{s} B_{\beta\sigma}(t, q_\sigma)\dot{q}_\sigma + B_\beta(q_\sigma) \equiv 0, \qquad \beta = 1,\dots,b. \tag{2.61}$$

Finally, we should emphasize that the value and significance of quasi-coordinates consists not only in the capacity of 'neutralizing' nonholonomic constraints. They can be very useful also when only holonomic constraints appear, as we shall see in derivation of the *Boltzmann–Hamel equations* in section 4.3.3. In order to have ready the formulae there, we shall introduce quasi-velocities according to the definition (2.54) or (2.56). Besides these we shall need a reverse relation between the generalized velocities $\dot{q}_\sigma$, $(\sigma = 1,\dots, s)$ and quasi-velocities $\overset{\circ}{\vartheta}_\rho$ $(\rho = 1,\dots, s)$. For this purpose let us write down equation (2.54) in the matrix form

$$\overset{\circ}{\vartheta} = \mathsf{A}\dot{\mathsf{q}}, \tag{2.62}$$

where

$$\overset{\circ}{\vartheta} = \left[\overset{\circ}{\vartheta}_1,\dots,\overset{\circ}{\vartheta}_s\right]^{\mathrm{T}}, \qquad \dot{\mathsf{q}} = \left[\dot{q}_1,\dots,\dot{q}_s\right]^{\mathrm{T}},$$

and $A = [a_{\rho\sigma}]$, ρ, $\sigma = 1,\dots, s$. Assuming that det $\mathsf{A} \neq 0$, equation (2.54) may be solved for $\dot{q}_\sigma$ $(\sigma = 1,\dots,s)$, and we get

$$\dot{\mathsf{q}} = \mathsf{B}\overset{\circ}{\vartheta}, \tag{2.63}$$

where

$$\mathsf{B} = [b_{\sigma\rho}], \tag{2.64}$$

and

$$\mathsf{B} = \mathsf{A}^{-1}. \tag{2.65}$$

Now, equation (2.63) in its 'working' shape is as follows:

$$\dot{q}_\rho = \sum_{\sigma=1}^{s} b_{\sigma\rho}\overset{\circ}{\vartheta}_\sigma, \qquad \rho = 1,\dots,s. \tag{2.66}$$