The problem now relies upon the expression of the spring extension in terms of generalized coordinates. The positions where the springs are attached depend on many coordinates and it may require quite serious calculation to determine the current extension of a spring. Additionally the spring attachment point can assume any arbitrary position in the body. It means that the body may have an arbitrary number of terminals and only current data concerning the position of vectors $\mathbf{a}_\nu$ (or $\mathbf{b}_\nu$) with respect to body B_i (or B_j), establish an actual terminal position.

From what we have said above it follows that the mechanical elements appearing in a kinematic chain of bodies don't fulfil the fundamental axiom of system theory, on which the linear graph method hinges.

Additionally, three-dimensional rotation between two bodies cannot be represented simply by three independent numbers; the numerical values depend also on the sequence in which the three rotations occur.

Both features of multi-body systems described above affect the coupling of equations of motions and their complexity. Moreover, since the models of separate mechanical elements are useless when they are interconnected to form a system, the linear graph method is not adequate in modelling problems involving many rigid bodies.

In the next section we will present a method which provides a useful aid for a modelling of certain classes of rigid-body systems.

5.4 MODELLING OF RIGID-BODY SYSTEMS

5.4.1 Introductory remarks†

The problem of multi-rigid body system modelling was investigated, among others, by **J. Wittenburg**, and he solved it using the Newtonian approach combined with graph-theoretical aids. Although we shall present another method, most of the introductory definitions and comments made by J. Wittenburg in his excellent monograph (Wittenburg (1977)) will be useful for us. We shall therefore follow his development and his definitions in this section.

Mechanical systems investigated in most student textbooks consist of either a single rigid body or several rigid bodies in some particularly simple geometric configuration. The important role they play in classical mechanics is due to the fact that their equations of motion can be integrated in closed form. However, the engineer in his everyday practice is confronted with an endless variety of much more complex systems. To mention only a few examples, one may think of linkages in machines, of steering mechanisms in cars, of railway trains consisting of elastically connected cars, of walking machines and manipulators, etc. The assumption that the individual bodies of such systems are rigid is an idealization which may or may not be acceptable, depending largely on the kind of problem under investigation. Thus, in a crank-and-slider mechanism, the seemingly rigid connecting rod has to be treated, as an elastic member when its forced bending vibrations are of concern. At the other extreme, the human body, which is composed of

† A substantial part of the reasoning in this section, and the introduction of the mathematical description of the interconnection structure (section 5.4.4) is based on items from the book *Dynamics of Systems of Rigid Bodies* by Jeans Wittenburg. The authors make grateful acknowledgment to Teubner Verlag for permission to quote these items from the above-named book.

obviously nonrigid members, may well be treated as a system of interconnected rigid bodies when only its gross motion is of interest. In this section, all bodies will be assumed rigid. In the joint connecting the bodies nonrigid members such as springs and dampers will be allowed. The goal of this investigation is a formulation of a system of exact nonlinear differential equations of motion. The mathematical formulae to be developed should satisfy two requirements which, in general, are not easily fulfilled simultaneously. First, they should be general enough to describe the dynamic behaviour of diverse mechanical systems. Second, their application to any particular mechanical system should be possible with only a minimum amount of preparatory work. The Lagrange equations of the second kind, (4.174), for example, satisfy only the first requirement, since when applying them to any particular mechanical system, a substantial amount of labour is required to formulate the Lagrangian L and its derivatives. The equations of motion to be developed in this chapter are considerably more explicit.

For the complete description of a multi-body system, a large number of parameters is required. They must specify the geometry and mass distribution, as well as the nature of forces acting from outside the system and internally in the joints between bodies. Those describing geometry and mass distribution can be subdivided into the following groups:

(1) the number of bodies;
(2) parameters specifying the interconnection structure of the system;
(3) parameters specifying the constraints imposed on the system bodies;
(4) parameters specifying the location of joints on the bodies;
(5) masses and inertia components of the bodies.

Before going into any detail, some definitions and introductory comments are required. Figure 5.26 illustrates a four-body system. Between certain pairs of bodies there is a direct interaction by internal forces. Thus, for example, between the bodies numbered 2 and 3 there exists direct force interaction caused by the constraints in the joint connecting these two bodies. Between bodies 3 and 4 there is a direct interaction by magnetic forces. Bodies 2 and 4, on the other hand, do not act directly upon each other. Their interaction is only indirect via another body. Two bodies are said to be **contiguous** if and only if they exert force on each other directly. The forces, by means of which two contiguous bodies interact, may be categorized into two classes:

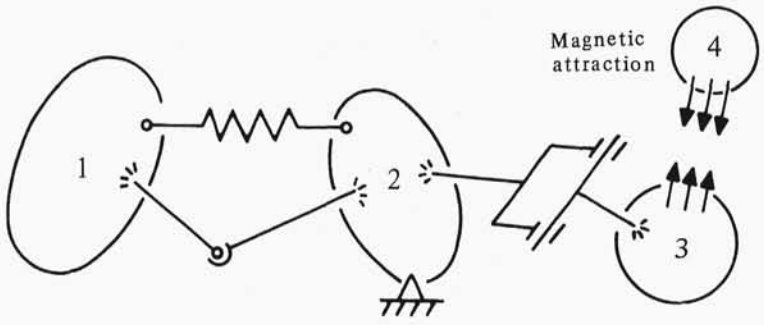


Fig. 5.26.

(1) those resulting from the existence of geometrical constraints between two bodies, and
(2) others, for instance exerted by springs or dampers, then gravitational forces, magnetic forces etc.

The geometrical constraints limiting the relative motion of two contiguous bodies will be alternatively called a **joint**, and two bodies connected by a joint will be referred to as **adjacent** bodies. A device or devices by means of which two contiguous bodies interact mutually will be called a **coupling**. It means that the coupling has a wider sense than the joint. A coupling may be a joint, a magnetic force, a spring, a damper etc. Various devices exerting forces between two contiguous bodies will not be considered here, but will be dealt with later when the potential energy of springs is considered.

In order to present the new method as simply as possible, we shall assume that the constraints in the joints are scleronomic and holonomic. All constraints must be ideal. In practice this means, among other things, that no dry friction is allowed in the joints. All these assumptions could be omitted, but the derivation and use of the equations would be more complicated.

Kinematic constraints are introduced not only by the individual joints but also through the interconnection structure of the system. This is illustrated by the plane crank-and-slider mechanism in Fig. 5.27 whose bodies are interconnected by three pin joints and one sliding joint. The body labelled 'base' is assumed to be fixed in inertial space. The total number of degrees of freedom is one. This remains unchanged if one pin joint is replaced by a ball-and-socket joint. On the other hand, it becomes zero if the axes of the three pin joints are not mounted parallel to each other. The crank-and-slider mechanism is a simple example of a broad and important class of multi-body systems categorized as systems with **closed kinematic chains**. In such systems the number of degrees of freedom depends on more than just the kinematic properties of the individual joints. In order to define a closed kinematic chain it is necessary to introduce first the notion of path between two bodies.

Consider any two bodies in a multi-body system, for instance bodies i and j (Fig. 5.28a). Proceed from body i to body j via sequence of bodies and joints in such a way that no joint is passed more than once. The set of joints thus defined is called the path between bodies i and j. If for all pairs of bodies the path between them is uniquely

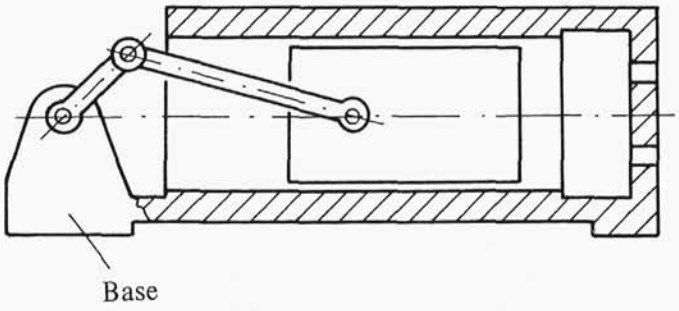
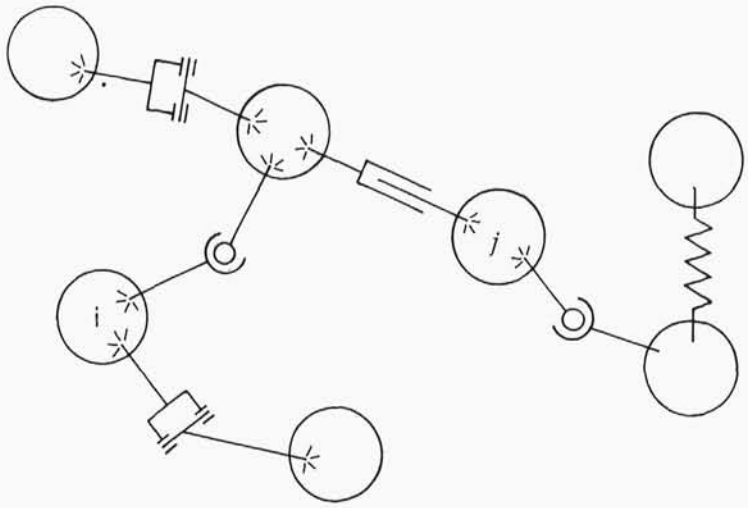
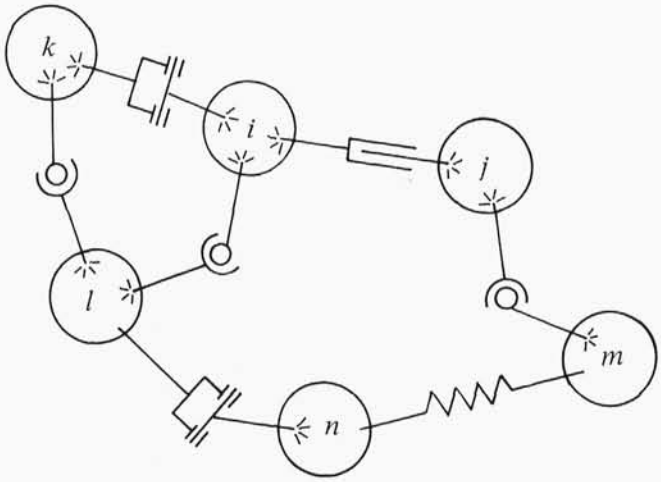


Fig. 5.27.



(a)



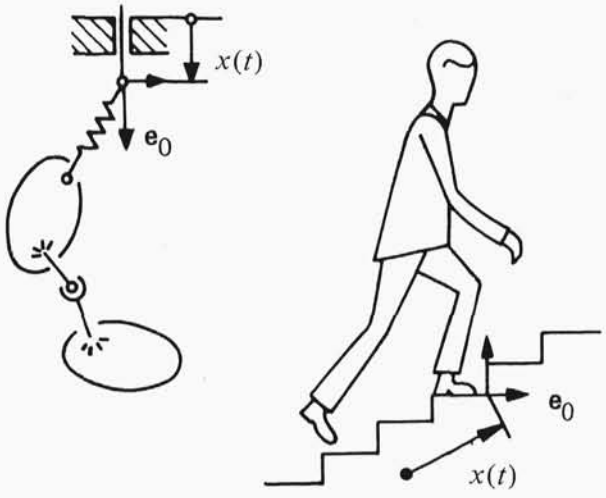
(b)

Fig. 5.28.

defined, as is the case for the system of Fig. 5.28a, then the system is said to have **tree structure**. If, on the other hand, between two bodies two different paths exist, then these two paths form a closed chain. The system of Fig. 5.28b contains two closed chains. The first consists of the bodies i, k, l and the second consists of the bodies i, j, m, n, l. If, in

particular, every coupling in a closed chain contains at least one constraint limiting the relative motion of contiguous bodies, i.e. a joint, then the closed chain is called a **closed kinematic chain**. Of the two closed chains in Fig. 5.28b only that consisting of the bodies i, k, l is a closed kinematic chain. In the second closed chain, for one of its couplings there are no constraints imposed on two contiguous bodies, therefore that chain is not a closed kinematic chain.

Multi-body systems are found in practice under two basically different conditions of operation. In most systems one or more bodies are connected by joints to an external body whose position in inertial space is a prescribed function of time. Typical examples are a double pendulum with a moving suspension point (Fig. 5.29a), the human body with one or both feet resting on an escalator (Fig. 5.29b) and most linkages in machines where the frame of the machine is the external body. It is obvious that the dimensions and inertial properties of the external body are irrelevant since its motion is prescribed. For this reason the external body system will not be counted as another body system, but will be represented by a moving base rigidly attached to it. In Fig. 5.29a, b, this base is called $\mathbf{e}_0$. The prescribed motion of the base as well as the properties of the joints between the base and the system will enter equations of motion to be developed. Comparatively rare is the mode of operation in which no body of a system is connected to an external body whose motion is prescribed. Typical examples for such systems are a flying helicopter (Fig. 5.30a) and the human body in a phase of motion without contact with the ground (Fig. 5.30b). For the formulation of the scalar differential equations of motion for such systems, some common frame of reference is needed in which vectors and tensors can be decomposed. Depending on the particular problem under consideration, this frame will be



(a) (b)

Fig. 5.29.

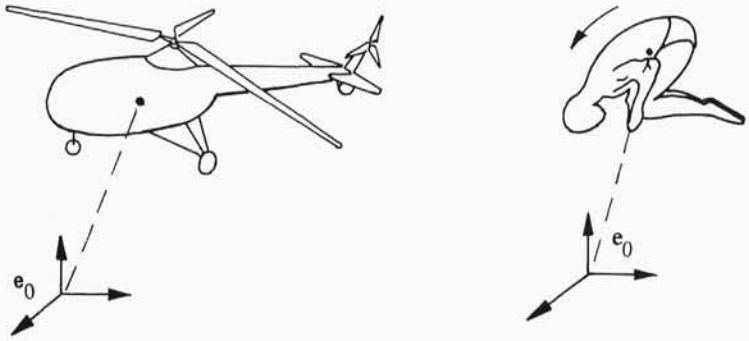


Fig. 5.30.

moving relative to inertial space according to some appropriately chosen function of time. In Fig. 5.30, the moving base is called $\mathbf{e}_0$. The position of a multi-body system in inertial space is uniquely specified if the position of adjacent bodies relative to each other is known for all joints and if, in addition, the position in $\mathbf{e}_0$ is known for one arbitrarily chosen body of the system. This suggests the introduction of a **fictitious joint** between the moving base and the one arbitrarily chosen body (indicated by a dashed line in Fig. 5.30). With this joint, in which, of course, there are no internal constraint forces, and with the moving base $\mathbf{e}_0$, the situation is now the same as for the systems in Fig. 5.29. The mathematical description of the interconnection structure of a system will be the same for both modes of operation.

Most multi-body systems found in engineering practice have an interconnected structure consisting of closed kinematic chains. Nevertheless, we shall investigate a system of rigid bodies comprising open kinematic chains, i.e. having a tree structure. There are two reasons for considering this class of systems; one is the greater simplicity of the mathematical description of the interconnection structure and of system kinematics, and the other is that any system with closed chains can be transformed into a system with tree structure by cutting suitably selected joints. Thus in order to obtain the equations of motion for a system with closed chains, all that is necessary is to add internal forces and geometric constraints for the cut joints to the equations of motion for a system with structure. The procedure applied in these cases is described in Wittenburg (1977).

Together with a system of bodies we shall consider a graph of the system. Such a graph comprises vertices, representing the bodies, and edges, representing the joints between the bodies. Obviously, the graph of an open kinematic chain has a tree structure.

Between systems with open kinematic chains, there may exist systems in which two or more parts are kinematically independent (i.e. are not kinematically interconnected) while at the same time these parts interact by means of certain forces (magnetic, elastic, etc.). An example of this kind of system is shown in Fig. 5.31a; the graph of this system is disconnected (Fig. 5.31b). For simplicity, say, we have two parts of system which are kinematically uncoupled. There are two ways of describing the motion of this system. In the first, the motion of one body of each part is described in an inertial frame base, $\mathbf{e}_0$. This is equivalent to the introduction of two fictitious joints, one for each part of a system

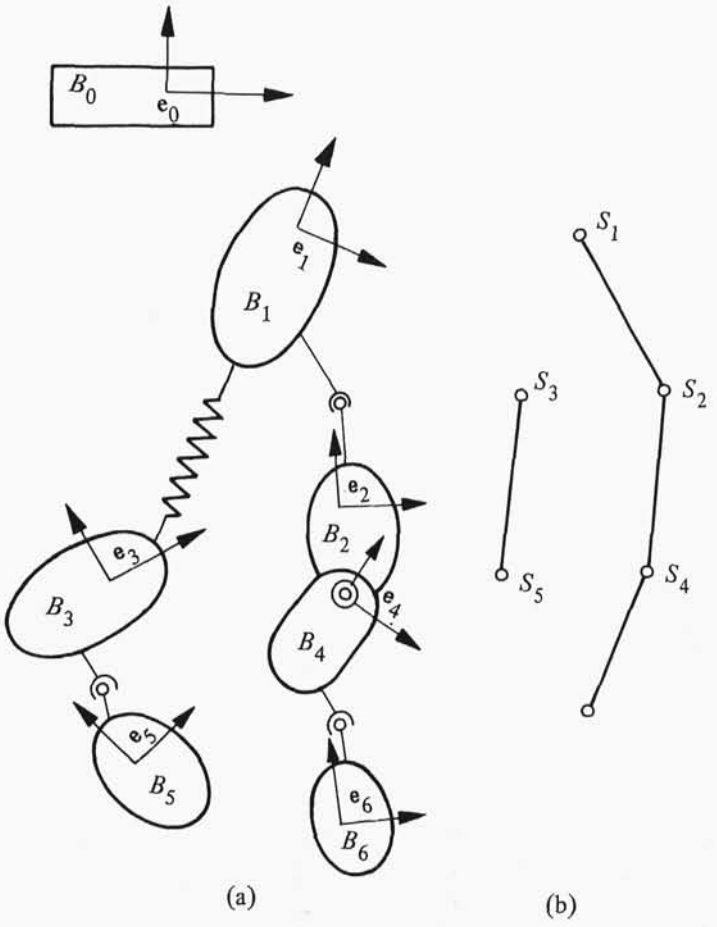


Fig. 5.31.

(Fig. 5.32a). In this case the kinematics of the two parts may be separated, and consequently in the task of kinetic energy determination the two parts of the system may be considered independent (Fig. 5.32a). In the second method, the description of the motion of such a system is to relate one body of one part to the inertial frame base $\mathbf{e}_0$ which, in the graph, produces an additional edge (s_0, s_1) representing the first fictitious joint. We then choose one body, say the jth, in the second part whose motion is described in the base of one, say the ith, body of the first part of a system. This is equivalent to the introduction of a second fictitious joint, and an additional edge (s_i, s_j) has to be added to the graph. In the example of Fig. 5.32b, the motion of a body 3 is described in base $\mathbf{e}_1$; therefore, an additional edge (s_1, s_3) appears in the graph. Of importance is the fact that, in both cases, the topological tree structure remains, and with the vertex s_0 representing the reference body, only one edge is incident.

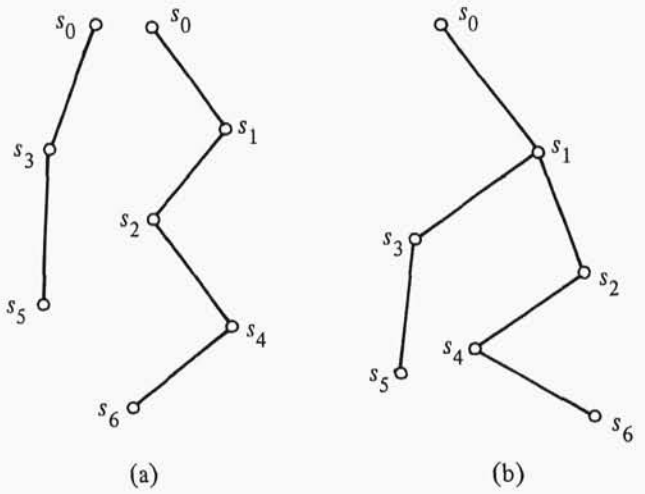


Fig. 5.32.

5.4.2 The key idea

For an introductory explanation of the key idea, let us consider first the simplest vibratory system shown in Fig. 5.33.

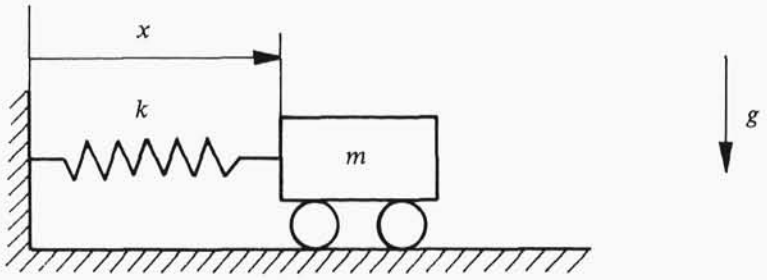


Fig. 5.33.

The equation of free motion of that system is

$$m\ddot{x} + kx = 0. \tag{5.23}$$

The Lagrangian of this system,

$$L = \tfrac{1}{2}(m\dot{x}^2 - kx^2), \tag{5.24}$$

is a function of variables $\dot{x}, x$ and two parameters m, k, so we can write

$$L = L(x, \dot{x}, m, k) = L(X, P), \tag{5.25}$$

where X is a set of variables consisting of x and $\dot{x}$, and P is a set of parameters consisting of m, and k.

Both parameters m, and k are **information carriers** about inertia and stiffness of two system elements, namely of a rigid body and of a linear spring, respectively. Obviously, the same equations (5.25) may be used for many different systems possessing the same topological structure. It is enough to insert the current numerical values of mass and stiffness into the equation (5.25), to obtain the equation valid for a particular data set.

Now we ask whether a similar procedure for a more complicated system is possible. To answer this question let us consider two mechanical systems shown in Fig. 5.34. Both systems have three translational degrees of freedom and consist of the same elements, i.e. three bodies and three springs. In both systems the same coordinates x_1, x_2, x_3 have been introduced. It is the **system structure** also called a **system topology** which differs them. The system topology influences the form of the Lagrangian and we have for the system in Fig. 5.34a

$$L_1 = \tfrac{1}{2}(m_1\dot{x}_1^2 + m_2\dot{x}_2^2 + m_3\dot{x}_3^2) - \tfrac{1}{2}\left[k_1(x_1 - l_1)^2 + k_2(x_2 - x_1 - l_2)^2 + k_3(x_3 - x_2 - l_3)^2\right] \tag{5.26}$$

and for the system in Fig. 5.34(b)

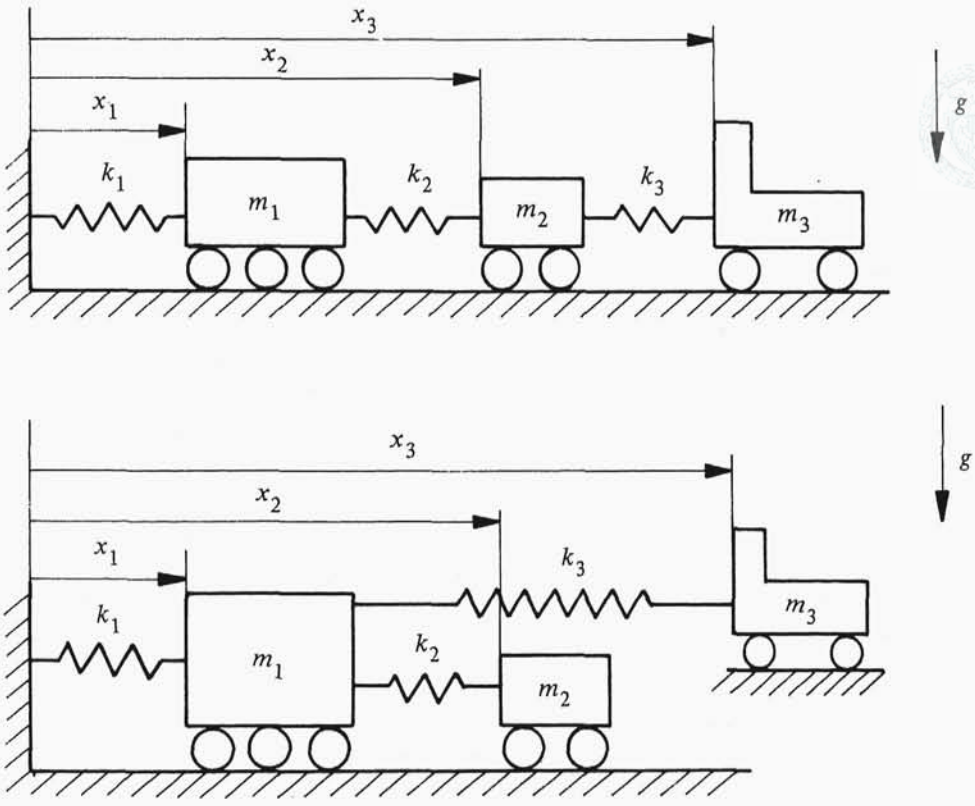


Fig. 5.34.

$$L_2 = \tfrac{1}{2}(m_1\dot{x}_1^2 + m_2\dot{x}_2^2 + m_3\dot{x}_3^2) - \tfrac{1}{2}\left[k_1(x_1 - l_1)^2 + k_2(x_2 - x_1 - l_2)^2 + k_3(x_3 - x_1 - l_3)^2\right], \tag{5.27}$$

where m_1, m_2, m_3 are masses of the bodies, k_1, k_2, k_3 are stiffnesses and l_1, l_2, l_3 are the natural lengths of the corresponding springs.

Suppose now that we can express the Lagrangian L as an explicit function of not only the sets X and P but also of a certain carrier of the structure information S, i.e.

$$L = L(X, P, S). \tag{5.28}$$

The carrier of structure information may, for instance, be an incidence matrix, path matrix or certain new characteristics. Differentiating (5.28) according to Lagrange's method we can obtain the equations of motion, in which in an evident form appear not only the variables and parameters but also a structure information carrier. Thus we get equations that are valid for the whole class of systems, rather than for only one particular system as in the case of the standard application of Lagrange's method. Having generated the equations of motion once, we can use them many times without the need for kinetic and potential energy calculation, and also without performing all the differentiations as in the classical approach. This is the key idea of the proposed method.

5.4.3 Basic notation conventions

Very many of the mathematical symbols which will be used in this section require certain ordering conventions. In further operations we shall deal with scalars, vectors, scalar matrices, and matrices whose elements are vectors of, more briefly, vector matrices. To distinguish all these quantities let us adopt the following convention:

Convention 5.1
Scalars will be denoted by roman italic letters, e.g. a, b or A, B;
vectors will be denoted by roman bold letters, e.g. $\mathbf{a}$, $\mathbf{b}$ or $\mathbf{A}$, $\mathbf{B}$;
scalar matrices will be denoted by univers letters, e.g. a, b or A, B; and
vector matrices will be denoted by univers bold letters, e.g. $\boldsymbol{\mathsf{a}}$, $\boldsymbol{\mathsf{b}}$ and $\boldsymbol{\mathsf{A}}$, $\boldsymbol{\mathsf{B}}$.

We shall need very often to express vector quantities as a scalar product of a unit vector and a length of a vector. Therefore let us introduce a second convention:

Convention 5.2. For an arbitrary vector $\mathbf{a}$ we shall denote a unit vector corresponding to the vector $\mathbf{a}$ by the symbol $\hat{\mathbf{a}}$ and we shall denote by a a projection of the vector $\mathbf{a}$ on the direction of the unit vector $\hat{\mathbf{a}}$. Thus we may write $\mathbf{a} = a\hat{\mathbf{a}}$. We shall apply an analogous convention for diagonal vector matrices, i.e. if

$$\mathbf{a} = \operatorname{diag} \mathbf{a}_i \quad \text{then} \quad \mathbf{a} = \mathsf{a}\hat{\mathbf{a}}, \qquad \text{where} \quad \hat{\mathbf{a}} = \operatorname{diag} \hat{\mathbf{a}}_i, \quad \mathsf{a} = \operatorname{diag} a_i.$$

We shall illustrate the usefulness of both conventions introduced above by considering different forms of vector expressions. An arbitrary vector $\mathbf{a}$ can be represented as a linear combination of three mutually orthogonal unit vectors $\hat{\mathbf{e}}_1$, $\hat{\mathbf{e}}_2$, $\hat{\mathbf{e}}_3$, i.e.

$$\mathbf{a} = a_1\hat{\mathbf{e}}_1 + a_2\hat{\mathbf{e}}_2 + a_3\hat{\mathbf{e}}_3. \tag{5.29}$$

The unit vectors are base vectors of a **vector basis** (also called **reference basis** or simply **basis**). The scalar quantities a_1, a_2 and a_3 in (5.29) are the coordinates of $\mathbf{a}$ in the basis $\hat{\mathbf{e}}_1$, $\hat{\mathbf{e}}_2$, $\hat{\mathbf{e}}_3$ or, more briefly, the coordinates of $\mathbf{a}$ in $\hat{\mathbf{e}}$, where $\hat{\mathbf{e}}$ is a column matrix consisting of unit vectors, i.e. $\hat{\mathbf{e}} = [\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3]^T$. The right-hand side of the equation (5.29) can be given the form of a matrix product. For this purpose the column matrix $\mathsf{a} = [a_1, a_2, a_3]^T$ of the coordinates of $\mathbf{a}$ in the base $\hat{\mathbf{e}}$ is introduced (a shorter name for a is **coordinate matrix** of $\mathbf{a}$ in $\hat{\mathbf{e}}$). With these matrices the equation (5.29) takes the form

$$\mathbf{a} = \hat{\mathbf{e}}^T \mathsf{a} = \mathsf{a}^T \hat{\mathbf{e}}. \tag{5.30}$$

Also a form of a vector representation given in Convention 5.2, i.e.

$$\mathbf{a} = a\hat{\mathbf{a}} \tag{5.31}$$

will be often used.

Note that we distinguish between a vector $\mathbf{a}$ as a certain physical quantity, a column matrix a which consists of the coordinate triple a_i ($i = 1, 2, 3$) and another column matrix $\mathbf{a} = [\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3]^T$, which in turn consists of three vectorial components $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$ of a given vector $\mathbf{a}$.

Another representation of a vector $\mathbf{a}$ we may obtain introducing an $(n \times 1)$ column matrix $\mathsf{1}_n$ comprising n unit entries, i.e. $\mathsf{1}_n = [1, 1, \ldots, 1]^T$. Thus we have

$$\mathbf{a} = \mathbf{a}^T \mathsf{1}_3 = \mathsf{1}_3^T \mathbf{a}. \tag{5.32}$$

By means of the column matrix $\mathsf{1}_n$, each diagonal matrix may be rearranged in the associated column matrix. For this purpose we shall adopt a subsequent convention:

Convention 5.3. If $\mathsf{a} = \operatorname{diag} a_i$, $i = 1, \ldots, n$, then $\tilde{\mathsf{a}}$ denotes the $(n \times 1)$ column matrix determined by the following relation:

$$\tilde{\mathsf{a}} = \mathsf{a}\mathsf{1}_n. \tag{5.33}$$

In numerous expressions we shall use row and column matrices. Some of them being *a priori* defined as the row or column matrices, the other being distinguished as rows or columns. To differentiate both kinds of row and column matrices we adopt the following:

Convention 5.4. The matrices defined *a priori* as either row or column matrices will be denoted simply by usual symbols as it was established by Convention 5.1, e.g. a, b or A, B. Sometimes, for better clarity, the column matrices appearing in formulae are denoted by the symbol $\{\ \}$, i.e. $\{\mathsf{a}\}$, means the column matrix $\mathsf{a} = [a_1, a_2, \ldots, a_n]^T$. For a notation of the row or column matrix being the ith row or ith column of a given matrix A, we use two methods. In the first, often used in the finite element method, the symbols $\lfloor \mathsf{A} \rfloor_i$ and $\{\mathsf{A}\}_i$ denote the ith row and ith column of a given matrix A, respectively, and in the second, the ith row and the ith column of a given $(n \times n)$ matrix A are the results of multiplications. Namely $\lfloor \mathsf{A} \rfloor_i = \varepsilon_i^T \mathsf{A}$ and $\{\mathsf{A}\}_i = \mathsf{A}\varepsilon_i$, where $\varepsilon_i = [0, \ldots 0, 1, 0, \ldots, 0]^T$ is an $(n \times 1)$ column matrix consisting of zero entries except for one unit entry in the ith row (ε_i is also called the **isolating vector**).

In mathematical formulae, we will use a new type of matrix product, namely a scalar product of two matrices whose elements are vectors. In order to perform this kind of operation let us adopt a following convention:

Convention 5.5. Let $\mathbf{a}$ be an $(m \times r)$ matrix with vectors $\mathbf{a}_{ij}$ $(i = 1,\ldots, m, j = 1,\ldots, r)$ as elements and $\mathbf{b}$ be an $(r \times n)$ matrix with vectors $\mathbf{b}_{ij}$ $(i = 1,\ldots, r, j = 1,\ldots, n)$ as elements. Then, the scalar product $\mathbf{ab}$ is a scalar $(m \times n)$ matrix with elements

$$(\mathbf{ab})_{ij} = \sum_{k=1}^{r} \mathbf{a}_{ik} \cdot \mathbf{b}_{kj}, \qquad i = 1,\ldots, m, \quad j = 1,\ldots, n. \tag{5.34}$$

5.4.4 The mathematical description of the interconnection structure

One step in the process is the establishment of the Lagragian, $L = T - V$, as a function of generalized coordinates q_σ and generalized velocities $\dot{q}_\sigma$ $(\sigma = 1,\ldots, \kappa; \kappa$ is a number of degrees of freedom), either through Lagrange's equations of the second kind or from Hamilton's principle. In the case of rigid-body systems, the requirement for the determination of kinetic and potential energies as a function of q_σ, $\dot{q}_\sigma$ means that the positions and velocities of each body have to be determined by means of the same variables i.e. q_σ and $\dot{q}_\sigma$. The actual expression for the position and velocity of any particular rigid body point depends on both the actual body state and generalized coordinates introduced.

Thus let us consider briefly what influences a given choice of coordinates. As we already know, systems of rigid bodies may be categorized into two fundamentally different classes. In the first class, one body of the system is connected to an external body performing prescribed motion, and in the second class none of the bodies in the system is connected to the external body. Thus in the first there is a real joint between a body of the system and the external body and in the second such a joint does not exist. However, for unification of the mathematical description of the joint structure of a system, a fictitious joint between the movable or immovable base $\mathbf{e}_0$ and one arbitrarily chosen body is introduced. Therefore, there always exists one distinguished body whose position can be fully described with the help of information about the location of base $\mathbf{e}_0$ and values of generalized coordinates associated with this body. Such a body will be called the **main carrier**. It is defined explicitly in the case when the system of bodies is joined with the external body, but otherwise its definition is arbitrary.

We usually begin the introduction of generalized coordinates for a system of bodies at the main carrier. Then the other generalized coordinates are successively introduced in accordance with constraints appearing between the bodies.

The body B_i is a **carrier** of the body B_j if a change in time of any one generalized coordinate associated with the body B_i causes a motion of the body B_j when all other generalized coordinates are frozen. In other words we may say that each body B_i which kinematically influences B_j is a carrier of the body B_j. The **direct carrier** of the body B_j is the carrier incident with B_j.

The interconnection structure of a system is conveniently displayed by a graph. Let the graph G related to the system bodies comprise $n + 1$ vertices representing the bodies (including the external body), and n edges representing the joints (including the fictitious joint). Bodies and joints are labelled separately. The order in which the numbers are

assigned is arbitrary, but a substantial clarification of notions as well as algorithms may be achieved if we introduce so-called **regular labelling** in the manner presented below.

Due to the isomorphism between the real system and its graph, the notions carrier, main carrier and direct carrier concerned with the bodies have their equivalents in the set of graph vertices. Thus among the vertices of the graph G, we will identify a vertex representing the external reference body B_0, and this vertex is called s_0. Since the vertex s_0 belongs to a tree structure graph, it is also referred to as a **root**. Then the vertex incident with s_0 is labelled s_1. It represents the main carrier. The edge connecting s_0 and s_1 is labelled 1. For a given graph with vertex s_0, a regular labelling can be achieved as follows. The graph contains at least one **peripheral vertex**. Peripheral vertices are all vertices except s_0, in which only one edge is incident. To these peripheral vertices, the highest numbers n, $n-1$, $n-2$ etc. are assigned. We give the same number to the corresponding edges incident to labelled vertices. Then all the vertices and edges already labelled (except s_0) are cut off from the graph. This results in a smaller graph with new peripheral vertices to which, in turn, the highest numbers still available are assigned. This recursive procedure is continued until all vertices and edges have been labelled. Proceeding in this manner, the only vertex which is incident to s_0 and the edge connecting these two vertices are labelled s_1 and 1, respectively as before. In Fig. 5.35 two directed graphs are shown. The graph in Fig. 5.35a is arbitrarily labelled, while a graph in Fig. 5.35b is regular labelled.

An arbitrary orientation of the edges may be assigned to any graph, turning them into arcs. If, in the regularly labelled graph G, the edge orientation has been introduced so that each arrow is directed to the vertex with the larger (or alternatively, smaller) number, then the obtained graph will be called a regularly directed graph with ascending (or descending) orientation.

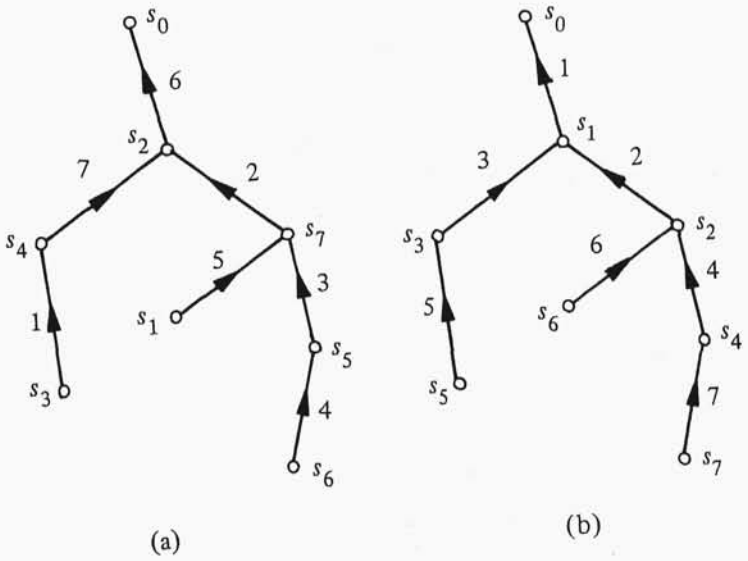


Fig. 5.35.

The regular labelled and regular directed graph will be called here a **regular graph** (this latter notion has a quite different meaning in graph theory).

In a graph with tree structure, a path between s_i and s_j is uniquely defined for each combination of i and j. In Fig. 5.35a, for instance, the path between s_6 and s_3 is the set of arcs 4, 3, 2, 7, 1. A vertex s_k is said to be on the path between s_i and s_j if at least one arc belonging to this path is incident with s_k. According to this definition, the vertices s_i and s_j themselves are on the path between s_i and s_j.

With the help of these ideas the following weak ordering relationship for vertices is defined. The symbol $s_i \leq s_j$ means that s_i is on the path between s_0 and s_j. The relationship $s_i < s_j$ means that s_i is on the path between s_0 and s_j but that it is not identical with s_j. Finally, $s_i \not\leq s_j$ is the negation of $s_i \leq s_j$. Note that for two vertices s_i and s_j both relationships, $s_i \not\leq s_j$ and $s_j \not\leq s_i$, can be satisfied simultaneously. Consider, for instance, s_3 and s_4 or s_6 and s_7 in Fig. 5.35b.

As we already know, the interconnection structure of the directed graph is uniquely represented by the incidence matrix $\hat{\mathsf{S}}$. In a specific case of a tree structure graph with $n+1$ vertices, the matrix $\hat{\mathsf{S}}$ has $n+1$ rows and n columns which correspond to the vertices and arcs, respectively. In further calculations we shall also use certain submatrices of $\hat{\mathsf{S}}$. The matrix $\hat{\mathsf{S}}$ can be partitioned into two submatrices S_0 and S, where S_0 is the row matrix

$$\mathsf{S} = [S_{01} \ldots S_{0n}] \tag{5.35}$$

and S is the square matrix

$$\mathsf{S} = \begin{bmatrix} S_{11} & \cdots & S_{1n} \\ \vdots & & \vdots \\ S_{n1} & \cdots & S_{nn} \end{bmatrix} \tag{5.36}$$

The matrices $\hat{\mathsf{S}}$ and S may be also expressed as the following differences:

$$\hat{\mathsf{S}} = \hat{\mathsf{S}}_+ - \hat{\mathsf{S}}_-, \qquad \mathsf{S} = \mathsf{S}_+ - \mathsf{S}_-, \tag{5.37}$$

where the matrices $\hat{\mathsf{S}}_+$, S_+ are obtained from $\hat{\mathsf{S}}$ and S respectively by preserving in them only positive, i.e. 1 entries, the matrices $\hat{\mathsf{S}}_-$, S_- are obtained from $\hat{\mathsf{S}}$ and S respectively by replacing the 1 entries by zeros and by changing the sign of all the –1 entries. For a directed graph of Fig. 5.35b the four matrices defined above are

$$\mathsf{S}_0 = [-1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0], \tag{5.38}$$

$$\mathsf{S} = \begin{bmatrix} 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \tag{5.39}$$

$$S_+ = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \tag{5.40}$$

$$S_- = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{5.41}$$

Although the matrix $\hat{S}$ provides complete information about the interconnection structure of a graph, in the case of tree structure regular graphs there exists a much more convenient, i.e. compact, form of algebraic graph representation. To present it, let us introduce the two integer functions i^+ and i^-, which establish relationships between arc indices and vertex indices. For arc indices $i = 1,\ldots, n$, $i^+(i)$ is the index of the vertex away from which the ith arc is pointing, and $i^-(i)$ is the index of the vertex toward which the ith arc is pointing.

For the graph of Fig. 5.35a the two functions read as follows:

i	1	2	3	4	5	6	7
i^+	3	7	5	6	1	2	4
i^-	4	2	7	5	7	0	2

For a regular graph with ascending orientation, $i^+(i) = i$, and with descending orientation, $i^-(i) = i$. From this, it follows that the interconnection structure of a regular graph may be uniquely defined by means of only one function, i.e. either i^- in the case of ascending or i^+ in the case of descending graph orientation. For example, for the graph in Fig. 5.35b we have

i	1	2	3	4	5	6	7
i^+	1	2	3	4	5	6	7
i^-	0	1	1	2	3	2	4

Note that the symbols i^+, i^- have to be considered in close conjunction with a given graph. So for instance, for the graph in Fig. 5.35a we have $4^+ = 6$, $4^- = 5$, and for the graph in Fig. 5.35b we have $4^+ = 4$ and $4^- = 2$.

In the cases of both functions i^+ and i^-, their arguments are from the set of arc indices and the values are from the set of vertex indices. Let us now define one more integer function $i^\sim$. For vertex indices $i = 1,\ldots, n$, $i^\sim$ is the index of its direct carrier. This means that the arguments of the function $i^\sim$ take values from the set of all vertex indices excluding the index 0, i.e. except for the root index. It is convenient to write down the functions i^+, i^-, $i^\sim$ by means of column matrices i^+, i^-, $\mathsf{i}^\sim$. For instance, for the graph in Fig. 5.35a we have

$$\mathsf{i}^+ = [3,7,5,6,1,2,4]^T, \quad \mathsf{i}^- = [4,2,7,5,7,0,2]^T, \quad \mathsf{i}^\sim = [7,0,4,2,7,5,2]^T,$$

and for the regular graph in Fig. 5.35b we have

$$\mathsf{i}^+ = [1,2,3,4,5,6,7]^T, \quad \mathsf{i}^- = [0,1,1,2,3,2,4]^T, \quad \mathsf{i}^\sim = [0,1,1,2,3,2,4]^T.$$

Note that for a regular graph with descending (or ascending) orientation the matrices i^- and $\mathsf{i}^\sim$ (or i^+ and $\mathsf{i}^\sim$) are equal.

In our further considerations, we shall employ the path matrix $\mathsf{P} = [P_{ij}]$, $i, j = 1,\ldots, n$ of the tree structure graph, where

$$P_{ij} = \begin{cases} 1 & \text{if arc } i \text{ belongs to the path between } s_j \text{ and } s_0 \text{ and is directed toward } s_0, \\ -1 & \text{if arc } i \text{ belongs to the path between } s_j \text{ and } s_0 \text{ and is directed away from } s_0 \\ 0 & \text{otherwise.} \end{cases}$$

For the graph of Fig. 5.35b, the path matrix has the form

$$\mathsf{P} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{5.42}$$

Because of regular labelling, P is upper triangular and because of the fact that the graph is regular directed, all nonzero entries of P have the same sign, in this case positive.

5.4.5 The kinetic energy

The kinetic energy for a point mass is defined as $T = \frac{1}{2} m\mathbf{v}^2$, where $\mathbf{v}$ is the absolute velocity of m, i.e. its velocity relative to an inertial reference base. For a rigid body, as for any deformable body, the kinetic energy is the integral

$$T = \tfrac{1}{2}\int_m \mathbf{v}^2 \, \mathrm{d}m, \tag{5.43}$$

where now $\mathbf{v}$ is the absolute velocity of mass particle $\mathrm{d}m$ of a body.

Let us now consider a rigid body in arbitrary motion (Fig. 5.36). The absolute velocity $\mathbf{v}$ of a mass particle $\mathrm{d}m$ is

$$\mathbf{v} = \mathbf{v}_P + \boldsymbol{\Omega} \times \mathbf{r}, \tag{5.44}$$

where $\mathbf{v}_P$ is the absolute velocity of the reference point P,

$\boldsymbol{\Omega}$ is the absolute angular velocity of the body, and

$\mathbf{r}$ is the radius vector from P to the mass particle.

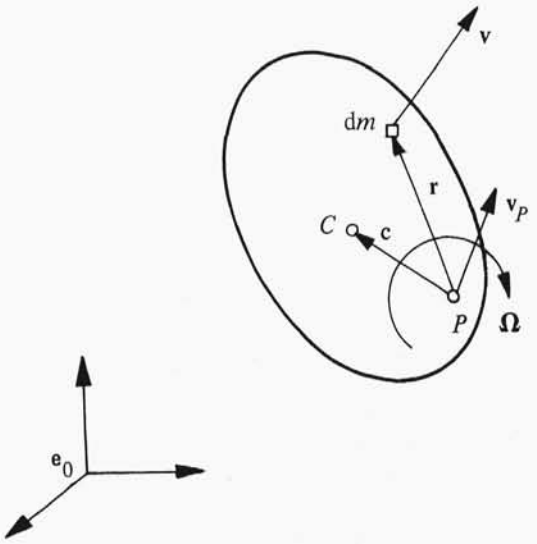


Fig. 5.36.

The radius vector $\mathbf{c} = \overrightarrow{PC}$ in Fig. 5.36 indicates the body mass centre C. Evaluation of the integral (5.43) yields

$$T = \tfrac{1}{2} m\mathbf{v}_P^2 + m\mathbf{v}_P\mathbf{u} + \tfrac{1}{2}\Omega^{\mathrm{T}} \mathsf{J}^P \Omega, \tag{5.45}$$