object considered, that is liquid density ρ and the pipe cross-section surface A, as well as quantities characterizing flow of the liquid, i.e. volume efflux Q.

Are these perceivable similarities of models incidental, or are they, perhaps, a rule? We shall try to answer this question in the subsequent section.

3.3 METHODOLOGY OF MODELLING BY MEANS OF BALANCE LAWS

3.3.1 The tetrahedron—a mnemonic aid in the modelling process

To every body the natural world seems immense and complex, the stage for a startling diversity of events and phenomena. These impressions are supported by estimates of the general order of magnitude of the values of some fundamental quantities such as the characteristic length of the universe, 10^{26}m, at one extreme, and of a nucleus, 10^{-15}m, at the other. These impressions are also supported by a great number of both animate and inanimate matter forms. More than 10^6 species have been described and named on our planet. About 100 different chemical elements form perhaps 10^6 or more identified and differentiated chemical compounds, and to this number may be added a vast number of liquid and solid solutions and alloys of various compositions having distinctive physical properties. Adding to these number innumerable phenomena which involve all man-made machines, mechanisms and tools take part, the impression of the complexity of the real world is fully justified.

However, thanks to a development of science and technology we have gained a remarkable understanding of some central and important aspects of the world. Three powerful theories may surely be mentioned here: classical and quantum mechanics, and classical electrodynamics. The theories just named, together with the theory of relativity and statistical mechanics, are perhaps the greatest intellectual achievements of mankind. It is remarkable that all the great theories mentioned above hinge on only a few fundamental laws. It is permanently the aim of a scientist to explain as much as possible with the simplest tools possible, particularly using the minimum set of physical laws and assumptions. Similarly we shall look for such a categorization of the great number of laws, relations and notions used in mechanics, which will lead to the formation of a suitable tool to aid in the modelling process.

The tetrahedron from Fig. 3.4 is a symbolic representation of a division of problems of mechanics into five groups of elements. These groups of elements are: **BLM**—basic laws of mechanics, **B**—body, **F**—forces, **M**—motion, and **SLM**—specific laws of mechanics. When modelling a complex thermodynamical phenomenon we have to draw from each group distinguished in Fig. 3.4, and the model itself has to contain elements from all five groups.

Before we pass over to more precise presentation of meaning of the breakdown introduced and the contents of the particular groups it is necessary to indicate that the names of groups are certain abbreviations (codewords) which should, as a rule, be understood more broadly. Thus, for instance, the codeword *force* should be understood as representing the description of mechanical interactions, that is—forces and torques and interrelations between these interactions—when we restrict ourselves to modelling of just purely mechanical phenomena. Then, if the scope of modelling is broadened, e.g. to include thermomechanical questions, this group would contain also thermal actions. We

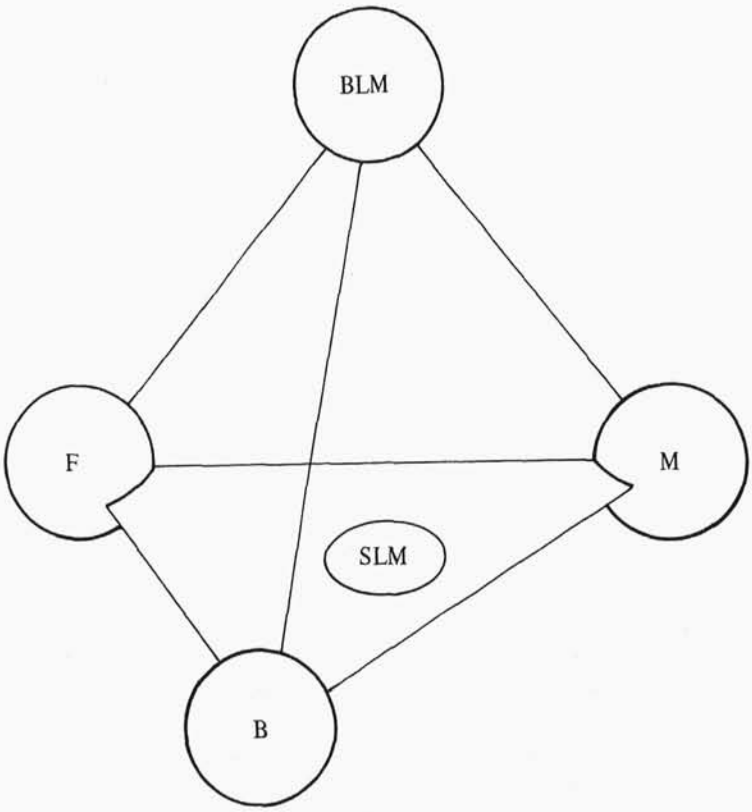

Fig. 3.4. Representation of a division of problems of mechanics.

shall therefore present particular groups in a broader manner, by giving examples of laws, formulae or problems belonging to these groups. Similarly the group denoted by the codeword *body* should be understood as representing problems concerning descriptions of body and system attributes as well as relations between them.

3.3.2 Basic laws of mechanics

Knowledge of fundamental laws of classical mechanics has essential significance in the establishment of causal models. In particular branches of classical mechanics these classical laws often appear in various forms, adapted to the object considered and accounting both for the specificity of notation and for the applied description of motion.

It is commonly recognized that Newton's three laws of motion create a basis for classical mechanics. For convenience, let us recall them in the traditional style.

First law. If the resultant force acting on particle is zero, the particle will remain at rest (if originally at rest) or will move with constant speed in a straight line (if originally in motion).

Second law. If the resultant force acting on particle is not zero, the particle will have an acceleration proportional to the magnitude of the resultant and in the direction of this resultant force, i.e.

$$m\mathbf{a} = \mathbf{F} \tag{3.21}$$

where m, $\mathbf{a}$ and $\mathbf{F}$ represent, respectively, the mass of the particle, the acceleration of the particle, and the resultant force acting on the particle.
Third law. The forces of action and reaction between interacting bodies are equal in magnitude, opposite in direction and collinear.

Looking at the above through the eyes of a modeller we can highlight their various aspects. Thus, whereas the first law may be considered only as an assumption of existence of the inertial frame, the third law gives rise to the formulation of algebraic relations between some forces appearing within a system under consideration; the second law may be reformulated as a differential equation relating the current position of the particle to a force acting on the particle.

For the purposes of modelling of dynamical processes, a vital role is played by these laws which, after mathematical processing, result in differential equations. The balance laws are those which can produce differential equations, and they will be called **generating laws**. As has already been stated (see section 3.1), equations resulting from balance laws appear in many different forms depending on which kind of objects is the subject of the modelling, which kind of motion description is used, etc. Because this text is concerned with discrete models, the basic laws of mechanics will be presented here mainly in forms suitable for the purposes of discrete model creation.

BLM-1. The **mass balance law** appears in the form of a mass conservation law. Considering systems of particles or of rigid bodies, whose masses remain constant, we may write an equivalence

$$m = \text{const}, \tag{3.22}$$

where m is the total mass of the system under consideration.

Consider now a fluid flowing through a pipe. Assuming that the flow is continuous, with no in- and out-flows in the range considered, the mass of the medium flowing through two arbitrary cross-sections in a certain time interval, t, is the same, i.e.

$$\rho_1 Q_1 t = \rho_2 Q_2 t \tag{3.23}$$

where ρ_1, ρ_2, Q_1, Q_2 are the density and the volume efflux at sections 1 and 2, respectively.

If the fluid is assumed to be incompressible, then $\rho_1 = \rho_2 = \rho = constant$ and (3.23) implies the equation

$$A_1 v_1 = A_2 v_2, \tag{3.24}$$

where A_1, A_2 are areas of pipe cross-sections 1 and 2, while v_1, v_2 denote the speeds associated with the stream velocities $\mathbf{v}_1$, $\mathbf{v}_2$ at sections 1 and 2, respectively.

In many practical applications, especially in establishment of discrete models of hydromechanical systems, the mass balance equation (3.8), from which the continuity equation is derived, is of little use, for it is too general, and equation (3.24) is too simplified, for it does not account for compressibility. In such a situation the law (3.8) should be given a working form.

To account for compressibility we may assume that mass is not produced within the region considered, i.e. $P(m) = 0$, and that pressure within this region may be assumed constant. Consider now the notion of **mass rate of flow** Q_m [kg/s]. Denoting the mass of liquid contained in the region by m, and density by ρ, we can represent equation (3.8) as

$$\frac{\mathrm{d}m}{\mathrm{d}t} = Q_m^{(\mathrm{i})} - Q_m^{(\mathrm{o})}, \tag{3.25}$$

where upper indices (i) and (o) respectively denote the input and output quantities.

Since $m = \rho\Omega$, where Ω is a volume of the mass considered, equation (3.25) can be given the form of

$$\frac{\mathrm{d}\rho}{\mathrm{d}t}\Omega + \rho\frac{\mathrm{d}\Omega}{\mathrm{d}t} = Q_m^{(\mathrm{i})} - Q_m^{(\mathrm{o})}, \tag{3.26}$$

which represents the most general equation for mass rates of flow. Note, though, that when the compressibility of a medium has to be taken into account, we often refer to **volumetric flow rate,** Q_Ω [m^3/s] in place of the mass rate of flow. The relation between these two rates is

$$Q_m = \rho Q_\Omega. \tag{3.27}$$

Using the volumetric flow rate from (3.27), equation (3.26) can be written in the form

$$\frac{\mathrm{d}\Omega}{\mathrm{d}t} + \frac{\Omega}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}t} = Q_\Omega^{(\mathrm{i})} - Q_\Omega^{(\mathrm{o})}, \tag{3.28}$$

which can be expressed in words as: *flow caused by change of volume and flow caused by compressibility is equal to the difference between inflow and outflow volumetric rates.*

BLM-2. The **linear momentum balance law** appears in either differential or integral form. The differential form

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = \mathbf{F} \tag{3.29}$$

can be stated as follows: *the rate of change of momentum* **p** *measured in an inertial space is proportional to the applied force and takes place in the direction of action of the force.*

The integral form of the linear momentum balance law is often called the *principle of linear impulse and momentum* and takes the form

$$\mathbf{p}_1 - \mathbf{p}_2 = \int_{t_1}^{t_2} F\,\mathrm{d}t \tag{3.30}$$

where the time integral of force **F** is known as the **impulse**. Thus (3.30) can be stated as follows: the change in the linear momentum of a system during a given interval of time is equal to the total impulse of the external forces acting on the system.

The alternative form of the linear momentum balance law is commonly used in the field of fluid mechanics; under certain circumstances some problems in fluid mechanics are adequately solved by an approximation in which viscous forces are neglected. With the assumption of an inviscid barotropic fluid, the linear momentum balance takes the form of the so-called **Euler equation** of motion (see Hunter (1983))

$$-\operatorname{grad} P+\rho\mathbf{b}=\rho\left[\frac{\partial\mathbf{v}}{\partial t}+(\mathbf{v}\cdot\operatorname{grad})\mathbf{v}\right], \tag{3.31}$$

where P, ρ are scalar functions of pressure and density, respectively, and $\mathbf{b}$, $\mathbf{v}$ are vectors of body forces and velocity, respectively. Note that the Euler equation is a field equation, i.e. a partial differential equation.

BLM-3. The **angular momentum balance law** also appears in two forms. The differential form can be formulated as follows: if the chosen reference point O is either (i) fixed in inertial space of (ii) at the centre of mass of the system, then the time rate of change of the angular momentum, $\mathbf{H}$, of a system about the given reference point is equal to the moment, $\mathbf{M}$, about that point of the external forces acting on the system, i.e.

$$\frac{\mathrm{d}\mathbf{H}}{\mathrm{d}t}=\mathbf{M}. \tag{3.32}$$

Integration of this equation with respect to time over the interval t_1 to t_2 results in the *principle of angular impulse and momentum*

$$\mathbf{H}_2-\mathbf{H}_1=\int_{t_1}^{t_2}\mathbf{M}\,\mathrm{d}t, \tag{3.33}$$

where $\mathbf{H}_1$, $\mathbf{H}_2$ are **angular momentum vectors** of a system about the given reference point O at time t_1 and t_2, respectively, while the time integral is known as the **total angular impulse** acting on the system due to external forces.

BLM-4. There are several forms in which the **balance of energy** is applied for modelling purposes. Probably the most commonly used form of the balance of energy is that known as the *principle of work and kinetic energy*: the change in the kinetic energy of a system in going from one position A to another position B is equal to the work done by external forces and internal forces acting on the system as it moves over the given interval, i.e.

$$T_B-T_A=W_{A\to B}. \tag{3.34}$$

For the case where the internal as well as the external forces are conservative, that is, their work $W_{A\to B}$ equals the difference in the positions A and B, i.e.

$$W_{A\to B}=V_A-V_B, \tag{3.35}$$

the principle of kinetic energy and work (3.34) takes the form

$$T_A+V_A=T_B+V_B. \tag{3.36}$$

The sum of the potential and kinetic energies is known as the **total mechanical energy** E, and from (3.36) we find that for a conservative system

$$E = T + V = \text{const.} \tag{3.37}$$

Equation (3.37) is a mathematical statement of the *principle of conservation of mechanical energy*.

Another form of the energy balance law is commonly applied in hydrodynamics problems. The Euler equation (3.32) admits a number of scalar integrals known collectively as the *Bernoulli equations*. The Bernoulli integrals takes various closely related forms which depend in detail on the specific assumptions employed, i.e. incompressible and/or irrotational and/or steady flow. For an incompressible, steady irrotational flow the energy balance per unit mass takes a form

$$\frac{P}{\rho_0} + V + \tfrac{1}{2}v^2 = f, \tag{3.38}$$

where f is a constant independent of time, P is the pressure, V is the potential energy of body forces $\mathbf{b}$, i.e. $\mathbf{b} = -\operatorname{grad} V$.

The above forms of energy balance (3.34)–(3.38) are not adequate when the environment surrounding the system is not solely mechanical, e.g. if thermal effects (heat streams) are involved as well as forces. In this case the general form of energy balance takes the form

$$\frac{\mathrm{d}E}{\mathrm{d}t} = N + \Phi_Q \tag{3.39}$$

where E is the total energy stored within the system, N is the power of the mechanical interactions, i.e. forces, and Φ_Q is the heat stream delivered to or produced within the system per time unit.

3.3.3 Body

For the modelling of dynamic phenomena, a certain set of information is necessary on properties of the bodies subject to modelling. Information of this kind encompasses not only such general statements as *deformability* of the body or *thermal conductivity*, but also the numerical values characterizing these properties, e.g. *Young's modulus* and *thermal conductivity coefficient*. In the process of modelling we sometimes make use of relations between various coefficients characterizing the properties of a given body. Such information, used in the modelling process, will be formally referred to as **body**. Because of the fact that properties of a body should be given or established during the introductory phase of modelling it is essential to be aware what is a physical property of a body. It is not easy, though, to define this notion. One cannot rely upon enumeration of all the properties of bodies, not only because there are too many, but first of all because of the relative nature of the notion of physical property itself. In order to confirm this statement let us consider a simple example of a uniform steel rod. Can the length L of this rod be considered a property? If this rod is hanging at one of its ends and thus plays the role of a physical pendulum moving in the surrounding in which temperature remains constant, then its length can be considered a geometrical property of this object. But when the same rod is subject to extension with a variable force $P(t)$ or is heated, then its length undergoes changes. It would not be justified when to accept length as physical property. Conclusions from the example is as follows: *a physical property of a body is a relative notion and depends upon the adopted physical model of the body.* The physical model, in turn depends on the phenomenon considered and on the purpose of modelling.

In spite of the difficulties in defining the notion of property of a body, mentioned before, we define the following statement: *the physical properties of a body are those physical quantities characterizing this body which do not depend upon its state* (e.g. upon position, velocity, temperature, deformation, etc.).

Thus within the framework of a rigid body model the following quantities can be regarded as physical properties: mass m, density ρ, volume Ω, while within the framework of a deformable body some examples are mass m, Young's modulus E, Poisson's ratio ν.

Similarly such characteristics as momentum or kinetic energy of a body, or volume of a compressible fluid cannot be regarded as physical properties since they depend upon the current state of the body.

Thus, the group known as *body* contains notions, definitions, relations and theorems concerning physical properties of the objects considered.

A good example for the elements of the group *body* is provided by considerations concerning inertia of a rigid body in general motion.

B-1. *Notion of the moment of inertia.* When considering the dynamics of particles or translational motion of bodies the only and sufficient measure of the inertia property of these objects is the mass. This quantity, though, is not sufficient for describing the inertia property in the case when we are dealing with rotational motion of a body.

Consider a small mass Δm mounted on a rod of negligible mass which may rotate freely about an axis l fixed in space (Fig. 3.5). If a constant couple is applied to the system, the rod and mass, assumed initially at rest, will start rotating about l. The time required for the system to reach a given speed of rotation is proportional to the mass Δm and to the square of the distance r. The product $r^2\Delta m$ provides a measure of the inertia of

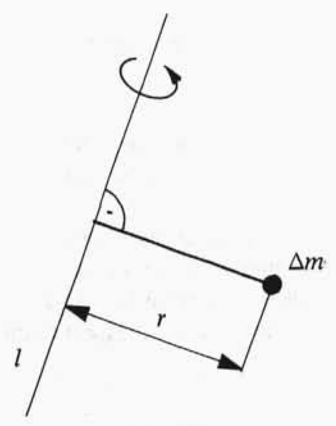

Fig. 3.5.

the system, i.e. of the resistance the system offers to motion. For this reason, the product $r^2\Delta m$ is called the *moment of inertia* of the mass Δm with respect to the axis l.

B-2. *Moments of inertia of a rigid body and relevant relations*. Consider now a body of mass m which is to be rotated about an axis l (Fig. 3.6). Dividing the body into elements of mass Δm_1, Δm_2, etc., we find that the resistance offered by the body is measured by the sum $r_1^2\Delta m_1 + r_2^2\Delta m_2 + \ldots$. This sum defines, therefore, the moment of inertia of the body with respect to the axis l. Increasing the number of elements, we find that the moment of inertia is equal, at the limit, to the integral

$$\int r^2 \,\mathrm{d}m, \tag{3.40}$$

where r denotes the perpendicular distance from the element of mass dm to the axis l.

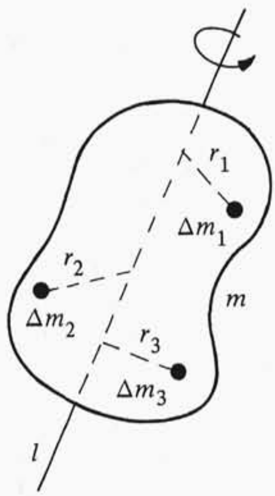

Fig. 3.6.

The radius of gyration k of the body with respect to the axis l is defined by the relation

$$I = k^2 m \quad \text{or} \quad k = \sqrt{(I/m)}. \tag{3.41}$$

The radius of gyration k therefore represents the distance at which the entire mass of the body should be concentrated if its moment of inertia with respect to l is to remain unchanged (Fig. 3.7).

The moments of inertia of a body with respect to the coordinate axes may easily be expressed in terms of the coordinates x, y, z of an element of mass dm. Noting, for example that the square of the distance r from the element dm to the x-axis is $y^2 + z^2$, we express the moment of inertia of the body with respect to the x-axis as

$$I_x = \int r^2 \,\mathrm{d}m = \int (y^2 + z^2)\,\mathrm{d}m. \tag{3.42a}$$

Similar expressions may be obtained for the moments of inertia with respect to the y- and z-axes. We write

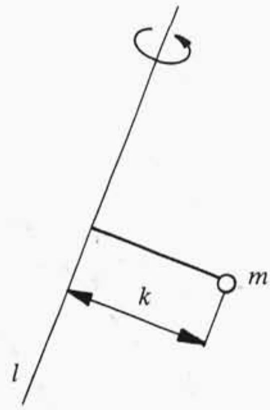

Fig. 3.7.

$$I_y = \int (x^2 + z^2)\ \mathrm{d}m, \tag{3.42b}$$

$$I_z = \int (x^2 + y^2)\ \mathrm{d}m. \tag{3.42c}$$

The moment of inertia of a body with respect to the origin of the coordinate system may also be expressed in terms of the coordinates x, y, z. Since the square of the distance r of the element of mass dm from the origin 0 is $x^2 + y^2 + z^2$, we have

$$I_0 = \int r^2\ \mathrm{d}m = \int (x^2 + y^2 + z^2)\ \mathrm{d}m. \tag{3.43}$$

Comparing (3.42a–c) and (3.43) we easily find that

$$2I_0 = I_x + I_y + I_z. \tag{3.44}$$

B-3. *Mass products of inertia.* In many instances not only moments of inertia but also other inertia characteristics are required. To introduce some of them, let us calculate the moment of inertia of a body with respect to an arbitrary axis l through the origin 0 (Fig. 3.8), in terms of the moment of inertia with respect to the three coordinate axes, as well as certain other quantities to be defined below.

The moment of inertia of the body with respect to l is represented by the integral $I_l = \int p^2\ \mathrm{d}m$, where p denotes the perpendicular distance from the element of mass dm to the axis l. But, denoting by $\boldsymbol{\lambda}$ the unit vector along l and by $\mathbf{r}$ the position vector of the element dm, we observe that the perpendicular distance p is equal to the magnitude $r \sin\theta$ of the vector product $\boldsymbol{\lambda} \times \mathbf{r}$. We therefore write

$$I_l = \int p^2\ \mathrm{d}m = \int (\boldsymbol{\lambda} \times \mathbf{r})^2\ \mathrm{d}m. \tag{3.45}$$

Expressing the square of the vector product in terms of its rectangular components, we have

$$I_l = \int [(\lambda_x y - \lambda_y x)^2 + (\lambda_y z - \lambda_z y)^2 + (\lambda_z x - \lambda_x z)^2]\ \mathrm{d}m, \tag{3.46}$$

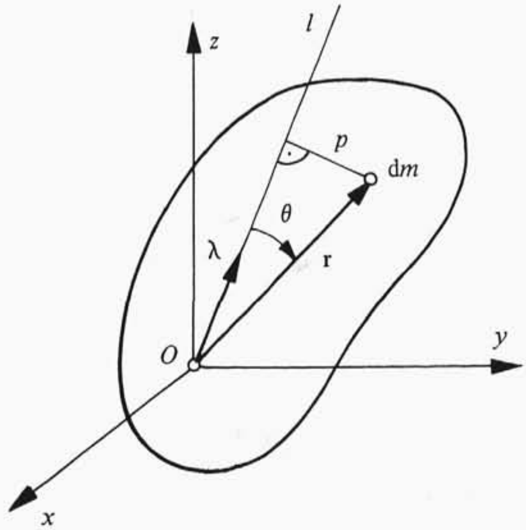

Fig. 3.8.

where the components $\lambda_x, \lambda_y, \lambda_z$ of the unit vector $\boldsymbol{\lambda}$ represent the direction cosines of the axis l, and the components x, y, z of $\mathbf{r}$ represent the coordinates of the element of the mass dm. Expanding the squares in the expression obtained and rearranging the terms, we write

$$I_l = \lambda_x^2 \int (y^2 + z^2)\, \mathrm{d}m + \lambda_y^2 \int (z^2 + x^2)\, \mathrm{d}m + \lambda_z^2 \int (x^2 + y^2)\, \mathrm{d}m$$
$$-2\lambda_x\lambda_y \int xy\, \mathrm{d}m - 2\lambda_y\lambda_z \int yz\, \mathrm{d}m - 2\lambda_z\lambda_x \int zx\, \mathrm{d}m. \qquad (3.47)$$

Referring to (3.42), we note that the first three integrals in (3.47) represent, respectively, the moments of inertia I_x, I_y, I_z of the body with respect to the coordinate axes. The last three integrals in (3.47), which involve products of coordinates, are called the *products of inertia* of the body with respect to the x- and y-axes, the y- and z-axes, and the z- and x-axes, respectively. We write then

$$I_{xy} = \int xy\, \mathrm{d}m, \quad I_{yz} = \int yz\, \mathrm{d}m, \quad I_{zx} = \int zx\, \mathrm{d}m. \qquad (3.48)$$

Substituting for the various integrals from (3.42) and (3.48) into (3.47), we have

$$I_l = I_x\lambda_x^2 + I_y\lambda_y^2 + I_z\lambda_z^2 - 2I_{xy}\lambda_x\lambda_y - 2I_{yz}\lambda_y\lambda_z - 2I_{zx}\lambda_z\lambda_x. \qquad (3.49)$$

B-4. *Inertia tensor*. For a rigid body which is free to move in three dimensions there are an infinite number of possible rotation axes. In the case of rotation about an arbitrary axis we need a complete way of characterizing the mass distribution of a rigid body. Here we introduce the *inertia tensor*, which for our purpose can be thought of as a generalization of the scalar moment of inertia of a body. The inertia tensor relative to a given coordinate system $0xyz$ is expressed in the matrix form as the 3×3 matrix

$$\mathsf{I} = \begin{bmatrix} I_x & -I_{xy} & -I_{xz} \\ -I_{xy} & I_y & -I_{yz} \\ -I_{xz} & -I_{yz} & I_z \end{bmatrix}, \tag{3.50}$$

where the scalar elements are given by (3.42) and (3.48).

Note that since the matrix (3.50) is symmetrical, only six out of nine elements of matrix I are independent. This set of six independent quantities will, for a given body, depend on the position and orientation of the frame in which they are defined. If we are free to choose the orientation of the reference time, it is possible to cause the products of inertia to be zero. The axes of the reference frame when so aligned are called the *principal axes* and the corresponding mass moment, are the *principal moments of inertia*. In this case the inertia tensor assumes the simple form of a diagonal matrix

$$\mathsf{I} = \begin{bmatrix} I_x & 0 & 0 \\ 0 & I_y & 0 \\ 0 & 0 & I_z \end{bmatrix}. \tag{3.51}$$

B-5. *Parallel-axes theorem.* Finally let us take an example of a theorem belonging to the set *body*. To do this consider a body of mass m. Let two parallel axes be given: an arbitrary axis n and a parallel centroidal axis l (Fig. 3.9). Denoting by d the distance between axes, by I the moment of inertia of the body with respect to n and by $\bar{I}$ its moment of inertia with respect to l, we have

$$I = \bar{I} + md^2. \tag{3.52}$$

This general relation is known as the *parallel-axes theorem.*

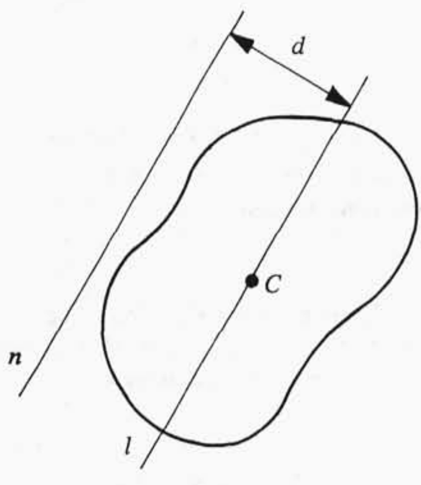

Fig. 3.9.