Note that the application of equation (4.301) requires expression of the acceleration energy through kinematic parameters $\ddot{e}_\lambda$ ($\lambda = 1,\ldots,l$). This can be done with the help of relation (4.296).

### 4.4.5 Case studies

#### *4.4.5.1 Constant speed drive*

In order to show the manner in which these three kinds of equations function we shall apply them, consecutively, to modelling of a mechanism which was first analysed by V. S. Novoselov (see Novoselov (1957)). This mechanism is shown in Fig. 4.19. Its purpose is to transmit the rotation of a driving shaft 1 (motor) to a drive shaft 2 (machine) by means of a disc mounted on the roller 3 so that it is free to rotate, and to have the speed of the driven shaft remain sensibly constant even though that of the driving shaft is not. The principle of functioning is as follows:

The vertical driving shaft has a rigidly attached horizontal disc. A intermediate horizontal shaft 3 has a thin disc of radius $a$. The disc can translate along its axis of rotation in both directions: to the left (towards the centre of the horizontal disc) owing to a centrifugal governor, and to the right owing to the spring with the stiffness $k_3$. The disc

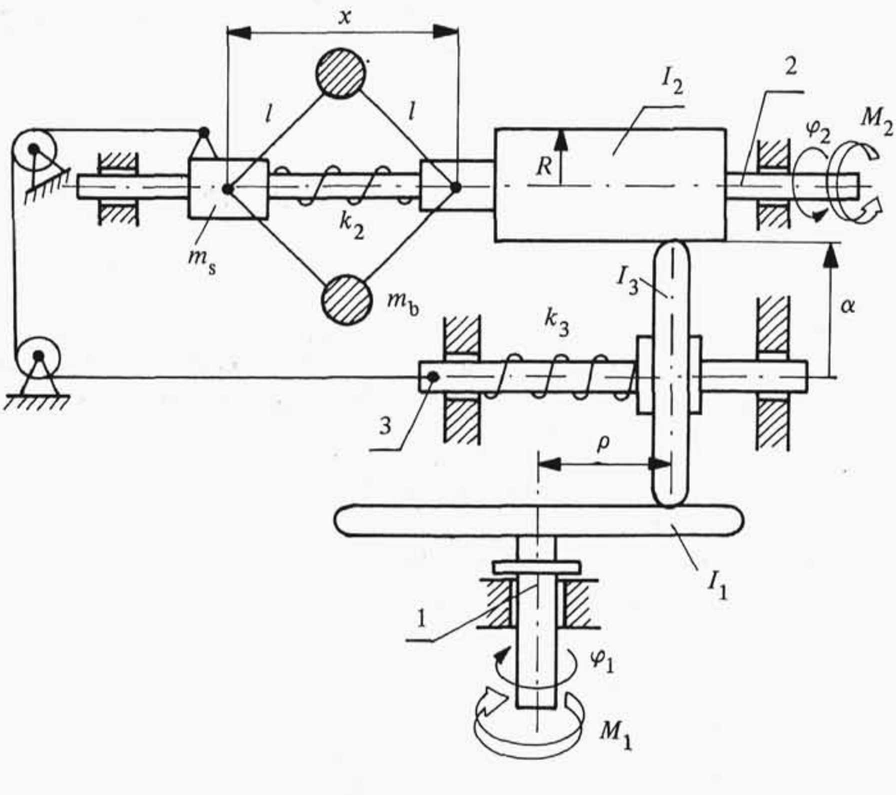


Fig. 4.19.

located on the shaft 3 drives a drum of radius $R$, which is rigidly attached to the driven shaft 2; mounted on 2 there is also a governor, whose two weights have mass $m_b$ each. When the sliding sleeve, $m_s$, translates, it does so against a spring with the stiffness $k_2$. This sleeve is connected by a cable, running over two pulleys $P_1$ and $P_2$, to the intermediate shaft 3 in such a way that the translation of this shaft is the same as that of the sleeve.

Given rotation speed of the shaft 2, that of the shaft 3 is determined by location of the pulley on shaft 3. It can be seen in the scheme of the mechanism that translation of pulley 3 counteracts the changes of rotation speed of shaft 2—an increase in the angular speed of shaft 2 causes translation of the pulley 3 towards the centre of the disc, due to the action of the centrifugal governor (the governor opens, $x$ and therefore $\rho$ are reduced, and this reduces the speed of shaft 2; in the case of a decrease in angular speed of the shaft 2, the spring moves the pulley 3 in the direction of the edge of disc 1.

Assume that:

(1) the cable is inextensible, thus of constant length, $c$, and value of $c$ depends upon the dimensions of construction elements of the governor (i.e. the positions of pulleys);
(2) the resistance of the pulley located on shaft 3 when moving along the disc of shaft 1 can be neglected;
(3) the rolling of pulley 3 between the cylinder 2 and disc 1 takes place without slipping;
(4) the reducer transmits power without losses (it is ideal);
(5) the springs are massless and linearily elastic (or rate $k$);
(6) the rods of the regulator, cable and pulleys are massless;
(7) the devices coupled to the motor and the machine are not accounted for.

The motion of the system can be described with three generalized coordinates:

$\varphi_1$ is the rotation angle of the driving shaft 1,
$\varphi_2$ is the rotation angle of the driving shaft 2,
$x$ is the translation of sleeve $m_s$ on shaft .

From assumption 3 we have

$$\rho\dot{\varphi}_1 = R\dot{\varphi}_2 \tag{4.304}$$

and taking into account assumption 1,

$$(x-c)\dot{\varphi}_1 - R\dot{\varphi}_2 = 0. \tag{4.305}$$

This is a nonholonomic constraint. The system has three generalized coordinates, but only two degrees of freedom ($s = 3$, $b = 1$, $l = 3 - 1 = 2$).

In order to follow the course of modelling let us specify the following parameters:

$I_1$ is the moment of inertia of the disc together with shaft 1 about its rotation axis;
$I_2$ is the moment of inertia of the cylinder together with shaft 2 about its rotation axis;
$I_3$ is the moment of inertia of the pulley about the axis of shaft 3;
$m_3$ is the mass of the pulley together with shaft 3;
$m_s$ is the mass of the sleeve of the governor;
$m_b$ is the mass of the ball of the governor;

$l$ is the length of the governor rod;
$k_2$, $k_3$ are the stiffnesses of the springs of governor and pulley, respectively;
$M_1$, $M_2$ are the external moments applied to shafts 1 and 2.

We now determine the quantities needed in all of these three kinds of equations, i.e. the Lagrange equations with multipliers, Maggi equations and Gibbs–Appell equations:

(1) *Kinetic energy of the system*

$$T = T_1 + T_2 + T_3 + T_g, \tag{4.306}$$

where:

$$T_1 = \tfrac{1}{2} I_1 \dot{\varphi}_1^2, \qquad T_2 = \tfrac{1}{2} I_2 \dot{\varphi}_2^2, \qquad T_3 = \tfrac{1}{2} m_3 \dot{\rho}^2 + \tfrac{1}{2} I_3 \dot{\varphi}_3^2$$
$$T_g = \tfrac{1}{2} m_s \dot{x}^2 + 2 \times \tfrac{1}{2} m_b v_b^2. \tag{4.307}$$

Some of the terms have to be expressed in terms of selected generalized coordinates. Note, for this purpose, that

— on the basis of assumption 1 we have

$$\dot{\rho} = \dot{x}; \tag{4.308}$$

— on the basis of assumption 3

$$\dot{\varphi}_3 a = \dot{\varphi}_2 R, \tag{4.309}$$

and hence

$$\dot{\vartheta}_3 = \frac{R}{a} \dot{\varphi}_2; \tag{4.310}$$

— on the basis of Fig. 4.20, presenting a detail of Fig. 4.19,

$$v_b^2 = (h\dot{\vartheta}_2)^2 + (l\dot{\vartheta})^2. \tag{4.311}$$

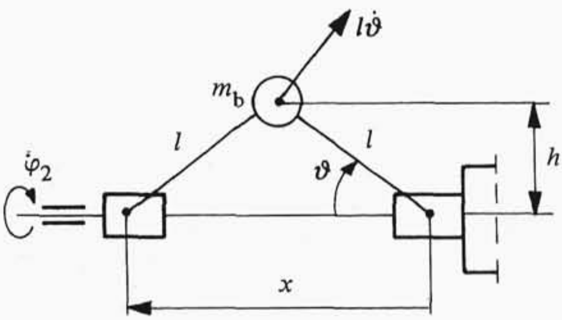


Fig. 4.20.

Since

$$h = \sqrt{\left(l^2 - \left(\frac{x}{2}\right)^2\right)} = \tfrac{1}{2}\sqrt{(4l^2 - x^2)}$$

$$x = 2l \cos \vartheta$$

then

$$\dot{x} = -2l\dot{\vartheta} \cos \vartheta, \quad \dot{\vartheta} = -\frac{\dot{x}}{\sqrt{(4l^2 - x^2)}} \tag{4.312}$$

and

$$v_b^2 = \tfrac{1}{4}(4l^2 - x^2)\dot{\varphi}_2^2 + \frac{l^2\dot{x}^2}{4l^2 - x^2}. \tag{4.313}$$

Having introduced the formulae (4.307)–(4.131) to equation (4.306) and grouped expressions at the corresponding generalized coordinates, we obtain the kinetic energy of the whole system in the form

$$\begin{aligned} T = \tfrac{1}{2} I_1 \dot{\varphi}_1^2 + \tfrac{1}{2}\left[ I_2 + \left(\frac{R}{a}\right)^2 I_3 + \tfrac{1}{2} m_b (4l^2 - x^2)\right]\dot{\varphi}_2^2 \\ + \left( m_3 + m_s + \frac{2l^2}{4l^2 - x^2} m_b \right)\dot{x}^2. \end{aligned} \tag{4.314}$$

(2) *Generalized forces*

We use, for potential forces, the formula (4.159). In order to do this we must calculate potential energy. It consists of the energy of elastic forces. (Assume that there is no contribution from the gravitational forces of the governor balls because of their symmetry with regard to the axis of rotation.) We assume an initial state for calculating the energy of elasticity forces. Assume, then, that there exists a steady state in which $\varphi_1 = \varphi_1^o$, $\varphi_2 = \varphi_2^o$ and $x = x_o$. Now, let us calculate the change of potential energy when this steady state is disturbed. From assumption (5) we have

$$V = \tfrac{1}{2}(k_2 + k_3)\,(x - x_o)^2. \tag{4.315}$$

Generalized potential force is expressed as

$$Q_x = -\frac{\partial V}{\partial x} = -(k_2 + k_3)\,(x - x_o). \tag{4.316}$$

We shall calculate the nonconservative generalized forces according to the method given in section 4.3.1.4. For this we first calculate the elementary work of nonconservative forces, which, in the case considered, are represented by external torques $M_1$ and $M_2$ acting upon shafts. We have

$$dA = M_1\,d\varphi_1 + M_2\,d\varphi_2 \tag{4.317}$$

and therefore, by application of the mnemonic rule (4.2.2.2)

$$\delta A = M_1\,\delta\varphi_1 + M_2\,\delta\varphi_2. \tag{4.318}$$

This means that the generalized potential forces are simply external torques. Thus

$$Q_{\varphi_1} = M_1, \qquad Q_{\varphi_2} = M_2. \tag{4.319}$$

(3) *Determination of the coefficients of nonholonomic constraints*
Because $s = 3$ and $b = 1$, on the basis of (4.258) we have

$$\sum_{\sigma=1}^{3} B_{\beta\sigma}\dot{q}_\sigma + B_\beta = B_{11}\dot{\varphi}_1 + B_{12}\dot{\varphi}_2 + B_{13}\dot{x}_3 + B_1 = 0. \tag{4.320}$$

Comparing coefficients of the corresponding generalized velocities in equations (4.305) and (4.320), we get

$$B_{11} = x - c, \qquad B_{12} = -R, \qquad B_{13} = 0, \qquad B_1 = 0. \tag{4.321}$$

We could now write down the Lagrange equations with multipliers. To preserve of the 'step-wise' nature of procedure, and to ensure clarity, we will not be doing this at this stage.

(4) *Determination of the coefficients of kinematic parameters*
Now turn to the coefficients $e_\lambda$, ($\lambda = 1,\ldots, l$) in equation (4.275). It would in fact be good to back-track a little and recall again the example of section 4.4.1. Since constraint equation has the form (4.305), as kinematic parameters we adopt one of the angular velocities and, specifically, that of the motion of the ring, i.e.

$$\dot{\varphi}_1 = \dot{e}_1, \qquad \dot{x} = \dot{e}_2. \tag{4.322}$$

Then

$$\dot{\varphi}_2 = \frac{x-c}{R}\dot{\varphi}_1 = \frac{x-c}{R}\dot{e}_1. \tag{4.323}$$

Now write down equations (4.275), taken into account that $s = 3$, $b = 1$, $l = s - b = 2$, and use

$$\begin{aligned}
\dot{q}_1 &\equiv \dot{\varphi}_1 = C_{11}\dot{e}_1 + C_{21}\dot{e}_2 + C_1 = \dot{e}_1 \\
\dot{q}_2 &\equiv \dot{\varphi}_2 = C_{12}\dot{e}_1 + C_{22}\dot{e}_2 + C_2 = \frac{x-c}{R}\dot{e}_1 \\
\dot{q}_3 &\equiv \dot{x} = C_{13}\dot{e}_1 + C_{23}\dot{e}_2 + C_3 = \dot{e}_2.
\end{aligned} \tag{4.324}$$

From the comparison of coefficients standing at corresponding $\dot{e}_\lambda$ ($\lambda = 1, 2$) we get

$$\begin{aligned}
&C_{11} = 1, && C_{21} = 0, && C_1 = 0, \\
&C_{12} = \frac{x-c}{R}, && C_{22} = 0, && C_2 = 0, \\
& && C_{23} = 1, && C_3 = 0. \\
&C_{13} = 0, && &&
\end{aligned} \tag{4.325}$$

(5) *Modified generalized forces*

Due to the formula (4.285) and taking into account that $\lambda = 1, 2$, we have

$$\begin{aligned}\varphi_1 &= C_{11}Q_1 + C_{12}Q_2 + C_{13}Q_3 = Q_{\varphi_1} + \frac{x-c}{R}Q_{\varphi_2} = M_1 + \frac{x-c}{R}M_2,\\ \varphi_2 &= C_{21}Q_1 + C_{22}Q_2 + C_{23}Q_3 = Q_x = -(k_2+k_3)(x-x_0),\end{aligned} \tag{4.326}$$

where the coefficients (4.325) and the forces (4.316) and (4.309) are used.

We have now all the information necessary for writing down the Maggi equations, but again we bypass these, and proceed to the Gibbs–Appell equations. Since at the previous stage we calculated forces $\phi_\lambda$ $(\lambda = 1, 2)$, it remains only to calculate the energy of acceleration. In view of the length of the procedure we perform this in two steps.

(6a) *The energy of acceleration in the generalized coordinates*

We will use the formula (4.301) given in section 4.4.4; similarly as in stage 1 (calculation of kinetic energy), we perform calculations separately for each component. Taking into account the relations (4.308) and (4.310), previously obtained, gives

$$S = S_1 + S_2 + S_3 + S_g, \tag{4.327}$$

where:

$$\begin{aligned}&S_1 = \tfrac{1}{2}I_1\ddot{\varphi}_1^2, \qquad S_2 = \tfrac{1}{2}I_2\ddot{\varphi}_2^2, \qquad S_3 = \tfrac{1}{2}m_3\ddot{\rho}^2 + \tfrac{1}{2}I_3\ddot{\varphi}_3^2,\\ &S_g = \tfrac{1}{2}m_s\ddot{x}^2 + 2\times\tfrac{1}{2}m_b w_b^2.\end{aligned} \tag{4.328}$$

We thus see that only the acceleration of the governor ball remains to be determined. We shall use *Coriolis'* theorem for that purpose and refer to Fig. 4.21a (see also Fig. 4.20):

$$\mathbf{w}_b = \mathbf{w}_r + \mathbf{w}_e + \mathbf{w}_c, \tag{4.329}$$

where

$$\mathbf{w}_c = 2\boldsymbol{\omega}_e \times \mathbf{v}_r.$$

We identify easily

$$\dot{\varphi}_2 = \omega_e, \qquad v_r = l\dot{\vartheta}, \tag{4.330}$$

due to which

$$w_c = 2\omega_e v_r \sin(\boldsymbol{\omega}_e, \mathbf{v}_r) = 2l\dot{\varphi}_2\dot{\vartheta}\cos\vartheta. \tag{4.331}$$

For the relative motion, which is the motion over the circle of radius $l$, we get

$$w_r^n = l\dot{\vartheta}^2, \qquad w_r^t = l\ddot{\vartheta}. \tag{4.332}$$

For the transport motion, being the motion about the axis 2, we have

$$w_e^n = h\dot{\varphi}_2^2 = l\dot{\varphi}_2^2\sin\vartheta, \qquad w_e^t = h\ddot{\varphi}_2 = l\ddot{\varphi}_2\sin\vartheta. \tag{4.333}$$

Adopting the system of orthogonal coordinates (Fig. 4.21b), we get

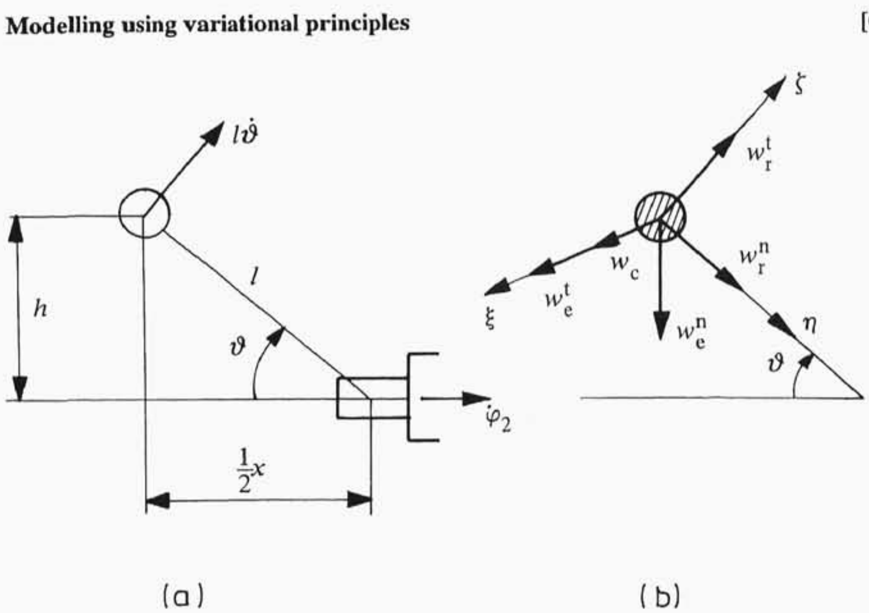


Fig. 4.21.

$$
\begin{aligned}
w_\xi &= w_c + w_e^t = l(2\dot{\varphi}_2\dot{\vartheta}\cos\vartheta + \ddot{\varphi}_2 \sin\vartheta) \\
w_\eta &= w_r^n + w_e^n \sin\vartheta = l(\dot{\vartheta}^2 + \dot{\varphi}_2^2 \sin^2\vartheta) \\
w_\zeta &= w_r^t + -w_e^n \cos\vartheta = l(\ddot{\vartheta} - \dot{\varphi}_2^2 \sin\vartheta\cos\vartheta) \\
w_b^2 &= w_\xi^2 + w_\eta^2 + w_\zeta^2 = l^2(\ddot{\varphi}_2^2 \sin^2\vartheta + 4\dot{\varphi}_2\dot{\vartheta}\ddot{\varphi}_2 \sin\vartheta\cos\vartheta + \ddot{\vartheta}^2 \\
&\quad -2\ddot{\vartheta}\dot{\varphi}_2^2 \sin\vartheta\cos\vartheta) + l^2\left[\dot{\vartheta}^4 + \dot{\varphi}_2^4 \sin^2\vartheta + 2\dot{\varphi}_2^2\dot{\vartheta}^2(1+\cos^2\vartheta)\right].
\end{aligned} \tag{4.334}
$$

After substitution of (4.334) into (4.328) and summation according to (4.327), we obtain

$$
\begin{aligned}
S &= \tfrac{1}{2} I_1\ddot{\varphi}_1^2 + \tfrac{1}{2}\left(I_2 + I_3\frac{R^2}{a^2}\right)\ddot{\varphi}_2^2 + \tfrac{1}{2}(m_3+m_s)\ddot{x}^2 \\
&\quad + m_b l^2(\ddot{\varphi}_2^2 \sin^2\vartheta + 4\dot{\varphi}_2\dot{\vartheta}\ddot{\varphi}_2 \sin\vartheta\cos\vartheta + \ddot{\vartheta}^2 - 2\ddot{\vartheta}\dot{\varphi}_2^2 \sin\vartheta\cos\vartheta) \\
&\quad + m_b l^2\left[\dot{\vartheta}^4 + \dot{\varphi}_2^4 \sin^2\vartheta + 2\dot{\varphi}_2^2\dot{\vartheta}^2(1+\cos^2\vartheta)\right].
\end{aligned} \tag{4.335}
$$

Note the following useful point: the energy of acceleration, expressed by formula (4.335), is expressed not only in terms of three generalized coordinates, i.e. $\varphi_1$, $\varphi_2$ and $x$ (more precisely by their second derivatives), but also in terms of the coordinate $\vartheta$. This provides a good illustration for the comments put forward in section 2.2.3 as to the generalized dependent variables—in the present case $\vartheta$ is related to $x$ through the relation (4.20)

$$
x = 2l\cos\vartheta. \tag{4.336}
$$

Replacement of coordinate $\vartheta$ and its derivatives would lead, however, to an even more complicated formula, and this is already sufficiently complex. On the other hand this is not necessary, for we ultimately aim at presenting function $S$ with the help of kinematic parameters $\ddot{e}_\lambda$ $(\lambda = 1, 2)$.

(6b) *Energy of accelerations in the kinematic parameters*

With the help of relations (4.322) and (4.323) we shall be able to express function $S$ in terms of $\ddot{e}_\lambda$ $(\lambda = 1, 2)$. For that purpose it is sufficient to differentiate these relations with regard to time, and obtain

$$\ddot{\varphi}_1 = \ddot{e}_1, \qquad \ddot{x} = \ddot{e}_2, \tag{4.337}$$

$$\ddot{\varphi}_2 = \frac{\dot{x}}{R}\dot{e}_1 + \frac{x-c}{R}\ddot{e}_1 = \frac{\dot{e}_2}{R}\dot{e}_1 + \frac{x-c}{R}\ddot{e}_1. \tag{4.338}$$

At this point we must replace the coordinate $\vartheta$ and its derivatives with parameters $e_\lambda$ and their derivatives. Precisely, we must replace $\sin\vartheta$, $\cos\vartheta$, $\dot{\vartheta}$ and $\ddot{\vartheta}$, and we use the following relations for this purpose (see Fig. 4.21a)

$$\sin\vartheta = \frac{h}{l} = \frac{\sqrt{\left(l^2 - (x/2)^2\right)}}{l} = \frac{\sqrt{(4l^2 - x^2)}}{2l}, \qquad \cos\vartheta = \frac{x}{2l}.$$

Hence

$$-\dot{\vartheta}\sin\vartheta = \frac{\dot{x}}{2l},$$

$$\dot{\vartheta} = -\frac{\dot{x}}{2l\sin\vartheta} = -\frac{\dot{x}}{\sqrt{(4l^2 - x^2)}} = -\frac{\dot{e}_2}{\sqrt{(4l^2 - x^2)}}$$

$$\ddot{\vartheta} = -\frac{x\dot{e}_2^2 + (4l^2 - x^2)\ddot{e}_2}{\sqrt{(4l^2 - x^2)^3}}. \tag{4.339}$$

Having substituted (4.339) into (4.335) we ultimately get

$$\begin{aligned} S = {} & \frac{1}{2}\left[I_1 + \left(I_2 + \frac{R^2}{a^2}I_3\right)\frac{(x-c)^2}{R^2} + \frac{(x-c)^2(4l^2 - x^2)}{4R^2}\right]\ddot{e}_1^2 \\ & + \frac{1}{2}\left(m_3 + m_s + m_b\frac{2l^2}{rl^2 - x^2}\right)\ddot{e}_2^2 \\ & + \left[\left(I_2 + \frac{R^2}{a^2}I_3\right)\frac{x-c}{R^2} + \frac{m_b}{2R^2}(x-c)(4l^2 - 3x^3 + 2cx)\right]\dot{e}_1\dot{e}_2\ddot{e}_1 \\ & + m_b x\left[\frac{1}{2}\frac{(x-c)}{R^2}\dot{e}_1^2 + \frac{2l^2}{(4l^2 - x^2)^2}\dot{e}_2^2\right]\ddot{e}_2 \\ & + \text{terms not depending on } \ddot{e}_\lambda. \end{aligned} \tag{4.340}$$

We can now write all the three equations:

(I) *Lagrange equations with multipliers*
On the basis of (4.274) we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \dot{\varphi}_1} - \frac{\partial T}{\partial \varphi_1} &= Q_{\varphi_1} + \lambda_1 B_{11} \\
\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \dot{\varphi}_2} - \frac{\partial T}{\partial \varphi_2} &= Q_{\varphi_2} + \lambda_1 B_{12} \\
\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \dot{x}} - \frac{\partial T}{\partial x} &= Q_x + \lambda_1 B_{13}
\end{aligned} \tag{4.341}$$

where $T$ was determined at stage 1, $Q_\sigma (\sigma = 1, 2, 3)$—at stage 2, and $B_{\beta\sigma} (\beta = 1)$—at stage 3. We have, therefore,

$$\frac{\partial T}{\partial \dot{\varphi}_1} = I_1 \dot{\varphi}_1, \qquad \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \dot{\varphi}_1} = I_1 \varphi_1,$$

$$\frac{\partial T}{\partial \dot{\varphi}_2} = \left[ I_2 + \frac{R^2}{a^2} I_3 + \tfrac{1}{2} m_\mathrm{b} (4l^2 - x^2) \right] \dot{\varphi}_2,$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \dot{\varphi}_2} = -m_\mathrm{b} x \dot{x} \dot{\varphi}_2 + \left[ I_2 + \frac{R^2}{a^2} I_3 + \tfrac{1}{2} m_\mathrm{b} (4l^2 - x^2) \right] \varphi_2,$$

$$\frac{\partial T}{\partial \dot{x}} = \left( m_3 + m_\mathrm{s} + m_\mathrm{b} \frac{2l^2}{4l^2 - x^2} \right) \dot{x},$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \dot{x}} = \frac{4l^2 m_\mathrm{b} x}{(4l^2 - x^2)^2} \dot{x}^2 + \left( m_3 + m_\mathrm{s} + m_\mathrm{b} \frac{2l^2}{4l^2 - x^2} \right) x$$

and

$$\frac{\partial T}{\partial \varphi_1} = 0, \quad \frac{\partial T}{\partial \varphi_2} = 0, \quad \frac{\partial T}{\partial x} = -\tfrac{1}{2} m_\mathrm{b} x \dot{\varphi}_2^2 + \frac{2l^2 m_\mathrm{b} x \dot{x}^2}{(4l^2 - x^2)^2}. \tag{4.342}$$

Having substituted (4.319), (4.321) and (4.342) into (4.341) we get

$$\begin{aligned}
& I_1 \ddot{\varphi}_1 = M_1 + \lambda_1 (x - c), \\
& \left[ I_2 + \frac{R^2}{a^2} I_3 + \tfrac{1}{2} m_\mathrm{b} (4l^2 - x^2) \right] \ddot{\varphi}_2 - m_\mathrm{b} x \dot{x} \dot{\varphi}_2 = M_2 - \lambda_1 R, \\
& \left( m_3 + m_\mathrm{s} + m_\mathrm{b} \frac{2l^2}{4l^2 - x^2} \right) \ddot{x} + \frac{2l^2 m_\mathrm{b}}{(4l^2 - x^2)^2} x \dot{x}^2 \\
& \qquad\qquad + \tfrac{1}{2} m_\mathrm{b} x \dot{\varphi}_2^2 = -(k_2 + k_3)(x - x_0).
\end{aligned} \tag{4.343}$$

Since it is a system of three equations with four unknowns: $\varphi_1$, $\varphi_2$, $x$ and $\lambda_1$, we complement it with the constraint equation (4.305).

*(II) Maggi equations*
From equation (4.284) we have

$$C_{11}\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \dot{\varphi}_1}-\frac{\partial T}{\partial \varphi_1}\right)+C_{12}\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \dot{\varphi}_2}-\frac{\partial T}{\partial \varphi_2}\right)+C_{13}\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \dot{x}}-\frac{\partial T}{\partial x}\right)=\phi_1,$$
$$C_{21}\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \dot{\varphi}_1}-\frac{\partial T}{\partial \varphi_1}\right)+C_{22}\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \dot{\varphi}_2}-\frac{\partial T}{\partial \varphi_2}\right)+C_{23}\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \dot{x}}-\frac{\partial T}{\partial x}\right)=\phi_2, \tag{4.344}$$

where the expressions

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \dot{\varphi}_\sigma}-\frac{\partial T}{\partial \varphi_\sigma}, \qquad \sigma=1,2,3,$$

were determined in (4.342), and the coefficients $C_{\beta\sigma}$ and forces $\phi_\lambda$ $(\lambda=1,2)$ were determined at stages 4 and 5. Thus, substituting appropriate expressions we get

$$I_1\ddot{\varphi}_1+\frac{x-c}{R}\left[I_2+\frac{R^2}{a^2}I_3+\tfrac{1}{2}m_\mathrm{b}(4l^2-x^2)\ddot{\varphi}_2-m_\mathrm{b}x\dot{x}\dot{\varphi}_2\right]=M_1+\frac{x-c}{R}M_2$$
$$\left(m_3+m_\mathrm{s}+m_\mathrm{b}\frac{2l^2}{4l^2-x^2}\right)\ddot{x}+\frac{2l^2m_\mathrm{b}}{(4l^2-x^2)^2}x\dot{x}^2+\tfrac{1}{2}m_\mathrm{b}x\dot{\varphi}_2^2=-(k_2+k_3)(x-x_0). \tag{4.345}$$

Thus, we have two equations (the same number as the number of degrees of freedom of the nonholonomic system), but with three unknowns $\varphi_1$, $\varphi_2$ and $x$. Again, we have to complement these equations with the constraint equation (4.305).

*(III) Gibbs–Appell equations*
From equation (4.301) we obtain

$$\frac{\partial S}{\partial \ddot{e}_1}=\phi_1,$$
$$\frac{\partial S}{\partial \ddot{e}_2}=\phi_2. \tag{4.346}$$

It now remains to differentiate function $S$ (4.340), since the forces $\phi_\lambda$ $(\lambda=1,2)$ have already been determined in stage 5. We have

$$\frac{\partial S}{\partial \ddot{e}_\lambda} = \left[ I_1 + \left( I_2 + \frac{R^2}{a^2} I_3 \right) \frac{(x-c)^2}{R^2} + \frac{(x-c)^2 (4l^2 - x^2)}{2R^2} \right] \ddot{e}_1$$
$$+ \left[ \left( I_2 + \frac{R^2}{a^2} I_3 \right) \frac{x-c}{R^2} + \frac{m_b}{2R^2} (x-c)(4l^2 - 3x^3 + 2cx) \right] \dot{e}_1 \dot{e}_2$$

and

$$\frac{\partial S}{\partial \ddot{e}_2} = \left( m_3 + m_s + m_b \frac{2l^2}{4l^2 - x^2} \right) \ddot{e}_2$$
$$+ m_b \left[ \frac{1}{2} \frac{(x-c)^2}{R^2} \dot{e}_1^2 + \frac{2l^2}{(4l^2 - x^2)^2} \dot{e}_2^2 \right] x. \tag{4.347}$$

Having introduced (4.347) into (4.346) and accounted for (4.325) we obtain

$$\left[ I_1 + \left( I_2 + \frac{R^2}{a^2} I_3 \right) \frac{(x-c)^2}{R^2} + \frac{(x-c)^2 (4l^2 - x^2)}{2R^2} \right] \ddot{e}_1$$
$$+ \left[ \left( I_2 + \frac{R^2}{a^2} I_3 \right) \frac{x-c}{R^2} + \frac{m_b}{2R^2} (x-c)(4l^2 - 3x^3 + 2cx) \right] \dot{e}_1 \dot{e}_2 = M_1 + \frac{x-c}{R} M_2,$$
$$\left( m_3 + m_s + + m_b \frac{2l^2}{4l^2 - x^2} \right) \ddot{e}_2 + m_b \left[ \frac{1}{2} \frac{(x-c)^2}{R^2} \dot{e}_1^2 + \frac{2l^2}{(4l^2 - x^2)^2} \dot{e}_2^2 \right] x$$
$$= -(k_2 - k_3)(x - x_0). \tag{4.348}$$

Finally we have yet again two equations with three unknowns, but different from those obtained by the Maggi method, for they are now $e_1$, $e_2$ and $x$. We now add the constraint equation (4.305) which entails the inclusion of the relations

$$\dot{\varphi}_1 = \dot{e}_1, \qquad \dot{x} = \dot{e}_2. \tag{4.349}$$

Let us look now at the results of modelling, that is, at equations (4.343), (4.345), (4.348) augmented by the constraint equation (4.305). At the first glance it seems that only the constraint equation is the same. Whereas the difference between (4.343) and (4.345) is natural, for there is the constraint multiplier (the equation with regard to $\ddot{x}$ is even identical), equation (4.348) may come as a surprise, for quite unexpectedly the model is expanded to five equations. We shall demonstrate that the differences are only apparent and not fundamental.

In the Maggi equations (4.345) the multiplier of constraints, $\lambda_1$, which appears in Lagrangian equations (4.343), is not obvious. We shall therefore try to eliminate it from the latter. To do this it is sufficient to determine $\lambda_1$ from the first equation and substitute it into the second. We obtain

$$\lambda_1 = \frac{I_1\ddot{\varphi}_1 - M_1}{x-c},$$

$$\left[I_2 + \frac{R^2}{a^2}I_3 + \tfrac{1}{2}m_\mathrm{b}(4l^2 - x^2)\right]\ddot{\varphi}_2 - m_\mathrm{b}x\dot{x}\dot{\varphi}_2 = M_2 - (I_1\ddot{\varphi}_1 - M_1)\frac{R}{x-c}.$$

If we multiply the second equation by $(x-c)/R$ and move the term $I_1\ddot{\varphi}_1$ to the left side, we obtain an equation identical to (4.341). This is not a coincidence. The method of Maggi does in fact eliminate the multiplier, but, in the case of a large number of constraint equations (and therefore also of multipliers) this is not as easy as in the case considered.

Regarding equations (4.348), it can be said that they constitute an excessive model (see section 1.2.5), for in fact we are interested only in those variables which describe motion, i.e. $\varphi_1$, $\varphi_2$ and $x$. In order to obtain an adequate model, it is sufficient to transform the first equation from the system (4.348) in the following manner: retain $\ddot{e}_1 = \ddot{\varphi}_1$ at the term $I_1$, and reduce the other terms using the transformed relation (4.338), i.e.

$$\ddot{e}_1 = \frac{R}{x-c}\ddot{\varphi}_2 - \frac{\dot{\varphi}_1\dot{x}}{x-c}.$$

In the second case it is sufficient to replace the magnitude $e_\lambda$ by corresponding generalized coordinates. After these relatively simple operations we exactly obtain the Maggi equations (4.345).

We shall try now to draw somewhat more general conclusions. It should first of all be emphasized that the simplicity of the Gibbs–Appell equations is quite misleading. The true nature of these equations becomes apparent when the acceleration energy is being determined, and again when it is differentiated.

The difference between modelling using the method of Lagrange with multipliers and the Maggi method is slight, and it in reality reduces, in the case of application of the Maggi method, to the additional determination of the coefficients of kinematic parameters (see stage 4). However, it may also be relatively simple to eliminate the constraint multipliers from the Lagrange equations, especially when the number of constraints is not too great. Our practice indicates that when there are two multipliers there is no need to apply the method of Maggi.

Although the Lagrange equations with multipliers make it possible to construct mathematical models for a majority of technically important nonholonomic systems, they become insufficient when we want to write down equations in simple form. This simplicity cannot consist solely in uncoupling the equations. Let us concentrate now on the fact that in all the three models of the reductor a common feature appears, that the dynamic equations of motion had to be complemented by the constraint equation to complete the model. In the case considered this equation had a very simple form. We known, however, at least from the example in section 4.4.1 (ball rolling without slipping), that the constraint equations may be more complicated. A question, then, arises as to whether it is possible to separate the dynamic equations from the kinematic ones. It turns out that there are situations when the answer is positive—namely for the so-called *Chaplygin systems*.

However, because of limited space we will not deal with these systems, nor with the *Chaplygin equations*.

There are many more modelling methods developed for specific cases of nonholonomic systems (see e.g. Neimark and Fufaev (1967)). However, we believe that the Lagrange equations with multiplier provide the most general method of nonholonomic system modelling. A proper application of these equations always gives a mathematical method, although the form of the model may be more complicated than for application of the other method. To exemplify the last statement we solve the problem of modelling a rotary hydropulsator using the Lagrange equations with multipliers. The mathematical model of this device is known, although for its derivation a sophisticated method, involving the so-called *Voronetz equations*, was used.

#### *4.4.5.2 Rotary hydropulsator*

Among the various devices for experimental research an important place is occupied by machines for performing fatigue tests of materials, and structure components, since fatigue often leads to failure and sometimes to dangerous accidents (e.g. the DC-10 catastrophe in 1978 was caused by fatigue failure of wing bolt). In order to reduce the time of lengthy and costly studies, the use of high-frequency oscillations is required. There are a number of methods of generating these oscillations, such as electromagnetic, magnetostrictive or piezoelectric. A widely used method of generation in the study of machine elements is hydraulic excitation, and this is most often achieved by the rotary type of hydropulsor. The fundamental parts of such a device are a rotor with sliding bolts and a movable disposer (see Fig. 4.22). The rotor and the disposer are set in rotational motion by means of two mutually independent electric motors. Due to the rotary disposer the fluid output changes from the maximum to minimum value depending upon the position of the shutter with respect to the line of centres of the rotor and the starter. Further rotation of the shutter again causes an increase in output, but in the opposite direction, causing a vibratory motion of the platform on which the object studied is standing.

An interesting phenomenon takes place in the device described—the resonant interaction of the vibratory system with the two sources of energy of limited power. We would like to draw the reader's attention to the fact that in the usual course on the theory of vibrations it is assumed that only unilateral influences exist, that is, an external nonreciprocating force acts upon the system. The first scholar who turned attention to this fact was Kononenko (1964). He considered the manner in which the properties of one source of energy of limited power influence the resonance behaviour of the vibrating system. The equation of motion of the pulsator is derived in the specialized literature by the so-called Voronetz method. We would like to demonstrate that the Lagrange equations with multipliers can be used to derive equations of motion. Thus, we are proposing the following objective of modelling: to build a mathematical model of the dynamics of the hydropulsator, suitable for the analysis of resonance of the vibrating system with two sources of energy of limited power.

We make the following assumptions:

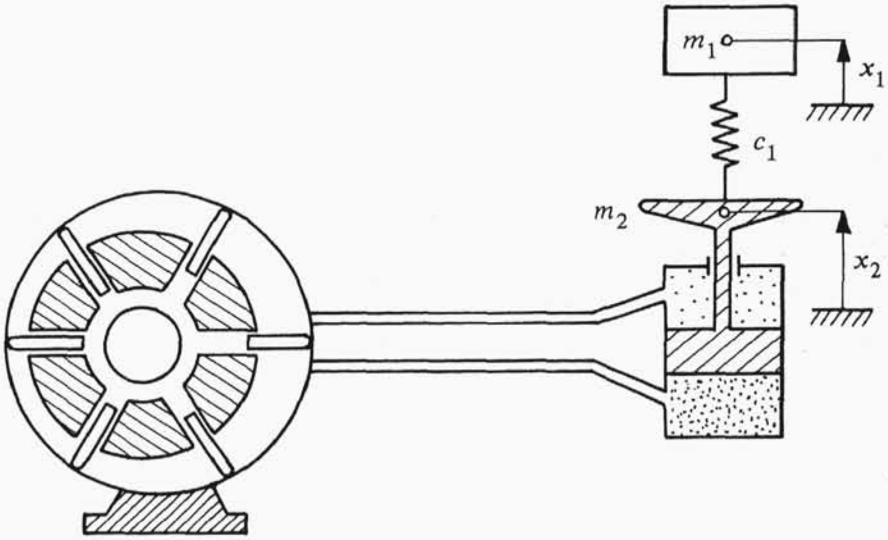


Fig. 4.22.

(1) the drives of the rotor and the disposer are mutually independent;
(2) the output of the hydropulsator is proportional to the angular velocity of the rotor;
(3) the fluid is incompressible (how to take account of compressibility has been shown in modelling the booster (see section 3.4.3);
(4) there are no leakages out of the installation;
(5) the characteristics of driving engines and resistance torques are known;
(6) any external and internal damping is neglected.

From the first assumption and when the discrete model of the platform and the object is considered, the whole system has four degrees of freedom: specifically the rotation of the rotor and of the disposer, and the displacement of the platform and the object. We therefore chose the following generalized coordinates: $\phi$, the rotation angle of the rotor; $\psi$, the rotation angle of the disposer, $x_1$, the displacement of the object, and $x_2$, the displacement of the platform.

Before we start to develop the mathematical model, we need to demonstrate that the functioning of the hydropulsator leads to the emergence of nonholonomic constraints, which is why the example has been introduced in the present section. On the basis of assumptions (2) and (3) we have

$$Q = c\dot{\phi}, \tag{4.350}$$

where $Q$ denotes a volume output efflux and $c$ is a known coefficient of proportionality, a characteristic of the geometry of the hydropulsator. The quantity of the fluid which is supplied to the cylinder of the platform depends upon the angle $\psi$: for full opening ($\psi = 0$) it is the maximum quantity, while for complete closure ($\psi = \pi/2$) there is no inflow. Thus, in the intermediate positions the flow is given by the formula

$$Q = \tfrac{1}{2} c\dot{\phi}(1 + \cos 2\psi). \tag{4.351}$$

Since

$$Q = \frac{\mathrm{d}\Omega}{\mathrm{d}t}, \tag{4.352}$$

where $\Omega$ denotes the volume of fluid, then as the rotor of the pulsator turns around by the angle $\mathrm{d}\phi$, the resulting flow will be

$$\mathrm{d}\Omega = Q\ \mathrm{d}t = \tfrac{1}{2} c(1 + \cos 2\psi)\ \mathrm{d}\phi \tag{4.353}$$

under assumption (4). The fluid of volume given by (4.353) flows through the input tube of cross-section surface $A$. Using assumptions (3) and (4), the relation

$$\mathrm{d}\Omega = A\ \mathrm{d}x_2 \tag{4.354}$$

takes place. Comparison of the equations (4.352) and (4.354) yields

$$\mathrm{d}x_2 = \frac{c}{2A}(1 + \cos 2\psi)\ \mathrm{d}\phi \tag{4.355}$$

or, in a differential form

$$\dot{x}_2 = \frac{c}{2A}(1 + \cos 2\psi)\dot{\phi}. \tag{4.356}$$

Since the constraint equation (4.356) is not integrable, it constitutes a nonholonomic constraint. We have thus demonstrated that the hydropulsator constitutes a nonholonomic system with three degrees of freedom ($s = 4$, $b = 1$, $l = s - b = 3$).

As previously state, we shall apply the Lagrange equations with multipliers (4.284) to derive the equations of motion. We have thus

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{x}_1}\right) - \frac{\partial T}{\partial x_1} = Q_1 + \lambda_1 B_{11}, \tag{4.357}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{x}_2}\right) - \frac{\partial T}{\partial x_2} = Q_2 + \lambda_1 B_{12}, \tag{4.358}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{\phi}}\right) - \frac{\partial T}{\partial \phi} = Q_3 + \lambda_1 B_{13}, \tag{4.359}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{\psi}}\right) - \frac{\partial T}{\partial \psi} = Q_4 + \lambda_1 B_{14}. \tag{4.360}$$

If we denote by $I_\mathrm{r}$ and $I_\mathrm{d}$ the moment of inertia of the rotor and the disposer, respectively, then, using the notation from Fig. 4.22, we can write

$$T = \tfrac{1}{2} m_1 \dot{x}_1^2 + \tfrac{1}{2} m_2 \dot{x}_2^2 + \tfrac{1}{2} I_\mathrm{r} \dot{\phi}^2 + \tfrac{1}{2} I_\mathrm{d} \dot{\psi}^2, \tag{4.361}$$

$$V = \tfrac{1}{2} k(x_1 - x_2)^2. \tag{4.362}$$

Constraint multipliers $B_{\beta\sigma}\,(\beta=1,\ \sigma=1,2,3,4)$ will as usually be determined from the comparison of the deployed form and the concrete form of constraints (4.356). We have, therefore,

$$\begin{array}{ccccl} B_{11}\dot{q}_1 + & B_{12}\dot{q}_2 + & B_{13}\dot{q}_3 + & B_{14}\dot{q}_4 + & B_1 = \dot{x}_2 - \dfrac{c}{2A}(1+\cos 2\psi)\dot{\phi}, \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \\ \dot{x}_1 & \dot{x}_2 & \dot{\phi} & \dot{\psi} & \end{array}$$

and hence

$$B_{11}=0, \quad B_{12}=1, \quad B_{13}=-\frac{c}{2A}(1+\cos 2\psi), \quad B_{14}=0, \quad B_1=0. \tag{4.363}$$

From the potential (strain) energy (4.362) and using (4.159) we get

$$Q_1 = -\frac{\partial V}{\partial x_1} = -k(x_1-x_2),$$
$$Q_2 = -\frac{\partial V}{\partial x_2} = k(x_1-x_2).$$

On the basis of assumptions (5) and (6) we have (4.364)

$$\begin{aligned} Q_3 &= M_r - H_r, \\ Q_4 &= M_d - H_d, \end{aligned} \tag{4.365}$$

where $M_r$ and $M_d$ as well as $H_r$ and $H_d$ are driving and resistance torques of the rotor and the disposer, respectively.

Now, having introduced expressions (4.361), (4.363)–(4.365) to equations (4.357)–(4.360), we obtain the following system of four equations with five unknowns $x_1$, $x_2$, $\phi$, $\psi$ and $\lambda_1$:

$$m_1\ddot{x}_1 + kx_1 = hkx_2 \tag{4.366}$$

$$m_2\ddot{x}_2 + kx_2 = kx_1 - \lambda_1 \tag{4.367}$$

$$I_r\ddot{\phi} + H_r = M_r - \lambda_1\frac{\mathrm{c}}{2\,\mathrm{A}}(1+\cos 2\psi) \tag{4.368}$$

$$I_d\ddot{\psi} + H_d = M_d. \tag{4.369}$$

The system (4.366)–(4.369), together with the equation of constraints (4.356), constitutes the complete model of the problem.

Before proceeding one must eliminate the unknown multiplier $\lambda_1$. For this purpose, multiply equation (4.367) by

$$\frac{c}{2A}(1+\cos 2\psi),$$

add the equation obtained to equation (4.368) and use equation (4.356); then

$$I_r\ddot{\phi} + m_2\left(\frac{c}{2A}\right)^2[(1+\cos 2\psi)\ddot{\phi} - 2\dot{\phi}\dot{\psi}\sin 2\psi](1+\cos 2\psi)$$
$$-k\frac{c}{2A}(x_1 - x_2)(1+\cos 2\psi) + H_r = M_r. \tag{4.370}$$

There are also equations

$$m_1\ddot{x}_1 + kx_1 = kx_2, \tag{4.371}$$

$$I_d\ddot{\psi} + H_d = M_d, \tag{4.372}$$

$$\dot{x}_2 = \frac{c}{A}\dot{\varphi}\cos\psi. \tag{4.373}$$

Finally, the model for the problem is composed of four equations (4.371)–(4.373) with four unknowns $x_1$, $x_2$, $\phi$, and $\psi$. One must also remember that $M_r = M_r(\dot{\phi})$, $M_d = M_d(\dot{\psi})$, $H_r = H_r(\dot{\phi}, \dot{\psi})$ and $H_d = H_d(\dot{\phi}, \dot{\psi})$; we are dealing with a coupled system (this remark concerns, in particular, equation (4.373) which, at a first glance, seems to be solvable independently of the other equations).

Thus, it has been shown that the Lagrange equations with multipliers may be substituted for more refined methods such as the Voronetz equations.