phenomenon. We draw the other group in different degrees, depending upon the purpose of modelling. Thus, if the purpose is the determination of the motion of a body (which is a common task in modelling), we have to complement the basis with information on the properties of the body, on forces and on how forces acting upon the body depend upon the characteristics of motion and properties of the body itself. Alternatively when the purpose is to identify the forces causing a motion of a body, then it is necessary for modelling to have information on the physical properties and motion of the body.

Information on the motion, forces and properties of a body entail the following:

— complementing the system of dynamic equations of motion with equations resulting from either kinetic relations such as (3.63), (3.66) or resulting from constraints,
— specification of forces, using for this purpose specific laws of mechanics,
— use of relations between the properties of a body.

This should bring the system of equations to the state in which the number of equations of the model is equal to the number of unknowns.

The identification of problems of mechanics means that in every mathematical model of dynamical phenomena there must appear at least one element from each set distinguished above, i.e. BLM, B, F, M and SLM. Thus if the system of equations is incomplete at a certain stage of modelling, one should verify that all the available information from all the sets indicated has been used.

We finish at this point with a survey of the tools of modelling used in mechanics. How they are used will be discussed in subsequent sections.

3.4 APPLICATIONS

3.4.1 Parachute with a payload

When we speak of resistance, our first reaction is to think of it as a disadvantageous phenomenon. A parachute is an example of a device in which the phenomenon of resistance to motion is exploited.

Parachutes are classified according to their various purposes: rescue, sports, military or transport (airborne supply of medical aid, food, etc.). For a parachute to perform its duties, it must behave in a stable manner. However, the reason for the various dynamic instabilities of a parachute have not been fully explained. One problem is that dynamic stability testing of parachutes is difficult. The theory of parachute instability requires a nonlinear three-dimensional analysis with several assumptions that are difficult to justify. Modelling itself is not easy, for because of the fact that the mass of parachute is usually small and rate of change of velocity great, one should consider the so-called *apparent masses*.

It is known that under appropriate conditions the apparent masses of the body can play an important role in the determination of dynamic characteristics, and this is certainly the case with parachute systems.

According to classification of airborne vessels a parachute is an aerodyne without an engine-driven propulsion unit, with the total aerodynamic force acting in the opposite direction to motion. Under the assumption that the parachute is already open—that is, omitting the initial opening shock from consideration—the general motion of a parachute can be separated into four stages: (1) steady vertical descent; (2) steady gliding; (3) large-angle pitching oscillation, and (4) large-angle coning motion.

The current objective is to derive the relevant equations of motion for the parachute and payload system. These equations, in turn, may arise from the adoption of a certain theory and be used to determine the effects of various parameters on the stability of a fully deployed parachute.

To reduce the problem to one which will yield useful results, the following assumptions have been made:

(1) the system consists of a symmetric parachute rigidly connected to the payload;
(2) the roll of the parachute about its axis of symmetry is negligible;
(3) the Earth is flat and there are no winds;
(4) the aerodynamic force of the payload is negligible;
(5) the centre of pressure of the canopy is taken at its mass centre.

Note that while the mass of the parachute and the payload, m, remain constant the apparent masses may change in time. Because of this, the location of mass centre of the entire system changes in time as well and this is why application of the equations of motion in the form commonly encountered in flight mechanics is impossible.

Derivation of equations of motion shall be based upon the theorem on the change of linear and angular momentum but in the form suitable in the case considered

$$\frac{\partial \mathbf{p}_o}{\partial t} + \Omega \times \mathbf{p}_o = \mathbf{F}, \tag{3.80}$$

$$\frac{\partial \mathbf{H}_o}{\partial t} + \Omega \times \mathbf{H}_o + \mathbf{v}_o \times \mathbf{p}_o = \mathbf{M}_o, \tag{3.81}$$

where

$$\frac{\partial \mathbf{p}_o}{\partial t} = \frac{\mathrm{d}p_x}{\mathrm{d}t}\mathbf{i} + \frac{\mathrm{d}p_y}{\mathrm{d}t}\mathbf{j} + \frac{\mathrm{d}p_z}{\mathrm{d}t}\mathbf{k}, \tag{3.82}$$

$$\frac{\partial \mathbf{H}_o}{\partial t} = \frac{\mathrm{d}H_x}{\mathrm{d}t}\mathbf{i} + \frac{\mathrm{d}H_y}{\mathrm{d}t}\mathbf{j} + \frac{\mathrm{d}Hz}{\mathrm{d}t}\mathbf{k}. \tag{3.83}$$

Adopting these equations to the system of coordinates and to notation of velocities used in mechanics of flight (see Fig. 3.22) and using the following notation:

$$\mathsf{V}_o = [U, C, W]^{\mathrm{T}}, \qquad \Omega = [P, Q, R]^{\mathrm{T}}, \tag{3.84}$$

$$\mathsf{F} = [X, Y, Z]^{\mathrm{T}}, \qquad \mathsf{M}_o = [L, M, N]^{\mathrm{T}}, \tag{3.85}$$

the components of the linear momentum vector, $\mathbf{p}_o$, may be expressed as

$$\begin{aligned} p_x &= (m + \alpha_{11})U + (mz_s - \alpha_{22}z_a)Q, \\ p_y &= (m + \alpha_{22})V - (mz_s - \alpha_{11}z_a)P, \\ p_z &= (m + \alpha_{33})W, \end{aligned} \tag{3.86}$$

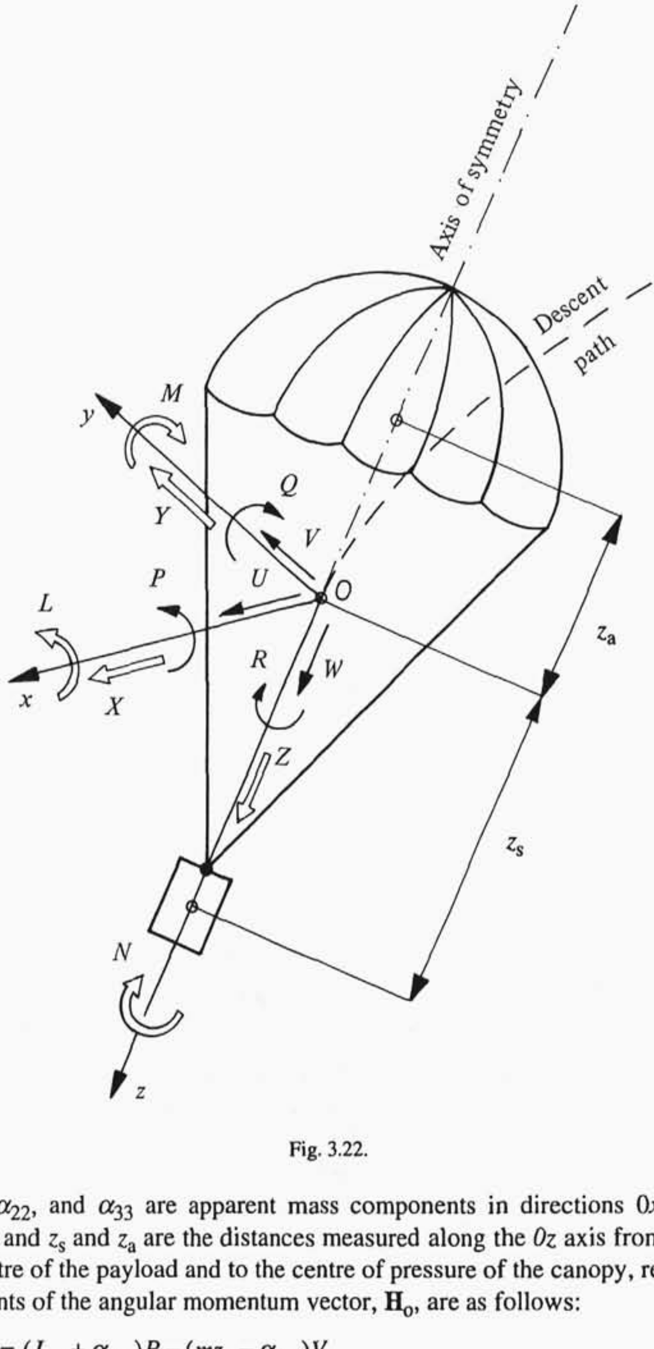

Fig. 3.22.

where α_{11}, α_{22}, and α_{33} are apparent mass components in directions $0x$, $0y$, and $0z$, respectively, and z_s and z_a are the distances measured along the $0z$ axis from the origin to the mass centre of the payload and to the centre of pressure of the canopy, respectively.

Components of the angular momentum vector, $\mathbf{H}_o$, are as follows:

$$H_x = (I_{xx} + \alpha_{44})P - (mz_s - \alpha_{22})V,$$
$$H_y = (I_{yy} + \alpha_{55})Q - (mz_s - \alpha_{11})U, \tag{3.87}$$
$$H_z = I_{zz}R,$$

where I_{xx}, I_{yy} and I_{zz} are the moments of inertia of the entire canopy and payload system, measured about the $0x$, $0y$ and $0z$ axes, respectively, $\alpha_{44} = \alpha_{55}$ are apparent moments of inertia about $0x$ and $0y$ axes, respectively.

Having performed all operations indicated in (3.80) and (3.81) we obtain a system of six equations of motion:

$$
\begin{aligned}
&(m+\alpha_{11})(\dot{U}-RV)+(m+\alpha_{33})QW+(mz_s-\alpha_{11}z_a)(\dot{Q}+RP)\\
&\qquad +\dot{\alpha}_{11}(U-Qz_a)=X,\\
&(m+\alpha_{11})(\dot{V}-RU)-(m+\alpha_{33})PW-(mz_s-\alpha_{11}z_a)(\dot{P}+RQ)\\
&\qquad +\dot{\alpha}_{11}(V-Pz_a)=Y,\\
&(m+\alpha_{33})\dot{W}-(m+\alpha_{11})(QU-PV)-(mz_s-\alpha_{11}z_a)(P^2-Q^2)\\
&\qquad +\dot{\alpha}_{33}W=Z,\\
&(I_{xx}+\alpha_{55})\dot{P}-(mz_s-\alpha_{11}z_a)(\dot{V}+RU-PW)+(I_{zz}-I_{yy}-\alpha_{55})QR\\
&\qquad +(\alpha_{33}-\alpha_{11})VW+\dot{\alpha}_{55}P+\dot{\alpha}_{11}z_aV=L,\\
&(I_{yy}+\alpha_{55})\dot{Q}+(mz_s-\alpha_{11}z_a)(\dot{U}-RV-QW)+(I_{xx}-I_{zz}+\alpha_{55})PR\\
&\qquad +(\alpha_{11}-\alpha_{33})UW+\dot{\alpha}_{55}Q-\dot{\alpha}_{11}z_aU=M,\\
&\qquad I_{zz}\dot{R}=N,
\end{aligned}
\tag{3.88}
$$

where the three following symmetry assumptions have been introduced $\alpha_{22} = \alpha_{11}$, $\alpha_{44} = \alpha_{55}$, and $I_{xx} = I_{yy}$.

The system of six differential equations (3.88) just obtained can be regarded as a starting point for further stability investigations. These may be performed either numerically by solving these equations and by subsequent simulation, or analytically after some necessary simplifications which reduce the nonlinear system (3.88) to its linearized form. For details of both procedures see Yavuz (1985).

3.4.2 Elevator hydraulic amplifier

A pilot can attain any arbitrary position of aircraft in space by rotating the craft about the three axes ξ, η, ζ shown in Fig. 3.19. Changes in the aircraft position are brought about by control surfaces, hereafter called controls. There are many types of controls, but the most frequently used ones are the ailerons, rudder and elevator. They are moved from the pilot's cockpit with the help of devices collectively known as flying controls. Inclination of the controls and maintaining them in this inclined position during flight sometimes requires a considerable effort on the part of pilot. In order to decrease this effort and resulting fatigue, a number of auxiliary devices are applied. Such devices, used especially in large aircraft flying at high speeds, are hydraulic amplifiers known as *boosters*.

Two types of booster are distinguished, namely reversible and irreversible ones. A reversible booster does not counter the entire hinge moment of the controls; a small part of it remains to be overcome by the pilot, so that he is aware of changes in the forces acting upon the controls. In other words the pilot is enabled to 'feel' the controls. In high-speed aircraft irreversible boosters are applied. We shall deal here with just such a booster. To focus attention let us assume that we are speaking of an elevator booster. The scheme of such a booster is shown in Fig. 3.23. This figure shows symbolically the so-called loading mechanism (composed of a spring and a damper), whose purpose is to ensure that the pilot feels the controls, since the feature of an irreversible booster is that it counters the whole hinge moment.

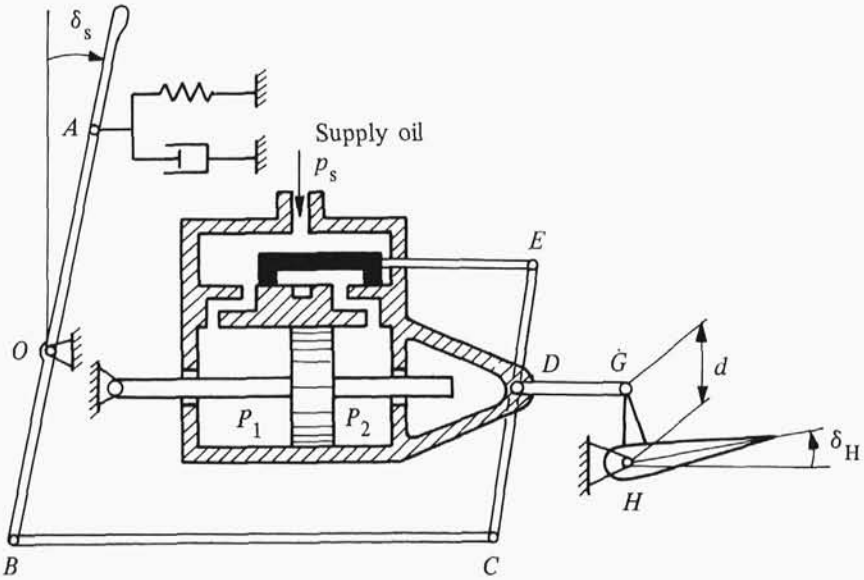

Fig. 3.23.

To model this complex system it is useful to explain the 'starting' of the system. Thus, inclination of the stick causes a displacement of the slider of the supply head and opening of the ports controlling the flow of oil. Due to this the oil from the pump, under pressure p_s, flows to the left chamber of the servo-motor, in which there is pressure p_1, which causes the booster body to move to the left, for the strap is immobile. Further functioning of the system should now be understandable from Figs. 3.24 and 3.25.

The purpose of modelling is to obtain the relation between the angle of inclination of the elevator, δ_H, and the angle of inclination of the stick, δ_s.

The following assumptions contribute to the physical model of the booster:

(1) pressures in each of the parts of the servo-motor and the supply head are homogeneous over the whole space (which does not imply that they are equal);
(2) the piston of the servo-motor is linked rigidly with the basis;
(3) the levers and the push-rods, excepting the rod *DG* joining the booster and the elevator, are perfectly rigid;
(4) there are no clearances on the joints;
(5) all the walls are undeformable;
(6) dry friction in the joints can be neglected;

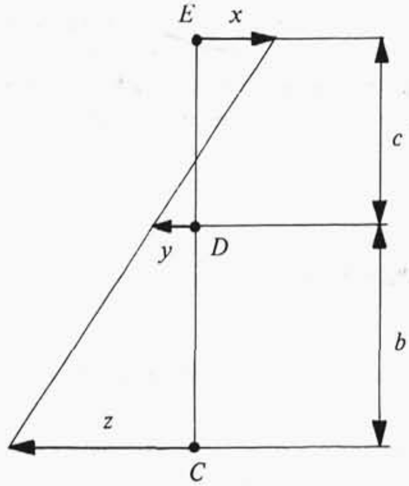

Fig. 3.24.

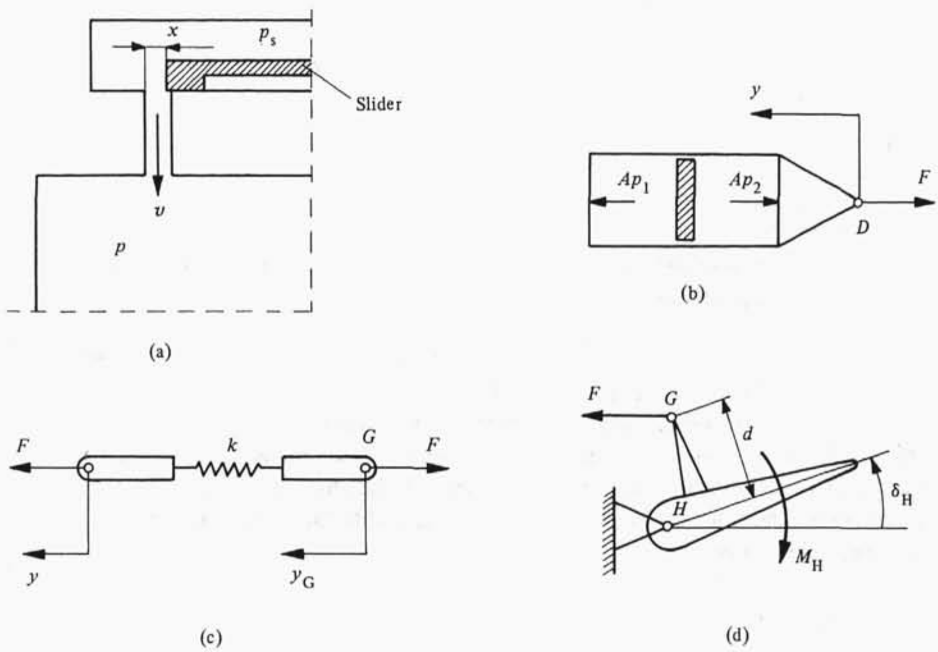

Fig. 3.25.

(7) there are no leakages in the installation;
(8) the neutral position of the stick corresponds to the neutral position of the elevator;
(9) inclinations are small;
(10) the flow around the elevator is steady.

Note that the most important is assumption (1), for it makes possible the development of a discrete model for a continuous medium such as oil.

In order to present this difficult task more clearly we shall separate from the whole system certain components, to be then described in sequence:

(1) the subsystem of rigid levers and joints;
(2) the supply head;
(3) the servo-motor;
(4) the elastic connecting rod DG;
(5) the elevator.

(1) Deflections of the stick, defined by the angle δ_s, are accompanied by the three following movements, shown in Fig. 3.24:

— displacement of the connecting rod BC by the distance z,
— displacement of the booster body by the distance y,
— displacement of the slide valve by the distance x.

On the basis of Fig. 3.24 we can establish the following relation between the three displacements mentioned above:

$$\frac{y+x}{z+x}=\frac{c}{c+b}, \tag{3.89}$$

whence

$$x=\frac{1}{b}[cz-(b+c)y]. \tag{3.90}$$

Under the assumption that inclinations of the control stick are small, we have, besides this, the following relation:

$$z=a\delta_s \tag{3.91}$$

where a denotes the length OB of the stick.

(2) Here the crucial question is the interrelation between the shift, x, of the slider of the supply head (which corresponds to opening of the channel port) and the volume efflux, Q, of flow through the channels supplying the chambers of the servo-motor. We postulate that both for compressible and incompressible fluids the relation between these two quantities is linear:

$$Q=Kx, \tag{3.92}$$

where K is a coefficient which depends upon the construction of the supply head and upon the parameters of the pump. For a compressible fluid K is established experimentally, while for an incompressible fluid it can be calculated on the basis of the Bernoulli equation:

$$p+\tfrac{1}{2}\rho v^2=\text{const}, \tag{3.93}$$

where p, ρ, v are respectively pressure, density and velocity of the fluid. In order to apply equation (3.93) to obtain the relation mentioned we shall consider the part of the system which is of interest to us, modelling it in the manner shown in Fig. 3.25a.

We shall assume the general form (3.92) distinguishing two cases:

$$Q_1 = K_1 x \qquad \text{and} \qquad Q_2 = K_2 x. \tag{3.94}$$

In order to obtain the general formula we shall take no account of quantities connected with the concrete chamber of the servo-motor. Similarly, we shall not be indicating in the figure the cross-sections for which we shall specify the integration constant in equation (3.93). Thus, let us consider two cross-sections—one is just ahead of the inflow and the second just ahead of the outflow. It is also assumed that the pressure in the outflow cross-section is the same as the pressure of the fluid in the chamber of the servo-motor. Thus, we have

$$p_s + \tfrac{1}{2}\rho v_s^2 = p + \tfrac{1}{2}\rho v^2. \tag{3.95}$$

Since the cross-section of the chambers of the supply heads is much greater than that of the supply channels, we can admit that $v_s \ll v$ or $v_s \approx 0$. Then, from equation (3.95) we can calculate the velocity of the flow in the channel,

$$v = \sqrt{(2\,\Delta p/\rho)}, \tag{3.96}$$

where $\Delta p = p_s - p$.

The volume efflux, according to equation (3.19), for incompressible fluids, can be expressed by the formula

$$Q = Av, \tag{3.97}$$

where A is the surface of the cross-section of the supply channel. As the valve ports are usually of rectangular cross-section with constant width h and variable length x (see Fig. 3.25a), formula (3.97), taking account of (3.96), takes the form

$$Q = hvx = h\sqrt{(2\Delta p/\rho)}x. \tag{3.98}$$

This relation is usually modified yet by introduction of a certain rectifying coefficient, $\beta < 1$, commonly called in hydraulics the *contraction coefficient*. In aeronautical hydraulics the coefficient ν of the flow through the supply head channels is used. We have then

$$Q = \gamma\sqrt{(\Delta p)}x, \tag{3.99}$$

where

$$\gamma = \nu h\sqrt{(2/\rho)}.$$

Thus we have obtained a relation analogous to (3.92), but appearing to have greater cognitive and practical value, when incompressibility of the fluid can be assumed.

Note at this moment that on the basis of (3.99) we have

$$Q_1 = \gamma\sqrt{(p_s - p_1)}x, \qquad Q_2 = \gamma\sqrt{(p_2 - p_o)}x, \tag{3.100}$$

where p_o is the pressure of the outflow (it is often assumed that $p_o = 0$). Thus, $Q_1 \neq Q_2$.

(3) Let us first take care of the adaptation of equation (3.28) for the servo-motor considered. For this purpose we shall transform the term $(\Omega/\rho)/\mathrm{d}\rho/\mathrm{d}t$ to a form more useful in practice. In order to do this we shall make use of a notion which serves in hydraulics to classify fluids. We are speaking, namely, of the *volume elasticity modulus* B, which is defined in the following way (see Guillon (1961) p. 5.3.2):

$$\frac{\mathrm{d}\rho}{\mathrm{d}p} = \frac{1}{B}\rho. \tag{3.101}$$

This relation can be obtained through linearization of the density–pressure curve around the chosen point. The curve itself is obtained experimentally under conditions of constant temperature. We should also remember that the value of B is constant only within a certain range of pressure changes. Using the language of modelling we can say that relation (3.101) is a *constitutive law*.

On the basis of (3.101) we have also another possible representation of the term $(\Omega/\rho)/\mathrm{d}\rho/\mathrm{d}t$:

$$\frac{\Omega}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}t} = \frac{\Omega}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}p}\frac{\mathrm{d}p}{\mathrm{d}t} = \frac{\Omega}{B}\frac{\mathrm{d}p}{\mathrm{d}t}, \tag{3.102}$$

due to which equation (3.28) can be represented as

$$\frac{\Omega}{B}\frac{\mathrm{d}p}{\mathrm{d}t} + \frac{\mathrm{d}\Omega}{\mathrm{d}t} = Q^{(\mathrm{i})} - Q^{(\mathrm{o})} \tag{3.103}$$

In order to describe phenomena occurring in the servo-motor we introduce the following notations:

- p_1, p_2 are pressures in the servo-motor chambers (see Fig. 3.23);
- Q_1, Q_2 are volume effluxes in channels supplying these chambers;
- A is a surface of the active cross-section of the chambers;
- y is the displacement of the booster body from the neutral position (see Figs. 3.24 and 3.25b).

The following relations then hold:

$$\Omega_1 = \frac{\Omega_0}{2} - Ay, \qquad \Omega_2 = \frac{\Omega_0}{2} + Ay, \qquad \Omega_1 + \Omega_2 = \Omega_0$$

$$Q^{(\mathrm{i})} = Q_1, \quad Q^{(\mathrm{o})} = 0 \qquad \text{for the first chamber,}$$

$$\mathrm{Q}^{(\mathrm{i})} = 0, \quad Q^{(\mathrm{o})} = Q_2 \qquad \text{for the second chamber.}$$

so that on the basis of (3.103) we obtain

$$\frac{\Omega}{B}\dot{p}_1 + A\dot{y} = Q_1, \tag{3.104}$$

$$\frac{\Omega}{B}\dot{p}_2 - A\dot{y} = -Q_2, \tag{3.105}$$

where Q_1 and Q_2 are given with formulae (3.100).

Under the influence of pressure differences the movement of the body occurs. Assuming that we account only for the mass of the booster (neglecting resistance and deformation) we obtain the form of the equation of motion as follows:

$$m\ddot{y} = A(p_1 - p_2) - F \tag{3.106}$$

where F denotes the reaction force of the connecting rod DG.

(4) Since in the real control system the path from the booster to the elevator is quite cumbersome, the connecting rod DG, which embodies the whole system of connections, is treated as deformable. As the physical model we shall use the popular discrete model (Fig. 3.25c) with the predefined elasticity k. In view of this we have

$$F = k(y - y_G), \tag{3.107}$$

with $y_G = d\delta_H$ (see Fig. 3.25d).

(5) Assuming that we do not account for elasticity of fixing of the elevator on the rudder and that we do neglect resistance we obtain the equation of motion in the form

$$I_H \ddot{\delta}_H = Fd - M_H \tag{3.108}$$

where M_H denotes the so-called *hinge moment* caused by aerodynamic load on the elevator.

If we assume that the flow is stationary, then we can admit that

$$M_H = C_H \delta_H, \tag{3.109}$$

where $C_H = \frac{1}{2}\rho_\infty v_\infty^2 S b_a m_H^\delta$ is the quantity characterizing a given elevator, with m_H^δ denoting the coefficient of the moment.

After introducing (3.109) to (3.108), we obtain

$$\ddot{\delta}_H + \omega_H^2 \delta_H = \frac{d}{I_H} F \tag{3.110}$$

where

$$\omega_H^2 = \frac{C_H}{I_H}.$$

Now let us perform certain introductory transformations, making use of relations at the points of contact between subsequent stages, and also within the framework of particular stages. And thus:

(1) we substitute (3.91) into (3.90), obtaining

$$x = \frac{1}{b}\left[ac\delta_s - (b + c)y\right]; \tag{3.111}$$

(2) we substitute (3.100) into equations (3.104) and (3.105), rspectively, and we get

$$\frac{\Omega}{B}\dot{p}_1 + A\dot{y} = K_1 x, \tag{3.112}$$

$$\frac{\Omega}{B}\dot{p}_2 - A\dot{y} = -K_2 x; \tag{3.113}$$

(3) we substitute formula (3.107) into equation (3.106) and we get

$$\ddot{y}+\omega_n^2 y = d\omega_n^2 \delta_{\mathrm{H}} + \frac{A}{m}(p_1 - p_2) \tag{3.114}$$

where

$$\omega_n^2 = \frac{k}{m}.$$

(4) we substitute formula (3.107) to equation (3.110):

$$\ddot{\delta}_{\mathrm{H}} + \left(\frac{d^2 k}{I_{\mathrm{H}}} + \omega_{\mathrm{H}}^2\right)\delta_{\mathrm{H}} = \frac{dk}{I_{\mathrm{H}}} y. \tag{3.115}$$

We can now balance the equations and unknowns. By manipulation of equations we obtained the system of five equations (3.111)–(3.115), with five unknowns x, y, p_1, p_2, δ_{H}, which constitutes the model of the booster dynamics. On the basis of this model we can obtain the required relation between the angle of inclination of the elevator, δ_{H}, and the angle of inclination of the stick, δ_s.

3.4.3 The ablation at the nose of the re-entry vehicle

A vehicle approaching a planetary atmosphere from space or from orbit possesses a large amount of total energy (kinetic energy due to its speed and potential energy due to its position above the planet). When it encounters the atmosphere, a shock wave forms ahead of the nose of the vehicle, heating the atmosphere in this region to a very high temperature. As the vehicle plunges into a deeper and denser atmosphere, the vehicle is increasingly heated by this enveloping layer of incandescent atmosphere, while the speed of the vehicle is continuously reduced by the braking force of the atmosphere. It is in this manner that the vehicle's kinetic energy is converted into heat. If all of the vehicle's energy were converted into heat energy within the vehicle itself, it would be more than enough to vaporize the vehicle, together with its payload and any cooling system it could carry.

There are several methods of dealing with this problem (see e.g. Loh (1963)). One of them, possibly the most interesting, for its similarity with natural 'cooling down' of meteorites, consists in application of a heat shield constructed of insulating layers of fibreglass and similar materials, properly applied and bounded. Under intense heat, the outer layer of the shield chars, melts and vaporizes. This process, known as *ablation*, in itself absorbs a good deal of heat. In addition, the vaporized material tends to block the flow of heat from the shock-layer into the body. The ablation heat shield is also one of the lightest, and certainly is the simplest, solutions of the re-entry problem that can be achieved.

The purpose of modelling is to determine the mass of the minimum protective cover which is lost in the process of ablation while protecting the vehicle from burning.

The following assumptions compose the physical model:

(1) mass losses are small compared with the mass of the entire vehicle, which means that we treat the mass of the vehicle as constant;

(2) the vehicle trajectory is very smooth (the vehicle slowly approaches the Earth);
(3) the shock-wave boundary-layer interaction may be neglected;
(4) the boundary layer remains laminar;
(5) we aim to protect only against convective heat transfer—in other words we do not account for the radiant heat transfer from the surface;
(6) the properties of the material are constant; this concerns, in particular, the following quantities: the density ρ_b, thermal conductivity K_b, specific heat C_p, latent heat of sublimation L_s, sublimation temperature T_a;
(7) the rate of accumulation of heat is small compared with the rate of disposal of heat;
(8) conduction of heat takes place along lines normal to the surface.

Before the vehicle re-enters the atmosphere the ablation shield is assumed to have uniform temperature T_∞ (see Fig. 3.26). When sublimation of material occurs, the conduction of heat within the material depends on the rate of mass loss from the surface. The problem is in general very complex, for it has a nonlinear, continuous and non-stationary character. Nevertheless, under the assumptions made we can treat the problem in an approximate way and describe it with the help of the heat energy balance as follows:

$$Q(t) = Q_s + Q_r, \tag{3.116}$$

where $Q(t)$ is the heat input at the surface, Q_s is the heat absorbed by sublimated material, and Q_r is the heat accumulated by the remaining material.

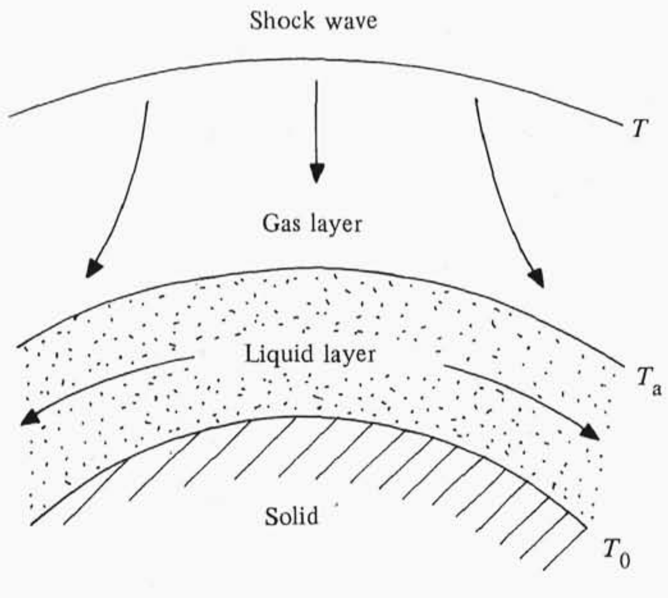

Fig. 3.26.

The following notation will be adopted:

T_∞ is the temperature of free stream,
T_a is the ablation temperature,
L_s is the latent heat of sublimation,
m is the mass ablated per unit area,
z is the outward normal distance from the ablation surface.

Under the assumptions made and with the notations adopted we have

$$Q_s = m\left[c_b(T_a - T_\infty) + L_s\right], \tag{3.117}$$

$$Q_r = c_b\rho_b \int_{-\infty}^{0} (T - T_\infty)\, dz, \tag{3.118}$$

where $T = T(t, z)$ is the current temperature. Let us now introduce the *integral thickness* θ defined as

$$\theta(t) = \int_{-\infty}^{0} \frac{T - T_\infty}{T_s - T_\infty}\, dz, \tag{3.119}$$

where $T_s(t)$ is the surface temperature. Then formula (3.118) takes the form

$$Q_r = c_b\rho_b(T_s - T_\infty)\theta(t). \tag{3.120}$$

Having introduced (3.117) and (3.120) into equation (3.116) and differentiated the latter with respect to time, we obtain

$$\frac{dQ}{dt} \equiv q(t) = \left[c_b(T_a - T_\infty) + L_s\right]\frac{dm}{dt} + c_b\rho_b\frac{d}{dt}\left[(T_s - T_\infty)\theta(t)\right]. \tag{3.121}$$

It is important to note that in equation (3.121) the heat-transfer rate $q(t)$ is that which the shield actually experiences and is itself a function of the rate of sublimation. For the reduction in heat-transfer rate due to the introduction of mass into the boundary layer a quasi-steady relation is used:

$$q(t) = q_0(t) - \alpha\tilde{c}_p(T_e - T_a)\frac{dm}{dt}, \tag{3.122}$$

where $q_0(t)$ is the heat-transfer rate experienced by a non-ablating body at the surface temperature T_a, α is the fractional temperature rise of gaseous material, $\tilde{c}_p$ is the effective mean value of specific heat at constant pressure, T_e is the temperature external to the boundary layer at stagnation point.

The unknown heating rate $q(t)$ is eliminated from equation (3.121) by use of formula (3.122) to give

$$q_0(t) = h_d + h_a, \tag{3.123}$$

where

$$h_{\mathrm{d}} = \left[c_{\mathrm{b}}(T_{\mathrm{a}} - T_{\infty}) + L_{\mathrm{s}} + \alpha\tilde{c}_{\mathrm{p}}(T_{\mathrm{e}} - T_{\mathrm{a}})\right]\frac{\mathrm{d}m}{\mathrm{d}t} \tag{3.124}$$

$$h_{\mathrm{a}} = c_{\mathrm{b}}\rho_{\mathrm{b}}\frac{\mathrm{d}}{\mathrm{d}t}\left[(T_{\mathrm{s}} - T_{\infty})\theta\right] \tag{3.125}$$

are the rates of disposal and accumulation of heat, respectively.

On the basis of assumption (7) equation (3.124) can be expressed as follows:

$$\frac{\mathrm{d}m(t)}{\mathrm{d}t} = \frac{q_0(t)}{\chi(t)}, \tag{3.126}$$

where

$$\chi(t) = c_{\mathrm{b}}(T_{\mathrm{a}} - T_{\infty}) + L_{\mathrm{s}} + \alpha\tilde{c}_{\mathrm{p}}(T_{\mathrm{e}} - T_{\mathrm{a}}). \tag{3.127}$$

Note that the goal of modelling, that is, determination of the mass of the cover necessary to protect the vehicle, could be realized if the right-hand side of equation (3.126) were a known function of time. Alas, neither $q_0(t)$ nor $\chi(t)$ (the latter because of the unknown temperature $T_{\mathrm{e}}(t)$ are such functions. That is why we have to make use of specific laws, provided in this case by experiment (Loh (1963)):

$$q_0(t) = C_1\sqrt{(\rho)}v^3 \tag{3.128}$$

$$T_{\mathrm{e}}(t) = T_{\infty} + C_2 v^2 \tag{3.129}$$

where C_1 and C_2 are certain constants depending upon the geometric, thermal and flow properties of the shield, while ρ and v denote the air density and velocity of the vehicle, respectively.

Adoption of formulae (3.128) and (3.129) does not close the problem, though, for it introduces new unknowns, namely air density and flight velocity. As far as density is concerned, we can accept the known law of variation of density with altitude y,

$$\rho = \rho_0 \exp(-\beta y), \tag{3.130}$$

where ρ_0 denotes the value of reference $\rho_0 \approx 1.25\ \mathrm{kg/m^3}$, β is the atmospheric density parameter, $\beta \approx 1.3 \times 10^{-6}\ \mathrm{m^{-1}}$, and therefore a new unknown, y, is dragged into the problem. This variable also needs description by an equation. This equation is the so-called kinematic equation

$$v_y = \frac{\mathrm{d}y}{\mathrm{d}t} = v\sin\theta, \tag{3.131}$$

where θ is the flight-path angle relative to local horizontal direction (positive for descent), see Fig. 3.27. As far as flight velocity is concerned, there is no alternative but to write down the dynamic equation of motion. Thus, under assumption (2), on the basis of Newton's second law expressed in natural coordinates, using the notation from Fig. 3.27, we have

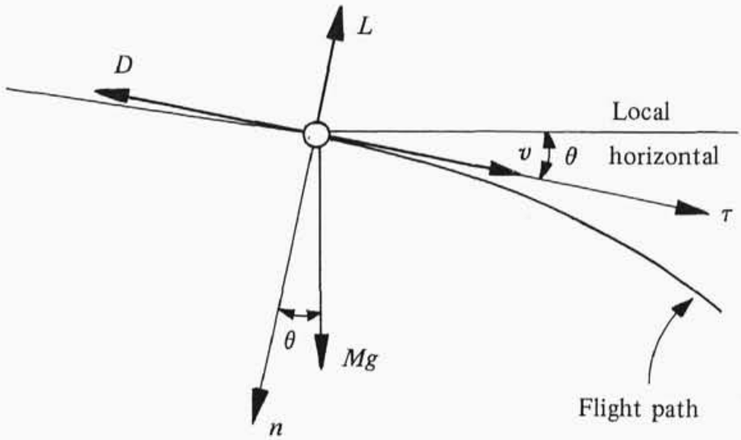

Fig. 3.27.

$$M\frac{\mathrm{d}v}{\mathrm{d}t} = -D + Mg\sin\theta, \tag{3.132}$$

$$M\frac{v^2}{\kappa} = -L + Mg\cos\theta, \tag{3.133}$$

where M is the mass of vehicle, D, L are drag and lift forces, respectively, and κ is the radius of curvature of the flight path.

From elementary aerodynamics we know that

$$D = \tfrac{1}{2}\rho v^2 C_\mathrm{D} A \qquad \text{and} \qquad L = \tfrac{1}{2}\rho v^2 C_\mathrm{L} A, \tag{3.134}$$

where C_D and C_L are the drag and lift coefficients, respectively, and A is the reference area for drag and lift.

Recalling (3.126) and substituting for q_0 and χ equations (3.128) and (3.127) (where equation (3.129) has been introduced) gives

$$\frac{\mathrm{d}m(t)}{\mathrm{d}t} = \frac{C_1\sqrt{(\rho)}v^3}{c_\mathrm{b}(T_\mathrm{a} - T_\infty) + L_\mathrm{s} + \alpha\tilde{c}_\mathrm{p} C_2 v^2}. \tag{3.135}$$

In the derivation of (3.135) an additional assumption has been made that $(T_\infty - T_\mathrm{a}) \ll C_2 v^2$. From (3.135) we see that the situation may be clarified if we perform a substitution

$$\frac{\mathrm{d}m}{\mathrm{d}t} = \frac{\mathrm{d}m}{\mathrm{d}v}\frac{\mathrm{d}v}{\mathrm{d}t}, \tag{3.136}$$

which would enable us to make use of equation (3.132). In fact, having introduced $\mathrm{d}v/\mathrm{d}t$ from equation (3.132) (with due account of formula (3.134) into the identity (3.136) gives

$$\frac{\mathrm{d}m}{\mathrm{d}t} = \frac{\mathrm{d}m}{\mathrm{d}v}\left(-\tfrac{1}{2}\rho v^2 \frac{C_\mathrm{D}A}{M} + g\sin\theta\right). \tag{3.137}$$

It should be remembered that, from the point of view modelling, equation (3.137) is not a new equation but only a transformation of the left-hand side of equation (3.135). Thus, having substituted (3.137) into (3.135) we obtain a modified energy equation (that is, modified through taking into account of the equation of motion):

$$\frac{\mathrm{d}m}{\mathrm{d}v} = \frac{C_1\sqrt{(\rho)}v^3}{\left[c_\mathrm{b}(T_\mathrm{a} - T_\infty) + L_\mathrm{s} + \alpha\tilde{c}_\mathrm{p}C_2v^2\right]\left(-\tfrac{1}{2}\rho v^2 \frac{C_\mathrm{D}A}{M} + g\sin\theta\right)}. \tag{3.138}$$

Equation (3.138), the equations of motion (3.132) and (3.133), and the additional equations (3.130) and (3.131) form the system of five equations with five unknowns, namely $m(v)$, $\theta(t)$, $\rho(y)$, $v(t)$, $y(t)$, whose solution can be determined numerically by means of computer.

An analytical solution is not necessary, since we are not asking for all the variables and not all of them are equally important. The most important one is of course the quantity $m(v)$ described by equation (3.138). We shall therefore analyse it. The magnitudes which couple this equation with the rest of the system are θ, ρ and v. The quantity v is indispensable, for it has replaced time, due to the use of (3.136). Density is not of concern and we can assume that formula (3.130) is independent of other equations. Thus, only the angle θ is left for consideration, and here there is no option but to make the mathematical assumption that

$$\theta \approx 0. \tag{3.139}$$

This can be justified: it corresponds to very flat flight paths (paths with low curvature), and thus from (3.131) $y \approx$ const. In order to decelerate from an extremely high velocity, the vehicle cannot return to Earth too steeply.

On the basis of assumption (3.139) equation (3.138) can already be integrated by separation of variables. To simplify notation let us introduce the dimensionless quantity

$$\vartheta = \frac{c_\mathrm{b}(T_\mathrm{a} - T_\infty) + L_\mathrm{s}}{\alpha\tilde{c}_\mathrm{p}}, \tag{3.140}$$

with whose help equation (3.138) can be expressed in a shorter form

$$\mathrm{d}m = \frac{C_3\mathrm{d}v^2}{(\vartheta + C_2v^2)\sqrt{\rho}}, \tag{3.141}$$

where $C_3 = C_1M/AC_\mathrm{D}\alpha\tilde{c}_\mathrm{p}$, and in which the transformation $v\,\mathrm{d}v = \frac{1}{2}\mathrm{d}v^2$ has been used.

We shall assume that we are interested in the loss of mass within the range of velocities v_0 and v_e for which ablation starts and terminates, and we can integrate equation (3.141) between these limits to get

$$m = C_3 \int_{v_0}^{v_e} \frac{dv^2}{(\vartheta + C_2 v^2)\sqrt{\rho}}. \tag{3.142}$$

Thus, the objective of the modelling has been realized.