2.3 SELECTED NOTIONS OF INTEGRATED MECHANICS

2.3.1 A physical system

Because an engineer in mechanics often encounters expressions such as *laws of mechanics, thermodynamics*, and *fluid mechanics*, there is a suggestion that one is dealing with various inviolable rules. In reality these are the same laws of nature, applied only to various systems. Thus, **physical system** shall be taken in this book to mean a mechanical system (see section 2.2.1), in which, besides the purely mechanical processes there take place also other processes, such as thermal, flow, electromagnetic, chemical, and so on.

The notion of *physical system* is close to that of *dynamical system*, the latter also appearing due to generalizations of various problems of theoretical mechanics. We shall not, however, be referring to this notion since in mathematical literature it is treated as an abstract object. One should perhaps only note that the beginnings of various theories generalizing the notion of system were made in 1912 by **George D. Birkhoff** (1884–1944), who defined a dynamical system as a single-parametric group of transformations of a closed region of n-dimensional space into itself. This definition is closely related to the system of n ordinary differential equations. Control theory introduced the notion of control function into this system of equations. The notion of dynamical system is here based upon the notion of input signal, state and output signal. The formal definition of a dynamical system as used in control theory (used, in any case, only for deterministic systems) was proposed in 1969 by **Rudolf E. Kalman** and is very complicated, stretching over several pages. Besides this, the definition mentioned is completely useless for our purposes. We also want to achieve uniformity of representation of various phenomena—not uniquely through formalization, though, but by indicating common roots of various branches of mechanics, as well as common elements in system modelling, for systems which traditionally belong to various branches of mechanics.

2.3.2 Generalized constraints

The concept of constraints as introduced in section 2.2.2 has for a long time been sufficient for constructing the models of phenomena and is still the subject of teaching in many courses. On the other hand this concept is presently not capable of meeting the requirements of contemporary technology. There are three cases which need generalizations in application of the methods of classical mechanics. First, we may encounter problems which generate constraints with much more complex analytical representation than (2.6) or (2.8). Such a situation arises, for instance, in an electrical motor, in which the commutator gives rise to complicated constraints, usually nonholonomic. But even when we remain within the domain of purely mechanical systems we encounter situations where obtaining constraints of the form (2.24) requires first an in-depth study of the phenomenon itself—for example, we must consider the influence of the deformation of tyres on the process of rolling of a pneumatic wheel. Such detailed knowledge is also needed to model the phenomenon of electric current flow in an electric generator at the point of contact of the rotor and commutator. This is confirmed by the difficulties encountered in the efforts to explain motion of the so-called *Barlow wheel*, undertaken at the end of the nineteenth century. In these situations constraint equations are not formulated as easily as in the classical scheme of a rigid body rolling without slipping. That is why such constraints will be referred to as **non-classical**.

In Volume 2, models of real bodies in the mechanics of continuous media will be shown to be very complicated and not to constitute a realistic basis for numerical analysis of a structure. Here, certain simplifying assumptions are introduced (at the stage of physical modelling), concerning stresses or deformations. Thus, for instance, in the elementary theory of beams there exists a routinely applied simplifying assumption, that is, the postulate of plane sections, which has been in use for a long time. *Kirchhoff's hypothesis* of indeformability of material fibres at the middle surface of the plate is considered one of the most significant discoveries in the theory of plates. Hypotheses of this kind can also be referred to as constraints. Since adoption of a hypothesis leads to a certain mathematical model, we shall call this type of constraint a **model constraint**. Because of the breadth and difficulty of this notion we shall not be considering problems involving it, although they are interesting and important. Notice only that the prototype of such constraints is constituted by the internal constraints of classical mechanics, which produce the model of a rigid body (see section 2.2.2).

Finally, many problems of contemporary technology are formulated as problems not of analysis but of synthesis. From this point of view we first determine what kind of constraints are imposed on the system, and only then do we consider the methods leading to satisfaction of these constraints. A strong motivation for such a generalization of the notion of constraints arose in problems of control of mechanical systems. The forerunners of such an approach were **Ivan V. Mieshtsherski** in Russia, at the end of nineteenth century, and **Henri Beghin**, in France, who introduced in 1922, the notion of **servoconstraints**. These constraints are not realized through direct contact, as occurs in classical mechanics, but through additional energy sources (of e.g. hydraulic or electromagnetic energy), which are controlled in such a way as to satisfy the constraints required. Action of this kind can be compared to the behaviour of a living organism, which functions so as to satisfy its wishes (constraints), such as the itinerary or programme of a tourist on an excursion. It is by analogy with the latter term that we introduce the notion of **programme constraints**, meaning here any analytical relation used to give the motion certain desired properties.

When speaking of control of mechanical systems it is worth noting that the motion equations in the form of (1.47) are, in modern control theory, also called constraints (nonholonomic constraints, for that matter).

2.3.3 Physical variables

The term 'physical variable' denotes the magnitudes used to describe the configuration of the physical system. Thus, they are in fact generalizations of generalized coordinates (see section 2.2.3). One should emphasize at once that generalized coordinates do not appear only in classical mechanics. Thus, for instance, electric charge is a typical generalized coordinate in electrodynamics (one could even suggest this is the prime example of a generalized coordinate). Why, therefore, do we introduce a new concept? We are doing it because in the 'mathematization' of a physical model the key question is not the nature of the physical system, but its transformation into a discrete or continuous model (see section 1.5). Thus, for a discrete model of any physical system we can effectively apply generalized coordinates, while for the continuous model such a notion is not applied even for mechanical systems! This statement will become more understandable after consideration of the issue of the number of describing functions—the counterpart of the number of degrees freedom in classical mechanics (see section 2.2.4).

We have to devote at least as much attention to physical variables as we did for generalized coordinates, since it is with the help of physical variables that we shall be studying the behaviour of physical systems. Because of a wide variety of kinds of system we do encounter varying definitions of variables. It should therefore be emphasized that in general we shall simply understand by physical variables those physical magnitudes which we are able to measure (directly or indirectly). Thus, some instances could be velocity, pressure, temperature, flow intensity, current intensity, electromagnetic field intensity, chemical potential or surface tension.

Depending upon the method of mathematical modelling physical variables are classified into certain groups. The most typical ones are listed below.

1. *Through variables and across variables*

Through variables are measures of something which flows through an element, e.g. a liquid through a tube or an electric current through a conductor. **Across variables**, on the other hand, are measures of the difference of states between two terminals of an element, e.g. the pressure fall between the ends of a tube or the difference in potential between the ends of a resistor.

2. *Extensive variables and intensive variables*

Extensive variables are quantities whose values depend upon the mass of the system in question, such as energy, entropy, electric charge. When values of the quantity do not depend upon the mass of the system, then it is an **intensive variable**, such as pressure, temperature or chemical potential. An important feature of extensive variables is their additive property, which intensive variables do not have.

3. *Input, state and output variables*

Input variables (excitations) represent the stimuli generated outside the system under consideration and influencing the behaviour of this system. **Output variables** (responses) describe the effects of functioning of the system, which are of interest for an observer or which can be measured. **State variables** (intermediate variables) describe dynamic behaviour of the system studied. If the system is represented by a black box with a certain number of terminals then the input and output terminals represent the set of input and output variables, respectively, while the state variables are 'contained' inside the box and are not accessible for observation or measurement.

Variables of this type are also called signals—especially in control theory. Thus input signals are those which can be changed in a predefined manner, as well as those which change randomly, irrespective of our will. Signals of the first type are called **control variables** (or for short, controls); those of the second are called disturbances.

State variables, because of frequent use of the notion of state in physics and technology (it appears in expressions like physical states of matter, states of nucleus, states of stress in a deformable body etc.), require a special treatment. Their concept, originating from control theory, was given above. One should remember, though, that they appeared much earlier in classical mechanics in order to denote generalized coordinates and momenta, as well as time (see section 2.2.5). State variables have a similarly long tradition in thermodynamics. Most probably every student of a secondary school has encountered the term *equations of state*. We would like now to take up just this question.

In classical thermodynamics, state variables encompass pressure, P, temperature, ϑ, and volume, Ω. The state equation there is the relation

$$f(P,\vartheta,\Omega)=0, \tag{2.67}$$

with the Clapeyron equation (1.11) as a special case.

Such an equation does not appear at all in contemporary thermodynamics. How can this be explained? The state equation defines the specifics of a given medium. The particular medium described by equation (1.11) is the perfect gas. When considering processes taking place in, say, composite materials or non-Newtonian fluids, it is not sufficient to use the variables appearing in equation (2.67) to describe them; thus, the state variables of a given medium cannot be identified with the variables bearing the same name as used in thermostatics.

We thus see that state variables form a subset of physical variables, and that the character and cardinality of this set depends upon the phenomenon studied and the purpose of modelling.

2.3.4 The number of describing functions

Two types of mathematical models were presented in section 1.5, classified according to the manner in which physical properties of a system were described: lumped and distributed models. Typical technical examples of such model types were also presented there. One can find in the literature the terms *system with one degree of freedom* to designate the model of Fig. 1.12 and the *system with infinite numbers of degrees of freedom* to designate the model of Fig. 1.14. We consider that this latter notion is incorrect and may be misleading. The reasons for such a view are as follows: position is described in discrete models by means of generalized coordinates, whose number corresponds to the number of degrees of freedom (see section 2.2.4). It is therefore reasonable that the model of Fig. 1.12 has one degree of freedom and that its mathematical model comprises one differential equation (1.25). Hence, it is natural to expect a model with an infinite number of degrees of freedom to be described by just such a number of differential equations. In reality, however, for the example considered the mathematical description consists of just one equation. What is the explanation of this contradiction?

To explain this let us think of what in fact has happened. Thus, to describe the position of a beam, we have to determine in some space (here one-dimensional space) the location of every element of this beam. The simplest way to do this is to choose as coordinates the translations of the beam elements. When, however, one is assuming the continuum hypothesis (and this is usually the case when modelling macroscopic phenomena), then material elements are not lumped, but distributed, in the sense that *they have continuous distribution over space*. That is why it is not possible to determine their position with a finite number of quantities, as is done for discrete models. This is why we think that instead of using the term the 'number of degrees of freedom' we should rather refer to the notion of the number of **describing functions**, which is more useful in modelling.

Let us now determine how this applies to the considered example of the longitudinal vibrations of a beam. In this case every element of the beam is alternately subject to tension and compression. Because the beam is, by assumption, thin, the location of each of its points in the equilibrium position may be described by just one coordinate, x. Now denote by u the deflection of a material element with respect to the equilibrium position, Fig. 1.14. This deflection depends upon the elements position and time, i.e.

$$u = u(x, t). \tag{2.68}$$

The function (2.68) is treated as a coordinate of the model, while the variable x was used to determine the location of a material element. Thus, this variable plays the same role as in classical mechanics for free particles. Now it can clearly be seen that because there are also infinitely many functions (2.68), the model could be considered as having infinitely many degrees of freedom. In the practice of modelling, though, only one function (2.68) is used to describe the motion of a beam, and that is why the notion of the degrees of freedom is not very useful at this stage of modelling.

The question of correspondence of the infinite number of degrees of freedom and the continuous model appears again at the stage of model analysis. If we want, for instance, to determine dynamic deformations of the beam, we first describe them with the function

$$u(x,t) = \sum_{j=1}^{\infty} \phi_j(x) q_j(t), \tag{2.69}$$

where the $\phi_j(x)$ are natural mode shapes and $q_j(t)$ are functions of time, called generalized coordinates. In such a case equations of motion of the construction represent an infinite system if ordinary differential equations in the form

$$m_j \ddot{q}_j + \omega_j^2 m_j q_j = F_j(t), \qquad j = 1, 2, \ldots, \infty \tag{2.70}$$

where $F_j(t) = \int \phi_j(x) f(x, t)\,\mathrm{d}x$ is the generalized force corresponding to the generalized coordinate q_j.

It is possible to achieve sufficient accuracy for practical purposes if a finite number of equations is accounted for (this number depending upon the accuracy required). This corresponds to replacement of a continuous model with a discrete one having a finite number of degrees of freedom.

As in section 2.2.4, it is also worth while here to mention abnormal situations. As in the ordinary case such situations emerge when we perform a far-reaching idealization, requiring, for instance, the motion of a supporting beam to be described as a system with one degree of freedom. In order to do this, we should assume that the beam is massless, although elastic, and that it is reduced to the mass concentrated at the beam end (see Fig. 2.11).

It can be demonstrated that the simplified model is described by the equation

$$\ddot{y} + \omega_0^2 y = 0, \tag{2.71}$$

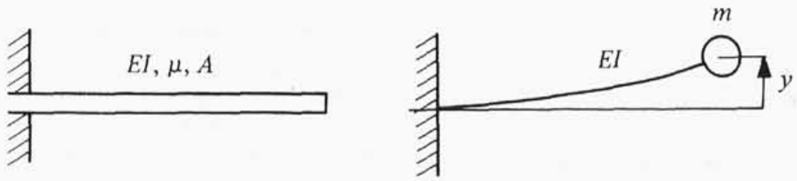

Fig. 2.11.

where $\omega_0 = (1.421/l)^2 \sqrt{(EIl/m)}$. Although this result is of practical value (for comparison—the exact value is $\omega_0 = (1.875/l)^2 \sqrt{(EIl/m)}$), the model itself is of little use, because its form precludes a wider perspective, since only one natural frequency can be deduced from it.

Finally, it should be emphasized that the foregoing considerations concern not only elastic bodies, but also all continuous media, such as the model of a plasma as a liquid.

2.3.5 Eulerian and Lagrangian description of the continuum motion

The present discussion is meant to provide only the introductory information necessary for a uniform formulation of the considerations of Chapter 3. Thus, the present discussion should not be understood as an introduction to the mechanics of continuous media, to be given in Volume 2.

Imagine, then, a continuous medium in motion. To concentrate attention let this be a fluid such as a cloud or water in a river, although it could also be a solid medium such as a deformable spacecraft or a hot-rolled steel band. This time, however, we shall assume that the medium considered, under the influence of forces acting upon it, may undergo deformations—that is, changes of locations of some parts of it relative to the other parts, these changes depending not only upon properties of these bodies, but also upon external factors (e.g. duration of the action of external forces, temperature, etc.). The examples of bodies quoted here cannot be described within the framework of the model of a rigid body. On the other hand, we would like to describe the motion of these bodies. How can we do this? Recall that in the mechanics of discrete systems it was sufficient for description to have certain magnitudes depending uniquely upon time—these could be, for instance, the Cartesian coordinates x, y, z of a mass particle (e.g. the centre of mass of a rigid body). In the mechanics of continuous systems the question is greatly complicated—besides the time coordinate one should introduce yet another magnitude, designed to reflect what occurs to individual particles of a body. Depending upon how this is done the motion of a continuum medium can be studied from two essentially different points of view. Thus, two methods of studying the motion are distinguished, both conceived by **Leonhard Euler** (1707–1783), but known under the names of the **Euler method** and the **Lagrange method**. The former consists in studying the velocity of an element in a predefined point of space, while the latter is based upon the study of motion of a selected element. Since distinction and application of these methods encounters certain difficulties, we shall use certain geometric illustrations to aid understanding.

Consider first the Lagrangian method, which, being more natural, results from the direct transformation of a description originating in classical mechanics. Suppose we are to determine the flow in a tube, and that we are interested in changes of velocity along the tube's axis. Assume further that at the initial instant t_0 the element considered occupies position (x_0, y_0) and had velocity v_0. After some time the element has moved and at time t_1 takes position (x_1, y_1) (compare Figs. 2.12a and b). If the motion is not steady, then the velocity will also change and become v_1. The latter value depends both on the time which has already elapsed and on the new location of the element. These two depend upon the earlier state of the element, and thus regressing we shall return to the initial instant and initial position (x_0, y_0). One can therefore say that the present state depends upon the history of the motion. That is why this method is sometimes called **historical**. It seems, however, that it is better to call the Lagrangian description the **wandering description**. The most popular name, however, is the **material description**.

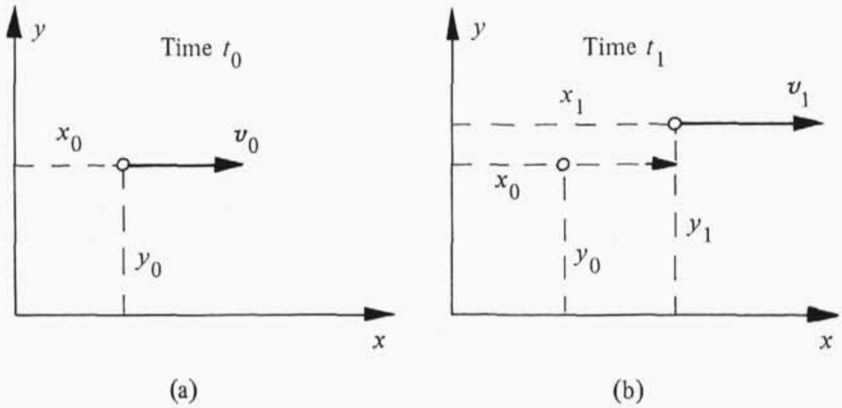

Fig. 2.12.

By application of the Lagrangian description method we obtain a description of the motion of the medium in the form of a function:

$$H = H(a, b, c, t) \tag{2.72}$$

where H is any (i.e. scalar, vector, or tensor) property of the medium (velocity in the example considered), while variables a, b, c and t are **Lagrangian variables** or coordinates. Letter t denotes time, and the triplet a, b, c denotes variables which pinpoint the element considered, similarly as x_0, y_0 in the planar case. They do not depend upon changes in the position of the element, nor upon time elapsing.

The method of Euler, on the other hand, is only indirectly applied in the study of element motion. It tells us only how the velocity v in a particular point (x, y) changes over time. Obviously, in various instances various elements are present at this point, but this method does not just take care of an element (as was the case with the Lagrangian description), it observes a point in space. If we could, in the example considered, take a photograph of particles passing through the point (x_0, y_0) at time t_0, and then at time t_1 (see Fig. 2.13a, b), we would obtain velocities v_0 and v_1 at the point considered. Knowing the exposure time, and observing the lengths of streaks on the film, being the images of the particle moving at time t_1 and t_2, one could calculate corresponding

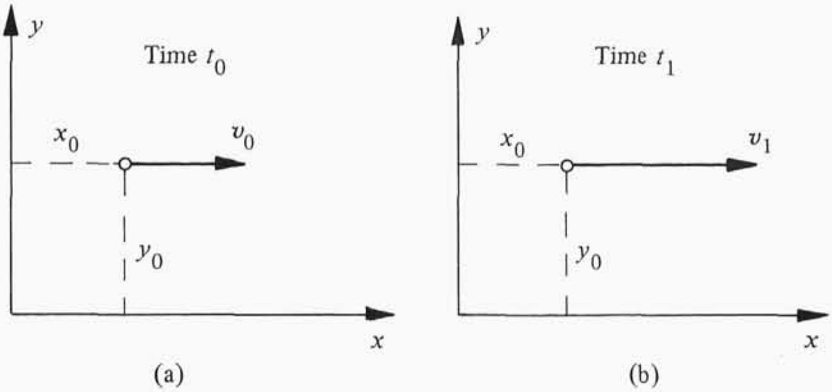

Fig. 2.13.

velocities. It should be noted that this type of photographing technique is used in experimental mechanics of liquids. That is why the method of Euler is often referred to as the **photographic** method. Other names, however are also in use, such as **local description** (in view of the choice of a given point and) **spatial description**.

Resulting from application of Euler's method the motion description takes the form of a function

$$H = H(x, y, z, t), \tag{2.73}$$

where H, as before, is any property of the medium motion, and variables x, y, z, t are called **Euler variables** or coordinates. Letter t denotes time, and the triplet x, y, z denotes the variables indicating a particular point of geometric space.

Transition from the system of Lagrangian variables to the system of Eulerian variables, and vice versa, is always possible provided that the correspondence between these two systems is mutually unique. Detailed information will be given in Volume 2. It is worth while to note yet that there is a popular view that Eulerian variables are applicable to problems of fluid mechanics, and Lagrangian variables to problems of solid mechanics. Such a view, though, is a simplified one.

To help memorize the differences between these descriptions let us compare them via another pair of illustrations (see Figs. 2.14 and 2.15). In the method of Lagrange we are tracking the motion of a definite particle M of a body, this particle passing through various points $P_1, P_2, P_3, \ldots$ of space; time t is here an independent variable. We are interested in the changes of physical magnitudes for a given particle M of a body.

In the method of Euler we are tracking the motion of various particles $M_1, M_2, M_3, \ldots,$ of a body, passing through a definite point P of space; independent variables are here the coordinates x, y, z of the point P, as well as time t. We are interested in the changes of physical magnitudes in a given point P of space.

From the mathematical standpoint the method of Lagrange differs from the method of Euler only by the fact that in the former case the independent variables are the parameters a, b, c identifying a particle of a continuum medium, and time t, while in the latter case the independent variables are the coordinates of a point in space, x, y, z, and time t. That

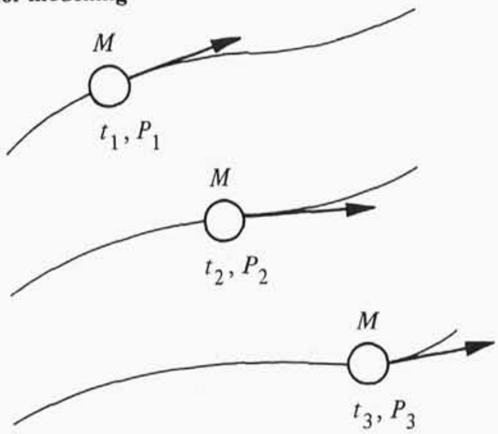

Fig. 2.14.

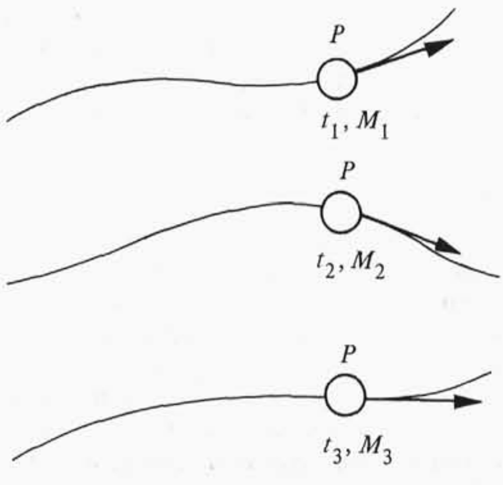

Fig. 2.15.

is why it is possible that the best way of referring to these coordinates are the expressions, respectively, of particle and field formulation.

In fluid mechanics and thermodynamics we often have to consider changes taking place in a certain volume of fluid. For purposes of description of such changes the notions of *fluid surface* and of *control surface* will be needed. These two notions can be easily introduced with the help of the notions of Eulerian and Lagrangian variables.

The fluid surface consists of any open or closed surface in the fluid velocity field, characterized by the fact that its location with respect to the Lagrangian coordinate system a, b, c does not change over time. This, therefore, means that the surface is constantly formed by the same elements of the fluid. According to such a definition the equation of the fluid surface has in Lagrangian coordinates the form of

$$F_{\mathrm{p}}(a,b,c)=0. \tag{2.74}$$

The region bounded by the closed fluid surface is called the fluid region. An example of such a region is provided by the space within a cylinder of a combustion engine during gas expansion (after the outlet valve has been opened).

By a **control surface** we mean any open or closed surface, as above, whose location does not undergo changes with respect to the Eulerian spatial coordinate system x, y, z. This definition leads to the following equation of the control surface (using Eulerian variables)

$$F_{\mathrm{k}}(x,y,z)=0. \tag{2.75}$$

The spatial region bounded by a closed control surface is called the *control region*. It is characteristic of such a region that it can contain, over time, various portions of matter. Consequently, in conditions of transient flow, the amount of matter in the region may change. An example of a control region is the space of a reservoir with inflow and outflow of gas.

2.3.6 The state space

The term of state, as a fundamental one for the description of a system, was introduced by **Alan. M. Turing** in 1936, and then applied by **Claude. E. Shannon** in his classical work on information theory. This notion is commonly used at present in system theory. Its use has come from a tendency to represent any physical system by a number of first-order differential (or difference) equations that interrelate an equal number of variables.

In section 2.2.5 we put forward the view that the *science of dynamics is the science of motion*. In classical mechanics the notion of *motion* is used in a narrow sense—to designate the change of object location over time. However, in integrated mechanics the concept of motion is ascribed a much wider meaning, since it refers to any change in the object state over time (see section 2.3.3). It is therefore, in fact, a traditional point of view of dialectics, and the well-known statement of such a general concept of motion is characteristic for cybernetics as well. Thus, motion will be understood by us to encompass such cases as change of temperature of a body, change of electric charge on the plates of a condenser, change of the gas pressure in a reservoir, etc. Even such a process as dissolution of a gas in a metal can be treated as a form of motion.

In this situation it becomes understandable that the notions of space, commented upon in section 2.2.5, are insufficient for the description of a dynamic system. The states of such a system can be described with complete accuracy by specifying values taken by the physical variables, characterizing the system's behaviour (see section 2.3.3).

Because of the importance of the notion of state space it would be best to give the definition of the state. The *state*, however, is treated in the theory of dynamic systems as a fundamental notion and that is why it cannot be defined in a more complete manner than the notion of 'set' in mathematics. The only thing one could do in the introductory chapter is to try to give the notion a more precise sense than the one commonly used, referring, however, to intuition, and amplifying by reference to the specific context.

Let us begin with etymology. The word *state* comes from the Latin *status*, meaning posture, location, relations. First, we will link the notion of state with that of motion, the latter being a more intuitive idea. The notion of 'system's motion' will mean here any change in the state of a system.

Then, the notion of state will be linked with the state variables, of which every one can be measured at any instant and expressed with a number. Note, by the way, that the quantities describing a system may be mutually interconnected through deterministic or stochastic dependences (see section 1.6).

It is perhaps best to illustrate the notion of state by means of a simple example, and not a purely mathematical one. Assume, then that we are dealing with a real reservoir having an inflow and an outflow of a liquid and that we are interested only in the current volume of the liquid contained in the reservoir. We can then say that this volume tells us everything of the state of the reservoir. We can say this because when we know how much liquid is there at a given instant in the reservoir we do not have to remember how and when this liquid got into the container and how much liquid has flowed out of the container. The knowledge of the current state is necessary if we are interested in the contents of the reservoir, i.e. if we want, for instance, to determine time, after which the reservoir will become empty in conditions of free outflow of the liquid.

This example makes evident a number of significant issues. The most important one is the statement that the choice of a state variable is in fact arbitrary. The contents of the reservoir (volume of liquid) specifies everything about the state, if we are interested only in the contents of the reservoir, and not, for example, in the process of mixing (when there are inflowing liquids of various concentrations or temperatures). The second important observation is that the container may get empty or there may be an overflow, implying that the state is subject to certain physical limitations. Third we would know even more about the system if we had the relations of the state variable with the other important variables. It can therefore be supposed that the key role in the description of the system will be played by the relation governing the behaviour of the state variable.

Let us emphasize yet that analogous reasoning can be performed, e.g. for the condenser charged with electric current, or for a radiator with the inflow and outflow of heat. Thus, the common feature of a physical system is the fact of storing (preservation, remembering) of a certain physical quantity, like mass, charge, heat, etc.

There are various ways of describing the state of a system. However, for purposes of modelling it is more convenient to represent the state of a system using the notion of state space, but doing so in a more complete way than in classical mechanics by accounting for a variety of physical variables, as presented above. Thus, while preserving the essence of motion of space from section 2.2.5, we shall say that the state space is a space in which each of the system can be represented by a point. If the system moves, the values of its variables change over time. The state point therefore moves in the state space, following a certain trajectory. The number of dimensions of the state space is equal the number of physical variables, defining the system's state. It is only within this context that one can speak of the degrees of freedom of a continuous system.

In order to get better acquainted with this important notion of integrated mechanics, let us consider two examples.

First we shall take a simple system such as a harmonic oscillator (see section 1.7) assuming only the case of $F(t) = 0$

$$\ddot{q} + 2h\dot{q} + \omega_0^2 q = 0, \tag{2.76}$$

where $2h = b/m$, and $\omega_0^2 = k/m$.

Introduce the notation

$$x_1 = q \qquad \text{and} \qquad x_2 = \dot{q}. \tag{2.77}$$

Then

$$\ddot{q} = \frac{\mathrm{d}\dot{q}}{\mathrm{d}t} = \frac{\mathrm{d}x_2}{\mathrm{d}t} = \dot{x}_2, \tag{2.78}$$

so that equation (2.76) can be represented in the form

$$\dot{x}_2 = -\omega_0^2 x_1 - 2hx_2. \tag{2.79}$$

Having defined the variables

$$\mathsf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \qquad \mathsf{A} = \begin{bmatrix} 0 & 1 \\ -\omega_0^2 & -2h \end{bmatrix}, \tag{2.80}$$

we can transform equation (2.76) into

$$\dot{\mathsf{x}} = \mathsf{A}\mathsf{x}. \tag{2.81}$$

This is a typical description of a system constructed with the help of the state variables, where x is the *state vector* and A is the *state matrix*.

Most probably examples in which state variables are identical with the coordinate and velocity have led to misunderstandings of the notions of phase and state variables, and even of generalized coordinates and velocities themselves. In order to demonstrate that the identity is true only in very particular cases, consider a system described by a single nth-order differential equation

$$y^{(n)}(t) = f(y^{(n-1)}, y^{(n-2)}, \ldots, y^{(1)}, y, t) \tag{2.82}$$

where f is some function, and $y^{(i)} = \mathrm{d}^i y/\mathrm{d}t^i$.

It is easy to show that this system could also be described by a set of n first-order differential equations. Define a set of variables $\{x_i(t)\}$, $i = 1, 2, \ldots, n$, so that

$$x_i(t) = y^{(i)}(t). \tag{2.83}$$

Then we have

$$\begin{aligned} \dot{x}_1(t) &= x_2(t), \\ \dot{x}_2(t) &= x_3(t), \\ &\vdots \\ \dot{x}_{n-1}(t) &= x_n(t), \\ \dot{x}_n(t) &= f(x_n, x_{n-1}, \ldots, x_1, t). \end{aligned} \tag{2.84}$$

This is just a specific case of the general system of first-order differential equations

$$\begin{aligned}\dot{x}_1(t) &= f_1(x_1, x_2, \ldots, x_n, t),\\ \dot{x}_2(t) &= f_2(x_1, x_2, \ldots, x_n, t),\\ &\vdots\\ \dot{x}_n(t) &= f_n(x_1, x_2, \ldots, x_1, t),\end{aligned} \tag{2.85}$$

which can be written more compactly as

$$\dot{\mathsf{x}}(t) = \mathsf{f}(\mathsf{x}(t), t), \tag{2.86}$$

where x is an n-dimensional 'state-vector' and f is an n-dimensional vector-valued function. This is the state-variable form for an nth order continuous-time system with no outside inputs. Note that if, at time t_0, values of the state variables $x_1(t_0), x_2(t_0), \ldots, x_n(t_0)$ are known, then the values of $x_1(t), x_2(t), \ldots, x_n(t)$ at any other time t can be determined.

We can easily generalize (2.86) to take the form

$$\dot{\mathsf{x}}(t) = \mathsf{f}(\mathsf{x}(t), \mathsf{u}(t), t), \tag{2.87}$$

where u is a control vector (or input vector—see section 1.7). A typical example of equation (2.87) is provided by the aircraft landing (see Merriam (1964))

$$\frac{\mathrm{d}^4 h}{\mathrm{d}t^4} + 2\gamma\omega_0 \frac{\mathrm{d}^3 h}{\mathrm{d}t^3} + \omega_0^2 \frac{\mathrm{d}^2 h}{\mathrm{d}t^2} = K\delta_H(t) \tag{2.88}$$

where h is the flight altitude, γ is the damping coefficient, ω_0 is the natural frequency, K is a coefficient depending on the flight velocity and amplification in control system, and δ_H is the angle of rotation of the elevator.

Let us introduce the notation

$$x_1 = h, \quad x_2 = \dot{x}_1 = \frac{\mathrm{d}h}{\mathrm{d}t}, \quad x_3 = \dot{x}_2 = \frac{\mathrm{d}^2 h}{\mathrm{d}t^2}, \quad x_4 = \dot{x}_3 = \frac{\mathrm{d}^3 h}{\mathrm{d}t^3}, \quad u = \delta_H. \tag{2.89}$$

Then, equation (2.86) takes the form

$$\dot{\mathsf{x}} = \mathsf{A}\mathsf{x} + \mathsf{b}u, \tag{2.90}$$

where

$$\mathsf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \quad \mathsf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\omega_0^2 & -2\gamma\omega_0 \end{bmatrix}, \quad \mathsf{b} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ K \end{bmatrix} \tag{2.91}$$

In this case the state variables do not correspond to any specific type of variables known from classical mechanics.