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Modelling by means of balance laws

Numerous separate university courses are being given on the several scientific domains whose methods shall be applied in this book (e.g. mechanical engineering, structure dynamics, classical elasticity, classical hydrodynamics, fluid mechanics, flight mechanics). In those separate courses it has not always been clear to students that many of the ideas and principles are common to all those topics. Modelling of the phenomena in which objects of different nature—such as mass particles, rigid and elastic bodies, and then liquids and gases—appear, requires a unified approach and possibly a similar methodology. This in turn requires the establishment of a common basis for creation of a causal model. The main aim of this chapter is to present this common basis for modelling of any kind of mechanical phenomena.

3.1 CONSERVATION LAWS VERSUS BALANCE LAWS

In the physical world there are a number of **conservation laws**, some of which are exact and some approximate. There are conservation laws related to energy, linear momentum, angular momentum, charge, number of baryons (protons, neutrons, and heavier elementary particles), strangeness, and various other quantities. In this section we discuss the relation between conservation and balance laws. The entire discussion at present will apply to the nonrelativistic regime, meaning speeds much lower than that of light, as well as the independence of mass and energy.

If all the forces in a problem are known, and if we are clever enough and have computers of adequate speed and capacity to solve the trajectories of all the particles, then conservation laws give us no additional information. But they are very powerful tools which a physicist uses every day. Why are conservation laws powerful tools?

(1) Conservation laws are independent of the details of the trajectory, and often of the details of a particular force. The laws are therefore a way of stating very general and significant consequences of the equations of motion. Sometimes conservation laws can identify impossibilities and hence prevent wasting time analysing an alleged perpetual motion device if it is merely a closed system of mechanical and electrical components.

(2) Conservation laws have been used even when the forces are unknown; this applies particularly in the physics of elementary particles.
(3) Conservation laws have an intimate connection with invariance. In the exploration of new and not yet understood phenomena conservation laws are often the only facts known. They may suggest appropriate invariance concepts: for example, the conservation of linear momentum could be interpreted as a direct consequence of the principle of Galilean transform invariance.
(4) Even when the force involved is known exactly, a conservation law may be a convenient aid in solving for the motion of a particle. Many physicists have a regular routine for solving unknown problems. First they use the relevant conservation laws one by one; only after this, if there is anything left to the problem, will they get down to real work with differential equations, variational methods, computers, intuition, and the other tools at their disposal.

The application of conservation laws is possible if all forces, or more generally all interactions appearing in the given phenomenon, are *conservative*, and this is the case at the level of elementary particles, i.e., as we shall call it, at the micro-level. However, considering systems appearing in engineering, i.e. on the macro-level, several concepts have been introduced as suitable aids for description of thermomechanical phenomena. For example frictional force, inelastic collision, plastic impact belong to these concepts, and these assume existence of either dissipative forces or energy dissipation in the form of heat. As dissipative interactions between systems are introduced, conservation laws cannot be used in every case. This is the reason why in engineering the **balance laws** or their differential forms, i.e. rate-of-change laws, are used rather than conservation laws.

In order to formulate the balance laws let us first answer the question of what kind of physical quantities may be the subjects of balance. Let us note that although such quantities as heat or mass may be subject to balance, temperature or density cannot. Thus, physical quantities may be classified into two basic groups, those known as extensive quantities because they possess an extensive property, and those known as intensive quantities because they possess an intensive property.

A geometric or physical quantity whose containment within a certain region composed of a sum of subregions is equal to the sum of containment contained in each constituent will be called an **extensive quantity**, or *EQ* for short. Some of *EQ*s, exemplified by mass, charge, momentum of mass, kinetic or internal energies and, entropy, can only be stored in a system filled with substance, and therefore they are called the **substantial quantities**. On the other hand, the momentum and energy of electromagnetic and gravitational fields may exist in a vacuum as well, so they belong to the **nonsubstantial quantities**. An *EQ* is denoted by a symbol Φ; in a given system the same symbol Φ will be used for denoting the amount of *EQ* stored in the system and then it would be called the **storage** of a given *EQ*.

Each physical quantity referring to the system points will be called an **intensive quantity**, or *IQ* for short. A value of *IQ* does not depend on the volume or mass under consideration. There are two types of *IQ*s; first, pressure and temperature, for example, are not explicitly dependent on the amount of mass involved, but have magnitudes that are representative of the overall state of the system. Second, there are the *IQ*s that are the specific values of *EQ*s; for example specific internal energy (internal energy per unit mass), specific entropy, density. This second type of *IQ* is generally denoted by a lower case letter φ.

For a material substance that satisfies the continuum postulate (see section 1.5) φ is defined by

$$\varphi = \lim_{\Delta m \to 0} \frac{\Delta \Phi}{\Delta m} = \frac{\mathrm{d}\Phi}{\mathrm{d}m} \tag{3.1}$$

and, the value Φ of the general *EQ* for a system is given by

$$\Phi = \int_{\text{system}} \varphi \, \mathrm{d}m. \tag{3.2}$$

Since density is defined by $\rho = \mathrm{d}m/\mathrm{d}\Omega$, equation (3.2) transforms to

$$\Phi = \int_\Omega \varphi\rho \, \mathrm{d}\Omega, \tag{3.3}$$

where integration is performed over the volume Ω occupied by the system.

The causes of a change in the amount of *EQ* stored in a system may be divided into two categories.

The first encompasses all phenomena occurring within a system exclusively, i.e. either the **creation** or the **annihilation** of the *EQ* under consideration at the expense or to the advantage of another *EQ* coexisting within the same system. Both processes have a common name—**production** of *EQ*. The rate of production, i.e. the production per unit time will be denoted by *P*.

Secondly the cause of change in *EQ* encompasses all interaction phenomena between the system and its surroundings, i.e. **inflow** and **outflow** of *EQ* through the system boundaries. Such changes in *EQ* will be called **transfer** of *EQ*. The **rate of transfer**, i.e. transfer per unit time, will be denoted by *T*.

It is easy to imagine the transfer between a system and its surroundings in cases of scalar *EQ*s such as mass or energy. It is simply the inflow/outflow of mass or energy between a system and its surroundings. When, however, the *EQ* under consideration is a vector quantity, such as momentum, then the transfer of this quantity ought to be understood as an interaction carried out by means of forces, for example surface forces.

The above considerations enable us to formulate the fundamental balancing axiom:

> *Change of the EQ stored within a system equals the sum of the EQ produced within the system and the EQ transferred between the system and its surroundings through the boundary* (Fig. 3.1).

Using the notation introduced before, the balancing axiom referring to the time interval $\mathrm{d}t$ and performed in an immobile reference system can be expressed as

$$\mathrm{d}\Phi = P \, \mathrm{d}t + T \, \mathrm{d}t, \tag{3.4}$$

where *P* is the **rate of production** of *EQ* and *T* is the **rate of transfer** of *EQ*.

Suppose now that the referential system moves with velocity *w*. In a mobile system, the motion of the boundary makes the storage of surroundings captured by the system, or vice versa. For that reason the transfer through the boundary of a reference system, T_w, is

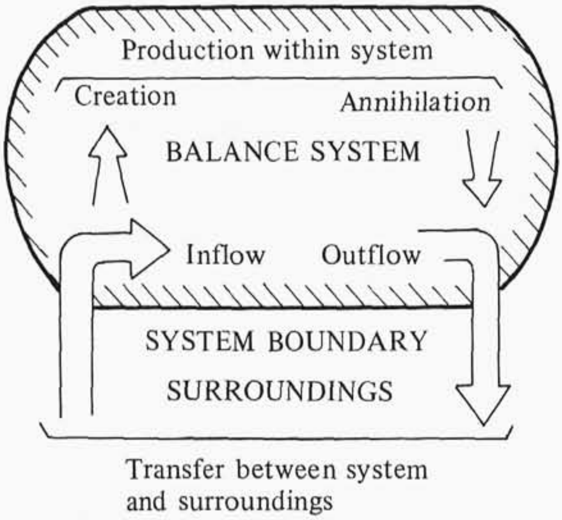

Fig. 3.1. Illustration of the fundamental balancing axiom.

not the same as transfer from/to the immobile system, T at the instant when both systems are coalescent.

Production and **storage** are not influenced by the motion of the system, so that they always keep their nonreferential character. In a mobile system, however, the change of storage during time $\mathrm{d}t$ is the relative increment $\mathrm{d}\Phi_\mathrm{w}$ due to displacement of boundaries $w\,\mathrm{d}t$. Thus the basic balance in mobile reference system is

$$\mathrm{d}\Phi_\mathrm{w} = P\,\mathrm{d}t + T_\mathrm{w}\,\mathrm{d}t. \tag{3.5}$$

Equation (3.5) is a most general form of a balance law of EQ. Note that if $w = 0$ then (3.5) reduces to (3.4).

Dividing (3.4) or (3.5) by $\mathrm{d}t$ we obtain the *differential forms* of balance laws, i.e.

$$\frac{\mathrm{d}\Phi}{\mathrm{d}t} = P + T \tag{3.6}$$

and

$$\frac{\mathrm{d}\Phi_\mathrm{w}}{\mathrm{d}t} = P + T_\mathrm{w}. \tag{3.7}$$

An example of the balance (3.4) could be the annual change of the number of vehicles in a country. This change results from industrial production (creation) and scrapping (annihilation) in the country and from imports (inflow) and exports (outflow), i.e. foreign trade. The balance (3.5) may be illustrated by the change of vehicles in wartime, when a part of the country boundaries are the moving fronts (boundaries of the reference region). Then the transfer volume should be supplemented with the effect of capturing the equipment won over from the enemy when advancing and/or of abandoning to the enemy when retreating.

For purposes of mathematical modelling of thermomechanical phenomena, the balances of five basic *EQ*s, i.e. mass, m, momentum $\mathbf{p}$, angular momentum $\mathbf{H}$, energy E, and entropy S are extremely important. Their general forms in an immobile referential system are as follows:

for mass

$$\frac{\mathrm{d}m}{\mathrm{d}t} = P(m) + T(m), \tag{3.8}$$

for linear momentum

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = P(\mathbf{p}) + T(\mathbf{p}), \tag{3.9}$$

for angular momentum

$$\frac{\mathrm{d}\mathbf{H}}{\mathrm{d}t} = P(\mathbf{H}) + T(\mathbf{H}), \tag{3.10}$$

for energy

$$\frac{\mathrm{d}E}{\mathrm{d}t} = P(E) + T(E), \tag{3.11}$$

and for entropy

$$\frac{\mathrm{d}S}{\mathrm{d}t} = P(S) + T(S). \tag{3.12}$$

Two comments concerning the above five balances seem to be worth making. First, the present form of these balances is too general for them to be used directly in practice. Before their application they have to be adapted to specific groups of modelling problems. The first stage of such an adaptation depends on the kind of physical models under consideration, i.e. whether they are discrete or continuous. The second stage of adaptation of balances (3.8)–(3.12) for continuum media depends on the kind of description of motion, i.e. whether it is spatial or material. Subsequent stages of implementing of balances depend on specific forms of interactions between the system and its surroundings, i.e. whether they are mechanical, taking place by means of force only, or also thermal, taking place by means of heat efflux. The next step in applying balance laws depends on the specific medium under consideration, i.e. whether it is a solid body, liquid or gas.

To illustrate the first stage of developing the application of a balance law, consider the balance of energy (3.11). Let us take into account the simplest physical model—a single mass particle moving in a gravitational field. The total energy of the particle consists of two components: kinetic energy $\frac{1}{2}m\mathbf{v}^2$, where $\mathbf{v}$ is the velocity of mass, and potential energy mgz, where z is the height above the Earth's surface. The total energy of a mass particle changes due to the action of forces: specifically, the time rate change of the total energy equals the power, N, of acting forces, i.e.

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{mv^2}{2}+mgz\right)=N. \tag{3.13}$$

Note that the general symbol $T(E)$ in (3.11) has been substituted in (3.13) by the power N.

Let us now take into account a continuous medium, enclosed within a control surface A. Assume that this continuum may interact mechanically and thermally with its surroundings. The balance may be expressed as follows:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega}\rho\left(u+\frac{v^2}{2}+gz\right)\mathrm{d}\Omega=N+\Phi_Q, \tag{3.14}$$

where u is a *specific internal energy*, and Φ_Q is a *heat efflux* through control surface A.

After expressing both aggregate quantities, i.e. power N and heat efflux Φ_Q, by means of given mechanical and thermal stimuli, equations (3.13) or (3.14) respectively may be used as valuable components of a mathematical model of a mass particle or of a fluid.

More detailed forms of balances will be given in section 3.3 for discrete systems and may be found in Hunter (1983) for continuous ones.

The second comment concerns the number of balances. One may ask why just five balances have been identified despite the well-known fact that each EQ can be balanced. To answer this question let us explain first why the number of balances is five, not four or six. These five balances form a greater set of **independent balances** for thermomechanical phenomena. It means that if we need to use, for any reason (say for convenience), a balance of any other EQ, we may use, for example, a balance of enthalpy, not in addition to all balances (3.8)–(3.12) but instead of either (3.11) or (3.12). The existence of five independent balances does not yet mean that all five have to be applied in the mathematical model of each thermomechanical phenomenon. It may be that only a certain subset of these five balances is used. For example, modelling of the motion of a mass particle requires using only one balance, namely that of linear momentum. Generally, the more complicated the physical model, the more balances have to be applied to form the mathematical model. The second problem appearing in the question may be reformulated as follows: why are only such quantities as mass, momentum, angular momentum, energy, and entropy the subject of balance laws? The answer is the following: apart from their independence, these quantities are traditionally related with the most important physical laws: the first four with conservation laws, and the last one with the famous *Clausius–Duhem inequality* about non-negative production of global entropy.

Finally, it should be stressed that when considering another class of phenomena, for instance, electromagnetic phenomena, one has to take into account other balances. Here balance of charge and magnetic flux provide the basis for constructing further equations.

3.2 TWO INTRODUCTORY EXAMPLES

Consider two examples, which are surely known to every student of mechanics, although they may be studied within the framework of different courses. They will serve to indicate certain similarities appearing in different models.

Example 3.1. A body of mass m is suspended vertically by a linear spring of stiffness k in the presence of a uniform gravitational field, the direction of the gravitational force being shown in Fig. 3.2a. Let the mass–spring system be driven by the sinusoidal force $f(t) = f_0 \sin \omega t$, where f_0 is the amplitude of the force and ω is its frequency. The problem is to determine the mathematical model of the body motion, assuming that the vertical displacement y of the body is measured from its position when the spring is unstressed.

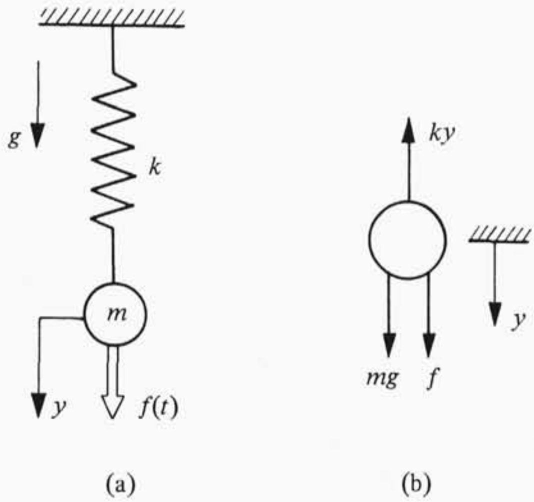

Fig. 3.2.

The equation of motion can be written with the aid of the free-body diagram shown in Fig. 3.2b. Using Newton's second law of motion, which may be treated as the result of a linear momentum balance adaptation to the case of a system of particles, we can write

$$m\ddot{y} = mg - ky + f_0 \sin \omega t, \tag{3.15}$$

or

$$m\ddot{y} + ky = mg + f_0 \sin \omega t. \tag{3.16}$$

Adding to (3.16) the initial conditions in their general form

$$y(0) = y_0, \qquad \dot{y}(0) = \dot{y}_0, \tag{3.17}$$

we obtain the mathematical model of forced body vibrations.

Example 3.2. Suppose that a liquid of density ρ flows along a curved pipe lying in a horizontal plane (see Fig. 3.3) and the flow itself is lossless. Assuming that the pipe cross-sectional area A is constant and the volume efflux Q is given, the hydrodynamical reaction R of the walls of the pipe on the liquid has to be determined.

The problem will be solved with the use of two balance laws, although in their integral forms. Denote by $\mathbf{v}_1$, $\mathbf{v}_2$ the stream velocities at sections 1 and 2, respectively. Let $v_1 = |\mathbf{v}_1|$, $v_2 = |\mathbf{v}_2|$ denote the associated speeds and let ρ be the density of the liquid. The

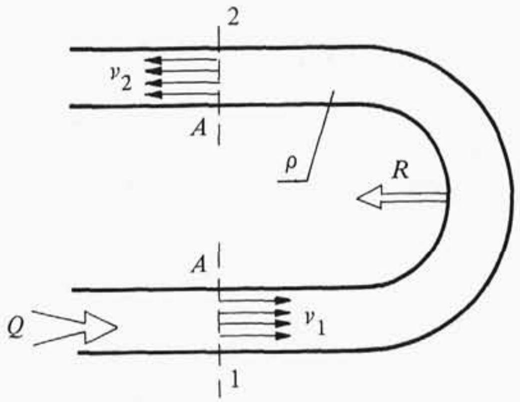

Fig. 3.3.

balance of mass: $m_1 = m_2 = \rho Q\,\Delta t$, i.e. mass of liquid inflowing m_1 is equal to mass m_2 of liquid outflowing, implies

$$Q = Av_1 = Av_2, \qquad \text{i.e.} \quad v_1 = v_2 = \frac{Q}{A}. \tag{3.18}$$

Consider a volume of the liquid which flows along a pipe in time interval (t_1, t_2). Its mass is constant and equal to $\rho Q(t_2 - t_1)$. Thus, its linear momentum at section 1 is $p_1 = \rho Q v_1(t_2 - t_1)$ and at section 2 is $p_2 = \rho Q v_2(t_2 - t_1)$.

The linear momentum balance equation yields

$$p_2 - p_1 = \int_{t_1}^{t_2} R\,\mathrm{d}t. \tag{3.19}$$

Since the flow considered is steady, the reaction R is constant. Then, taking into account that $\mathbf{v}_2 = -\mathbf{v}_1$ and considering rate of change of the linear momentum of the liquid, we can write

$$\rho Q v_2 + \rho Q v_1 = R. \tag{3.20}$$

Finally, taking into account (3.18), we obtain $R = 2\rho Q^2/A$.

We have considered two examples traditionally located in various branches of mechanics, such as structure dynamics and fluid mechanics. Do we perceive what connects them? Do we see common, or at least similar elements among the two models considered? Note that in order to formulate the models we had to apply various forms of the law of the rate of linear momentum change. Let us also turn our attention to the fact that in model (3.16) there appear quantities, i.e. coefficients, characterizing the object considered, such as mass m, spring stiffness k. Besides that, particular terms in equations (3.16) depend upon kinematic characteristics such as acceleration $\ddot{y}$, or position y. Similarly, the hydrodynamic reaction R depends upon the quantities characterizing properties of the object considered, that is liquid density ρ and the pipe cross-section surface A, as well as quantities characterizing flow of the liquid, i.e. volume efflux Q.

Are these perceivable similarities of models incidental, or are they, perhaps, a rule? We shall try to answer this question in the subsequent section.

3.3 METHODOLOGY OF MODELLING BY MEANS OF BALANCE LAWS

3.3.1 The tetrahedron—a mnemonic aid in the modelling process

To every body the natural world seems immense and complex, the stage for a startling diversity of events and phenomena. These impressions are supported by estimates of the general order of magnitude of the values of some fundamental quantities such as the characteristic length of the universe, 10^{26}m, at one extreme, and of a nucleus, 10^{-15}m, at the other. These impressions are also supported by a great number of both animate and inanimate matter forms. More than 10^6 species have been described and named on our planet. About 100 different chemical elements form perhaps 10^6 or more identified and differentiated chemical compounds, and to this number may be added a vast number of liquid and solid solutions and alloys of various compositions having distinctive physical properties. Adding to these number innumerable phenomena which involve all man-made machines, mechanisms and tools take part, the impression of the complexity of the real world is fully justified.

However, thanks to a development of science and technology we have gained a remarkable understanding of some central and important aspects of the world. Three powerful theories may surely be mentioned here: classical and quantum mechanics, and classical electrodynamics. The theories just named, together with the theory of relativity and statistical mechanics, are perhaps the greatest intellectual achievements of mankind. It is remarkable that all the great theories mentioned above hinge on only a few fundamental laws. It is permanently the aim of a scientist to explain as much as possible with the simplest tools possible, particularly using the minimum set of physical laws and assumptions. Similarly we shall look for such a categorization of the great number of laws, relations and notions used in mechanics, which will lead to the formation of a suitable tool to aid in the modelling process.

The tetrahedron from Fig. 3.4 is a symbolic representation of a division of problems of mechanics into five groups of elements. These groups of elements are: **BLM**—basic laws of mechanics, **B**—body, **F**—forces, **M**—motion, and **SLM**—specific laws of mechanics. When modelling a complex thermodynamical phenomenon we have to draw from each group distinguished in Fig. 3.4, and the model itself has to contain elements from all five groups.

Before we pass over to more precise presentation of meaning of the breakdown introduced and the contents of the particular groups it is necessary to indicate that the names of groups are certain abbreviations (codewords) which should, as a rule, be understood more broadly. Thus, for instance, the codeword *force* should be understood as representing the description of mechanical interactions, that is—forces and torques and interrelations between these interactions—when we restrict ourselves to modelling of just purely mechanical phenomena. Then, if the scope of modelling is broadened, e.g. to include thermomechanical questions, this group would contain also thermal actions. We