### 3.3.4 Force

Existence of stimuli is the **causal agent** of each physical phenomenon. Considering purely mechanical phenomena the role of stimuli is played by forces, whereas in thermo-mechanical processes both forces and heat streams can be regarded as stimuli and this must be reflected within each mathematical model of a thermomechanical process.

The effectiveness of modelling relies upon the concise and accurate expression of the effects of the action of stimuli, and therefore different measures, characteristic of forces and heat streams are to be considered in this group.

Several notions, relations, rules and theorems listed below may be considered as examples of elements belonging to the group known as **force**.

**F-1**. *Resultant of several concurrent forces.* Consider a particle $A$ acted upon by several concurrent $\mathbf{F}_1, \mathbf{F}_2, \ldots, \mathbf{F}_n$, i.e. by forces which are all directed through the same point $A$ (Fig. 3.10). The force

$$\mathbf{R} = \sum_{i=1}^{n} \mathbf{F}_i \tag{3.53}$$

acting through the point of concurrency is called the *resultant force* or just the *resultant*.

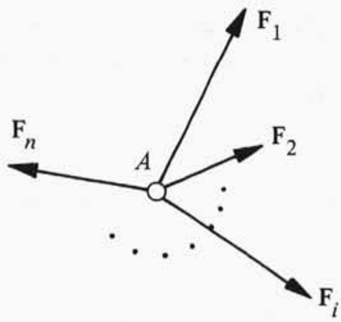


Fig. 3.10

**F-2**. *Moment of force about a point.* Let $\mathbf{F}$ be a force acting at point $A$ whose position vector with respect to a chosen origin $O$ is given by $\overrightarrow{OA} = \mathbf{r}$ (see Fig. 3.11). Then the *moment* $\mathbf{M}$ of the force $\mathbf{F}$ about $O$ is defined to be

$$\mathbf{M} = \mathbf{r} \times \mathbf{F} \tag{3.54}$$

**F-3**. *The work of force.* Suppose a force $\mathbf{F}$ acts on particle constrained to move along a curve $c$, joining points $A$ and $B$ (Fig. 3.12). The **work** $W$ done by the force $\mathbf{F}$ as it moves along the path from $A$ to $B$ is given by the line integral

$$W = \int_A^B \mathbf{F} \cdot \mathrm{d}\mathbf{r}. \tag{3.55}$$

**F-4**. *The impulse of force.* The **impulse** of the force $\mathbf{F}$ is the time integral of the force, i.e.

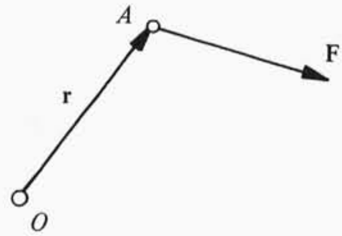


Fig. 3.11.

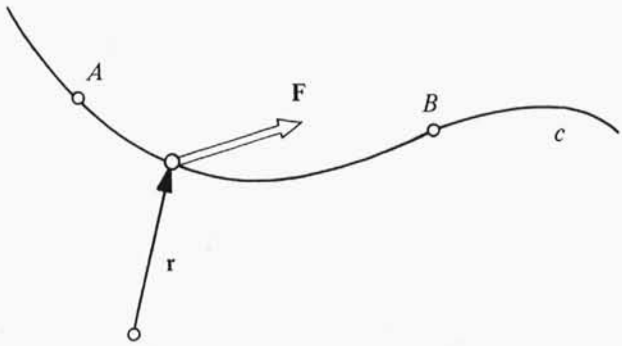


Fig. 3.12.

$$\mathbf{Imp} = \int_{t_1}^{t_2} \mathbf{F}\,\mathrm{d}t. \tag{3.56}$$

There are various equivalent representations of a system of forces and couples, listed in the following paragraphs.

**F-5**. *The polygon rule*. The resultant of several concurrent forces may be easily obtained graphically by use of the so-called *polygon rule* (compare Figs. 3.13a and 3.13b). The resultant vector **R** is drawn from the origin of the first vector **A** to the terminus of the last vector **D**, closing the polygon.

**F-6**. *Resolution of a given force into a force through a point and a couple*. Any force **F** acting on a rigid body may be moved to an arbitrary point $O$ provided that a couple is added, of moment, $\mathbf{M}_O = \mathbf{r} \times \mathbf{F}$, equal to the moment of **F** about $O$ (see Fig. 3.14).

**F-7**. *Reduction a system of a forces to one force and one couple*. Any system of forces, however complex, may be reduced to an equivalent force–couple system acting at a given point $O$ (Fig. 3.15). The equivalent force–couple system is defined by the equations

$$\mathbf{R} = \sum \mathbf{F}, \qquad \mathbf{M}_O^{\mathbf{R}} = \sum \mathbf{M}_i = \sum (\mathbf{r}_i \times \mathbf{F}_i), \tag{3.57}$$

which express that the force **R** is obtained by adding all the forces of the system, while the moment $\mathbf{M}_O^{\mathbf{R}}$ of the couple, called the *moment resultant* of the system, is obtained by adding the moment about $O$ of all the forces of the system.

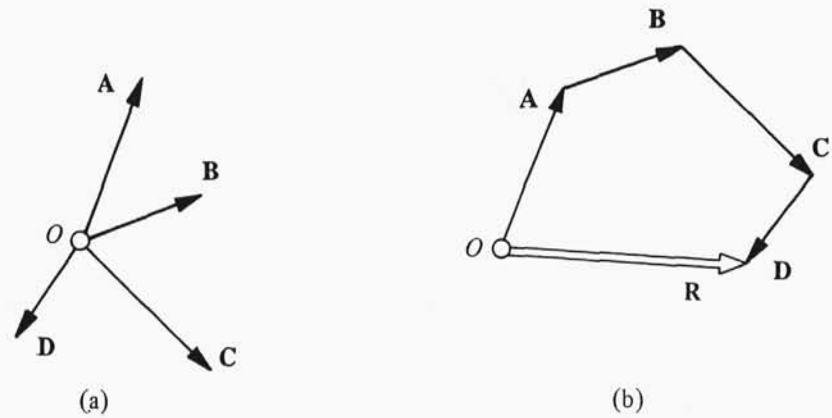


Fig. 3.13.

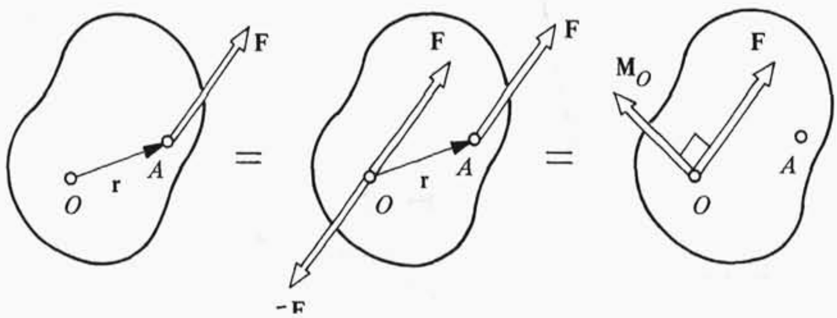


Fig. 3.14.

**F-8.** *Varignon's theorem.* If several forces $\mathbf{F}_1, \mathbf{F}_2, \ldots, \mathbf{F}_n$ are applied through a point $A$ (Fig. 3.16), and if we denote by $\mathbf{r}$ the position vector of $A$, then the moment about a given point $O$ of the resultant of several concurrent forces is equal to the sum of the moments of the various forces about the same point $O$, i.e.

$$\mathbf{r} \times (\mathbf{F}_1 + \mathbf{F}_2 + \ldots + \mathbf{F}_n) = \mathbf{r} \times \mathbf{F}_1 + \mathbf{r} \times \mathbf{F}_2 + \ldots + \mathbf{r} \times \mathbf{F}_n \tag{3.58}$$

This property was originally established by the French mathematician **Varignon** (1654–1722), long before the introduction of vector algebra, and is known as *Varignon's theorem.*

### 3.3.5 Motion

The famous Heraeklite's statement, *panta rhei*, is not accidental. Motion surely is one of the most frequently observed physical phenomena. Motion, or more generally a state change of a physical system, is often considered as a result of application of stimuli to the

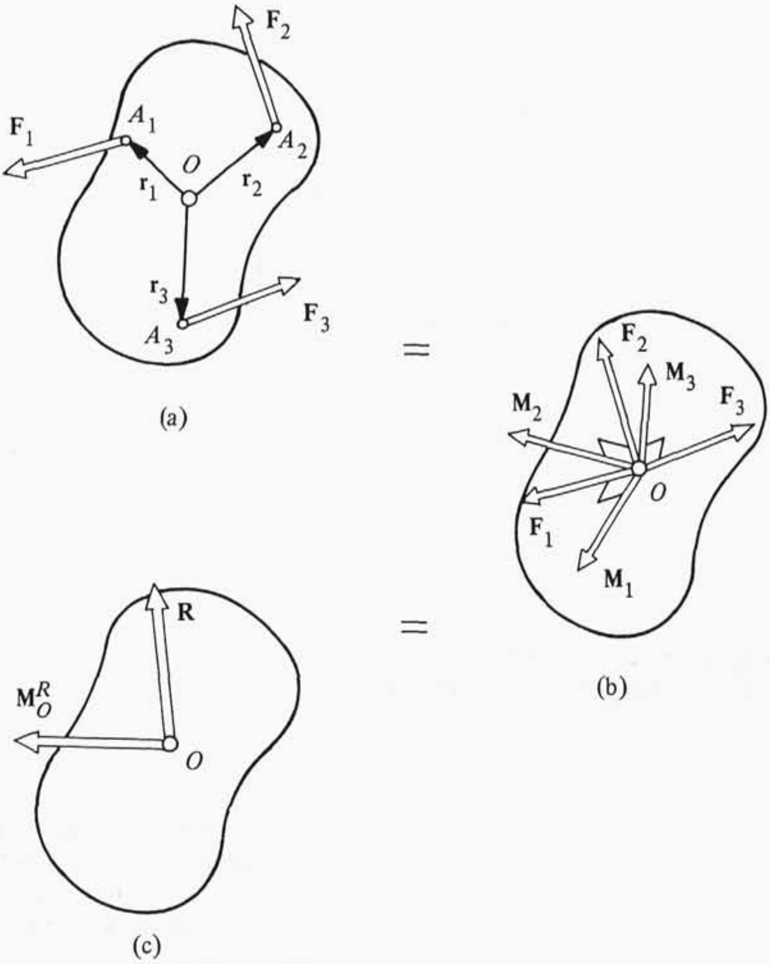


Fig. 3.15.

system. Numerous kinematical problems are provided in, for example, the process of calculating flight trajectories for aircraft, missiles, and spacecraft, and also in the design of cams, gears, and linkages in order to control or to produce certain desired motions. Thus, a concise mathematical description of motion is a fundamental requirement of modelling. Kinematics is just concerned with this kind of problem, and this part of mechanics may be identified with the group known as **motion**. (Since kinematics deals with position in space as a function of time it is often referred to as the *geometry of motion*).

The examples of kinematic relations listed below belong to this group.

**M-1**. *Definition of velocity and acceleration.* The velocity of the particle $A$ is defined as

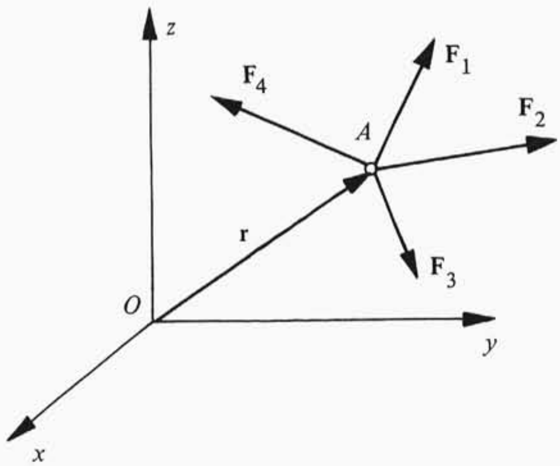


Fig. 3.16.

$$\mathbf{v} = \lim_{\Delta t \to 0} \frac{\Delta \mathbf{r}}{\Delta t} = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = \dot{\mathbf{r}}, \tag{3.59}$$

where $\mathbf{r}$ is a position vector of the particle $A$.

The acceleration of the particle $A$ is defined as

$$\mathbf{a} = \lim_{\Delta t \to 0} \frac{\Delta \mathbf{y}}{\Delta t} = \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \dot{\mathbf{v}} = \ddot{\mathbf{r}}. \tag{3.60}$$

**M-2**. *Absolute velocity and acceleration of a particle.* Consider a fixed frame of reference $Oxyz$ and a frame $A\xi\eta\zeta$ moving in a known, but arbitrary, fashion with respect to $Oxyz$ (Fig. 3.17). Let $P$ be a particle moving in space. The position of $P$ is defined at any instant by the vector $\mathbf{r}$ in the fixed frame, and by the vector $\boldsymbol{\rho}$ in the moving frame. Denoting by $\mathbf{r}_A$ the position vector of $A$ in the fixed frame, we have

$$\mathbf{r} = \mathbf{r}_A + \boldsymbol{\rho}. \tag{3.61}$$

The absolute velocity, $\mathbf{v}$, of the particle is

$$\mathbf{v} = \mathbf{v}_A + \boldsymbol{\omega} \times \boldsymbol{\rho} + \mathbf{v}_R, \tag{3.62}$$

where $\boldsymbol{\omega}$ is the angular velocity of the frame $A\xi\eta\zeta$ at the instant considered, while $\mathbf{v}_R = (\dot{\boldsymbol{\rho}})_{A\xi\eta\zeta}$ is the velocity of $P$ relative to the frame $A\xi\eta\zeta$.

The absolute acceleration, $\mathbf{a}$, of the particle $P$ may be expressed as

$$\mathbf{a} = \mathbf{a}_A + \boldsymbol{\epsilon} \times \boldsymbol{\rho} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{\rho}) + 2\boldsymbol{\omega} \times \mathbf{v}_R + \mathbf{a}_R, \tag{3.63}$$

where $\boldsymbol{\epsilon}$ is the angular acceleration of the frame $A\xi\eta\zeta$ at the instant considered, while $\mathbf{a}_R = (\dot{\boldsymbol{\rho}})_{A\xi\eta\zeta}$ is the acceleration of $P$ relative to the frame $A\xi\eta\zeta$.

The absolute acceleration, $\mathbf{a}$, may be also expressed in the following concise form:

$$\mathbf{a} = \mathbf{a}_T + \mathbf{a}_C + \mathbf{a}_R, \tag{3.64}$$

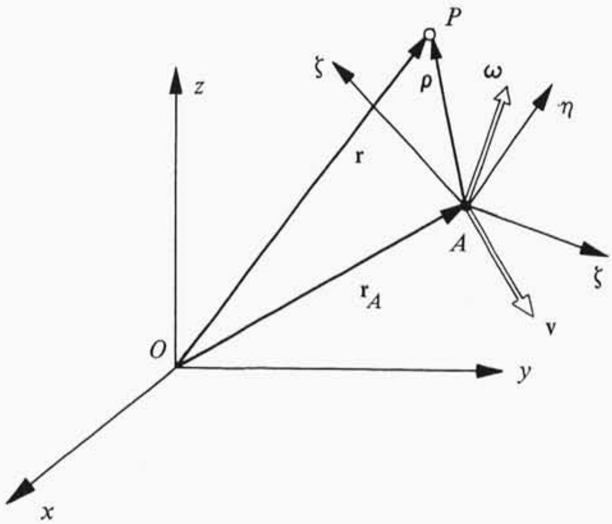


Fig., 3.17.

where $\mathbf{a}_T = a_A + \boldsymbol{\epsilon} \times \boldsymbol{\rho} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{\rho})$ represents the acceleration of the $P'$ of the moving frame which coincides with $P$ at the instant considered and $\mathbf{a}_C = 2\boldsymbol{\omega} \times \mathbf{v}_R$ is called the **complementary**, or **Coriolis acceleration**.

**M-3**. *Decomposition of angular velocity* $\boldsymbol{\omega}$. The reader who has met the Euler angles in a first course in mechanics may have been frustrated by the fact that the final formulae are not the same in all books on mechanics. It seems that there are two reasons for this state of affairs. First, the same name is assigned to essentially different sets of three angles with identical notation $\phi$, $\theta$, $\psi$. Euler angles are thus differently defined for a rigid body spinning about its axis which itself rotates about a fixed point (see Fig. 3.18) and for an aircraft flying along a certain trajectory (Fig. 3.19. Second, if we consider cases when the meaning of the Euler angles remains the same, the sequence of carrying out the Euler rotation is not always the same, and so far no standard sequence has been agreed upon. Several different types of Euler angle systems are in common use. Two of them will be presented in what follows.

Let us first consider the rigid body rotating in any manner about a fixed point $O$ (Fig. 3.18). For this type of motion it is convenient to introduce two sets of coordinates to specify the motion. In Fig. 3.18 the coordinates $x$, $y$, $z$ are fixed in space, and plane $A$ contains the $x$-, $y$-, $z$-axes and the fixed point $O$ on the rotor axis. Plane $B$ contains point $O$ and is always normal to the rotor axis. Then intersection of these two planes is the so-called *line of nodes*, $n$, which is located at an angle $\psi$ from the $x$-axis. Angle $\theta$ measures the inclination of the rotor axis from the vertical $z$-axis and is also the measure of the angle between planes $A$ and $B$. The angles $\psi$ and $\theta$ completely specify the position of the rotor and turn with it about axis $\zeta$. The angular displacement of the rotor in plane $B$ is specified by the angle $\phi$ between the $\xi$-axis attached to the rotor and the $n$-axis.

The three angles $\psi$, $\theta$, $\phi$ completely specify the position of the rotor and are known as

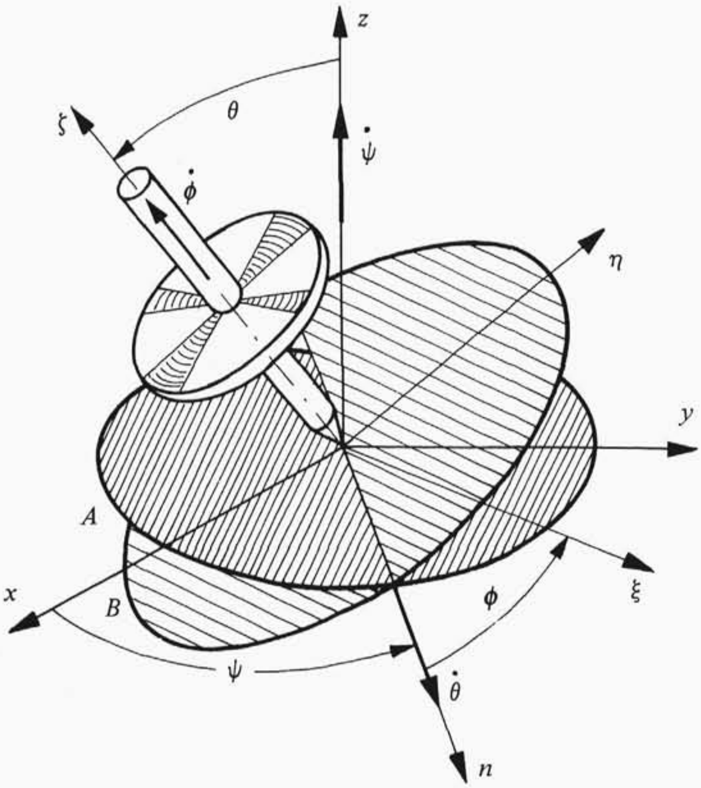


Fig. 3.18.

**Euler's angles**. The time rates of change of these angles $\dot{\psi}, \dot{\theta}, \dot{\phi}$ specify, respectively, the **precession, nutation** and **spin** of the rotor. The angular velocity of the rotor is then

$$\boldsymbol{\omega} = \dot{\psi}\mathbf{e}_z + \dot{\theta}\mathbf{e}_n + \dot{\phi}\mathbf{e}_\zeta. \tag{3.65}$$

In many instances the components of angular velocity $\boldsymbol{\omega}$ on a body-fixed $\xi$, $\eta$, $\zeta$ coordinate system in terms of Euler angles are required. Inspection of Fig. 3.20 yields

$$\begin{aligned}
\omega_\xi &= \dot{\psi}\sin\theta\sin\phi + \dot{\theta}\cos\phi, \\
\omega_\eta &= \dot{\psi}\sin\theta\cos\phi - \dot{\theta}\sin\phi, \\
\omega_\zeta &= \dot{\psi}\cos\theta + \dot{\phi}.
\end{aligned} \tag{3.66}$$

Now consider the Euler angles used in flight mechanics for description of the angular position of an aircraft. In order to describe the motion of the aircraft with respect to the Earth or inertial space, it is necessary to be able to specify the orientation of one axis system with respect to another. For this purpose we introduce three coordinate systems

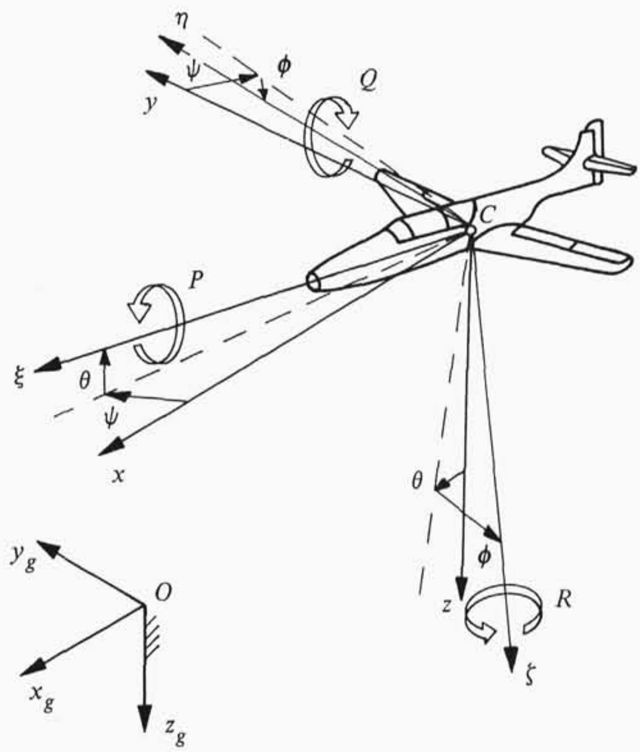


Fig. 3.19.

The origin of the first one is, by definition, located at the centre of gravity, $C$, of the aircraft. In general, the axis system $C\xi\eta\zeta$ is fixed to the aircraft and rotates with it. Such a set of axes is referred to as *aircraft axes*. The axis is taken with $C\xi$ forward, $C\eta$ out the right wing, and $C\zeta$ downward as seen by pilot to form a right-handed axis system (see Fig. 3.19).

Since the frame $C\xi\eta\zeta$ is fixed to the aircraft and moves with it, the position and orientation of the aircraft cannot be described relative to it. For this purpose we introduce an *Earth-fixed frame* of reference $Ox_gy_gz_g$. Let $Oz_g$ be taken vertically downwards, and $Ox_g$ horizontal in the vertical plane containing the initial vector of the mass centre. The origin $O$ is assumed to coincide with $C$ at $t = 0$.

The origin of the third coordinate system $Cxyz$ is assumed to be at the centre of gravity, $C$, and its axes remain parallel to the respective axes of the Earth-fixed frame (see Fig. 3.19).

Orientation of the aircraft is then given by a series of three consecutive rotations, whose order is significant (Fig. 3.19). The aircraft is imagined first to be oriented so that its axes are parallel to $Ox_gy_gz_g$. It is then in the position $Cxyz$. The following rotations are then applied:

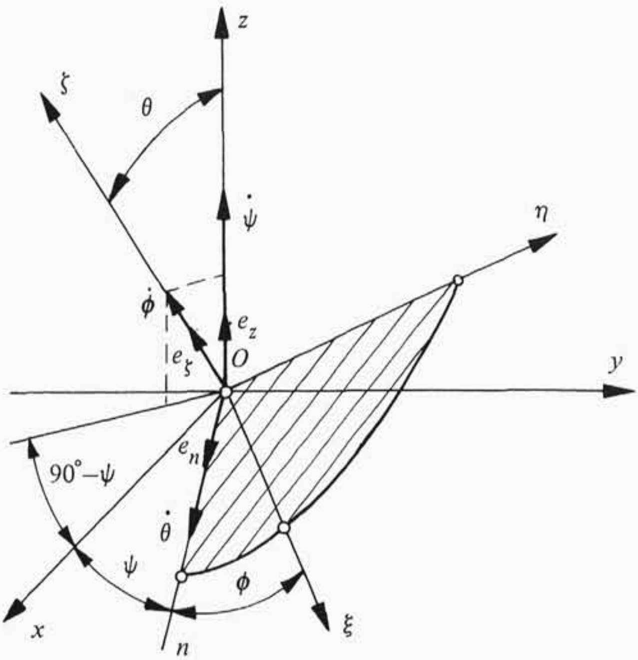


Fig. 3.20.

(1) a rotation $\psi$ about $Cz$, carrying the axes to $Cx_1y_1z_1$ (bringing $C\xi$ to its final *azimuth*);

(2) a rotation $\theta$ about $Cy_1$, carrying the axes to $Cx_2y_2z_2$ (bringing $C\xi$ to its final *elevation*);

(3) a rotation $\phi$ about $Cx_2$, carrying the axes to their final position $C\xi\eta\zeta$ (giving the final angle of *bank* to the wings).

In flight mechanics the angles $\psi$, $\theta$, $\phi$ are referred to as **yaw**, **pitch** and **roll**, respectively.

In flight mechanics, the components of the angular velocity $\boldsymbol{\omega}$ on the aircraft-fixed axes $\xi$, $\eta$, $\zeta$ are traditionally denoted by $P$, $Q$, $R$, respectively. Thus we have

$$\boldsymbol{\omega} = \mathrm{P}\mathbf{e}_\xi + \mathrm{Q}\mathbf{e}_\eta + \mathrm{R}\mathbf{e}_\zeta, \tag{3.67}$$

and

$$\begin{aligned} P &= \dot{\phi} - \dot{\psi}\sin\theta, \\ Q &= \dot{\theta}\cos\phi + \dot{\psi}\cos\theta\sin\phi, \\ R &= -\dot{\theta}\sin\phi + \dot{\psi}\cos\theta\cos\phi. \end{aligned} \tag{3.68}$$

**M-4**. *Euler's theorem.* The most general displacement of a rigid body with a fixed point $O$ is equivalent to a rotation of the body about an axis through $O$.

### 3.3.6 Specific laws of mechanics

Each material object can interact with its surroundings both when it is at rest and when it moves. The measure of interaction used in mechanics is force, and the vast majority of the specific laws of mechanics are devoted simply to specification of force in dependence upon the physical properties of the object, its surrounding with which it interacts, and possibly also the kinematic characteristics of the object's motion.

As before, in order to better present this set, whose elements enter mathematical models, we shall refer to examples.

**SLM-1**. *Newton's law of gravitation.* This law states that two particles of mass $m_1$ and $m_2$ are mutually attracted with equal and opposite forces $\mathbf{F}$ and $-\mathbf{F}$ of magnitude $F$, given by the formula

$$F = G\frac{m_1 m_2}{r^2}, \tag{3.69}$$

where $r$ is the distance between the two particles and $G$ is the gravitational constant.

It may seem somewhat strange that such an important law as the law of universal gravitation be classified as a specific law of mechanics. Note, though, that this law specifies just one kind of force—force of attraction.

**SLM-2**. *Resistance laws.* There exist many formulations of resistance laws and their form depends upon such factors as the kind of object in motion, the medium with regard to which the motion takes place, or the purpose of application of a given law. Consider the following:

(a) *Stoke's law for the force acting on a moving sphere.* The resistance force $F$ acting on a rigid sphere moving in a incompressible viscous fluid is

$$F = 6\pi\mu v_\infty r, \tag{3.70}$$

where $\mu$ is the coefficient of dynamic viscosity, $v_\infty$ is the velocity of free stream flow, and $r$ is the radius of the sphere.

(b) The drag acting on an aerofoil segment of width d$y$ is

$$\mathrm{d}D = c_\mathrm{D}\rho_\infty \frac{v_\infty^2}{2} l(y)\,\mathrm{d}y, \tag{3.71}$$

where $c_\mathrm{D}$ is drag coefficient, $\rho_\infty$, $v_\infty$ are respectively the density and velocity of the undisturbed air, $l(y)$ is the width of a chord of a wing.

**SLM-3**. *Constitutive laws.* A broad group of specific laws is made up of constitutive laws for various media. We shall cite some of them:

(a) *Hooke's law.* Probably the best known among the constitutive laws is Hooke's law in its simplest form relating the tensile stress, $\sigma$, in a uniform wire to the tensile strain, $\varepsilon$, through the equation

$$\sigma = E\varepsilon, \tag{3.72}$$

where the constant $E$ is Young's modulus of the wire.

(b) *Constitutive law of ideal fluid.* The simplest constitutive equations encountered in continuum mechanics are those for an ideal fluid. These equations are

$$\sigma_{ij} = -P(\rho, \vartheta)\delta_{ij}, \tag{3.73}$$

i.e.

$$\sigma = -P(\rho, \vartheta)\mathsf{I}, \tag{3.74}$$

where the positive quantity $P$, a scalar function of density $\rho$ and temperature $\vartheta$, is called the 'pressure.

### 3.3.7 Characteristics of bodies in motion

We have distinguished five fundamental groups whose elements are used in modelling. The question arises of whether, once we have selected some specific laws and the basic laws of mechanics, and some kinematic relations, and perhaps introduced the interactions and relevant complete information on the properties of the modelled object, we obtain a **complete model**, that is one in which the number of unknowns is equal to the number of equations. However, the model obtained at this stage is, as a rule, incomplete. Why is that? When using fundamental laws in their general formulation, e.g. (3.9), (3.10) or (3.11), we refer to the quantities appearing there, such as momentum $\mathbf{p}$, angular momentum, $\mathbf{H}$, and energy, $E$, which are aggregate quantities and in reality depend in a unique manner upon the properties of the object and upon its current state. Simplifying, we may talk of characteristics of bodies in motion. This simplification results from the fact that energy of a system may not only consist of kinetic energy, but also of potential, thermal, electromagnetic energies, etc., which do not or may not depend upon the kinematic state of a body.

Let us therefore create a subset of characteristics of bodies in motion, denoted **BM**. This subset is not shown in the diagram of Fig. 3.4, but we can image that it is located on the edge of the tetrahedron linking sets **B** and **M**. Now we shall present several examples of relations belonging to subset **BM**.

**BM-1**. *Linear momentum of a particle.* Consider a particle of mass $m$ moving with a velocity $\mathbf{v}$. The vector

$$\mathbf{p} = m\mathbf{v} \tag{3.75}$$

defines the linear momentum or simply momentum of the particle at the instant.

**BM-1**. *Angular momentum of a particle.* Let us denote the position vector of the particle with respect to a fixed reference point $O$ by $\mathbf{r}$. The moment of momentum or angular momentum about $O$ is given by the vector

$$\mathbf{H} = \mathbf{r} \times m\mathbf{v}. \tag{3.76}$$

**BM-3**. *The angular momentum of a rigid body rotating about a fixed point.* Consider a rigid body rotating about a fixed point with the angular velocity $\boldsymbol{\omega}$. Let us assume that the reference point is at the origin $O$ of the Cartesian coordinate system (Fig. 3.21). Then the angular momentum of the body may be expressed as

$$\mathsf{H} = \mathsf{I}\omega, \tag{3.77}$$

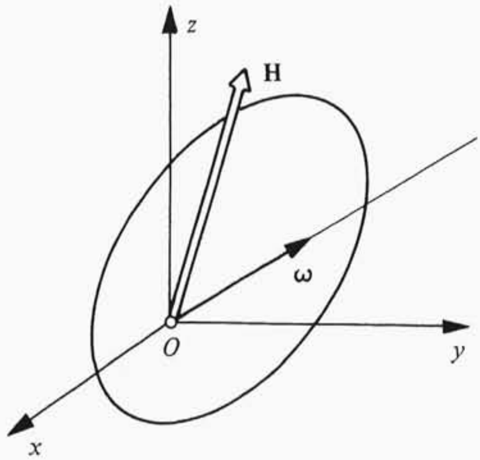


Fig. 3.21.

where I is the inertia tensor defined by (3.50), and $\omega = [\omega_x, \omega_y, \omega_z]^T$ is the column matrix representing the angular velocity vector in the $x$, $y$, $z$ coordinate system.

**BM-4**. *Kinetic energy*. The kinetic energy of the particle is

$$T = \tfrac{1}{2} m v^2, \tag{3.78}$$

where $v$ is the absolute speed of the particle.

Kinetic energy of the rigid body relative to an inertial system is

$$T = \tfrac{1}{2} m v_c^2 + T_{\text{rot}}, \tag{3.79}$$

where $v_c$ is the speed of the centre of mass of the body, and $T_{\text{rot}} = \frac{1}{2}\omega^T \mathsf{I} \omega$ where, in turn, I is the inertia tensor defined by (3.50), and $\omega$ is the angular velocity of the body, and the origin of the $x$, $y$, $z$ coordinate system is at the centre of mass.

### 3.3.8 Final hints

We have classified problems of mechanics into five fundamental groups described in five sections 3.3.2–3.3.6. How is this classification used in modelling? Let the starting point for the answer to this question be the observation that when building a mathematical model we usually have in mind one of four purposes mentioned below, i.e.

(1) determination of the motion of the object/body,
(2) identification of forces causing a given phenomenon,
(3) determination of properties of the object/body,
(4) establishment of specific laws governing a given phenomenon or characterizing the object (e.g. constitutive laws of the media).

It is essential that in each case we make use of the basic laws of mechanics. These laws constitute the foundation for the mathematical model of any complex mechanical phenomenon. We draw the other group in different degrees, depending upon the purpose of modelling. Thus, if the purpose is the determination of the motion of a body (which is a common task in modelling), we have to complement the basis with information on the properties of the body, on forces and on how forces acting upon the body depend upon the characteristics of motion and properties of the body itself. Alternatively when the purpose is to identify the forces causing a motion of a body, then it is necessary for modelling to have information on the physical properties and motion of the body.

Information on the motion, forces and properties of a body entail the following:

— complementing the system of dynamic equations of motion with equations resulting from either kinetic relations such as (3.63), (3.66) or resulting from constraints,
— specification of forces, using for this purpose specific laws of mechanics,
— use of relations between the properties of a body.

This should bring the system of equations to the state in which the number of equations of the model is equal to the number of unknowns.

The identification of problems of mechanics means that in every mathematical model of dynamical phenomena there must appear at least one element from each set distinguished above, i.e. BLM, B, F, M and SLM. Thus if the system of equations is incomplete at a certain stage of modelling, one should verify that all the available information from all the sets indicated has been used.

We finish at this point with a survey of the tools of modelling used in mechanics. How they are used will be discussed in subsequent sections.

## 3.4 APPLICATIONS

### 3.4.1 Parachute with a payload

When we speak of resistance, our first reaction is to think of it as a disadvantageous phenomenon. A parachute is an example of a device in which the phenomenon of resistance to motion is exploited.

Parachutes are classified according to their various purposes: rescue, sports, military or transport (airborne supply of medical aid, food, etc.). For a parachute to perform its duties, it must behave in a stable manner. However, the reason for the various dynamic instabilities of a parachute have not been fully explained. One problem is that dynamic stability testing of parachutes is difficult. The theory of parachute instability requires a nonlinear three-dimensional analysis with several assumptions that are difficult to justify. Modelling itself is not easy, for because of the fact that the mass of parachute is usually small and rate of change of velocity great, one should consider the so-called *apparent masses*.

It is known that under appropriate conditions the apparent masses of the body can play an important role in the determination of dynamic characteristics, and this is certainly the case with parachute systems.

According to classification of airborne vessels a parachute is an aerodyne without an engine-driven propulsion unit, with the total aerodynamic force acting in the opposite direction to motion. Under the assumption that the parachute is already open—that is, omitting the initial opening shock from consideration—the general motion of a parachute