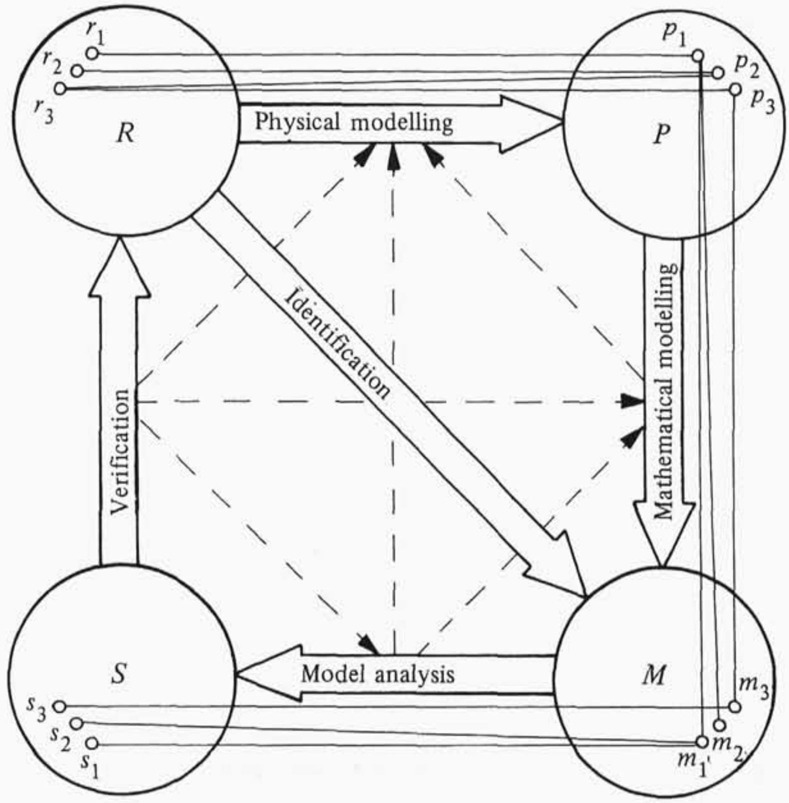


Fig. 1.5. Scheme of the process of investigation of the phenomena.

which are symbolized by broken lines will be called **modifications**. It should be stressed that in practice most modelling does not take the precise form shown in Fig. 1.5. The figure is there just to give some idea of the underlying relationship between real-world problems and the mathematical techniques used to find solutions to them.

1.3 AN EMPIRICAL AND A CAUSAL MODEL

Each observed phenomenon may be described in many different ways depending on factors such as required exactness, the purpose of modelling, calculation possibilities at the stage of model analysis, etc. Even if all the mentioned factors have been established, two different approaches to mathematical model formulation may still be distinguished. These two approaches result in different kinds of models. However, before naming them let us consider an example in which two methods of mathematical model formulation will be presented.

Example. A simple rigid beam AB is loaded by a force F applied at point C (see Fig. 1.6). The direction of action of the force is known and given by the angle α. The reaction Y_B of the support B has to be determined.

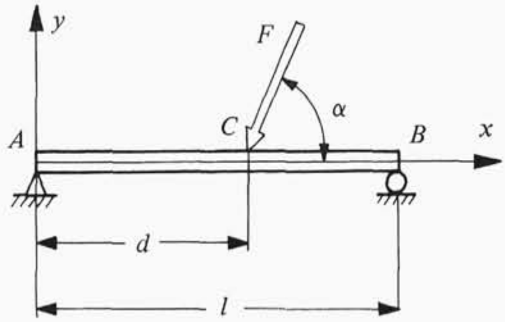


Fig. 1.6.

Two different approaches to this problem solution may be used. The experimental approach is simply to weigh the loading of the support. By performing a series of measurements of the reaction Y_B for different loadings F at fixed angle α, and then drawing a graph (as for example in Fig. 1.7), we may easily establish a relation between the reaction Y_B and the loading F. This relation has the form

$$Y_B = k_1 F,$$

where $k_1 = \tan\gamma$ is a positive constant, and γ is the angle shown in Fig. 1.7.

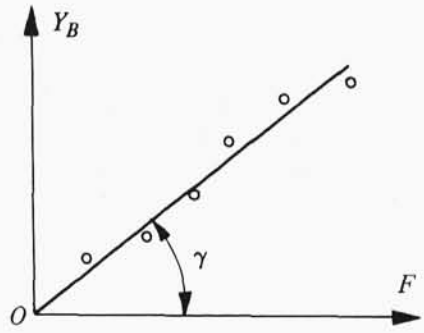


Fig. 1.7.

If we perform measurements for different loadings F and for different angles α, we can also, by further examination of experimental data, find out how the coefficient k_1 depends on the angle α. It would have the form $k_1 = k \sin\alpha$, and, consequently, reaction Y_B may be expressed as follows:

$$Y_B = kF \sin\alpha \tag{1.9}$$

where k is a positive constant.

The theoretical approach to problem solution requires the writing of only one equilibrium equation, namely that for moments about point A. Thus we have

$$\sum M_A = Fd\sin\alpha - Y_B l = 0$$

Hence

$$Y_B = \frac{d}{l} F\sin\alpha. \tag{1.10}$$

The example considered has shown that different ways of model formulation may result in different models. It is reasonable to distinguish the two kind of model. We shall call the model obtained via experimental procedure an **empirical model**, and the one obtained via a theoretical procedure a **causal model**. Thus a causal model is one obtained by means of certain physical laws. The experimental method of model formulation is also referred to as identification.

When evaluating the usefulness of models with respect to an *a priori* settled modelling purpose, we cannot perceive substantial differences between the two kinds of models. An important advantage of a causal model is that its formation requires usually much less work than in the case of an empirical one. In carrying out the experiments and then establishment of data, the obligatory stages of elaboration of any empirical model, are usually extremely laborious and expensive. But the main advantage of the causal model is constituted by its generalization possibilities. It is sufficient to change the point in which the force F is applied, and the result obtained (1.9) will not be valid any more. A great part of the work invested in the empirical model preparation is then lost. At the same time the causal model (1.10) after the same change remains still valid. This preservation of model validity despite some changes of the parameters is really a great advantage of the causal model.

The above considerations may suggest that there is always a clear, sharp division between an empirical and a causal model. Such an impression is, however, not true. The notion of 'causal model' is to some extent relative and depends on the investigator's enquiries and on the current state of knowledge about the class of phenomena considered. Moreover, an empirical model can have the same form as the causal one. A good illustration for the last two sentences is provided by the following well-known historical example.

Suppose we are interested in how the pressure P of a perfect gas changes when its volume Ω varies at constant temperature ϑ, i.e. during a so-called isothermal process. If we knew nothing about the laws governing this process, we should perform an experiment and put relevant measurements on the P–Ω chart. The next stage of modelling would be an approximation of the experimental data by means of a curve, whose equation would provide an empirical formula for the isothermal process. That is what **Robert Boyle** (1627–1691) did, and in 1661 he announced his discovery, known today as **Boyle's law**:

$$P\Omega = \text{constant} \qquad (\vartheta = \text{constant}).$$

Note that to some extent this simple model may already be treated as a causal model, and if we were to ask why two state variables, P and Ω, of a perfect gas in isothermal conditions vary in an exactly unique manner, we might answer that the isothermal process is governed by Boyle's law. But this answer would be insufficient for an inquiring mind. **Benoit Clapeyron** (1799–1864), who gave a more general formula

$$P\Omega = nR\vartheta \tag{1.11}$$

(where n is the number of moles in the system, and R is the universal gas constant), known today as **Clapeyron's equation**, certainly thought so. Thus, Boyle's law is a particular case of (1.11). Although Clapeyron's equation (1.11) was first obtained in experimental way, the same result may be obtained by means of theoretical considerations within the frames of statistical mechanics. In this case, fundamental laws of statistical mechanics together with assumptions specifying interaction forces between gas molecules, have to be used. Thus (1.11) may be treated as a causal model of perfect gas behaviour. But now we may also ask about the ultimate cause of the fundamental laws of statistical mechanics, and we shall find ourselves in a similar position to the time before Clapeyron's equation had been announced.

The above considerations give rise to the following conclusion: a notion of causal model is relative and depends on the actual state of knowledge and the enquiry of the investigator. Despite this relativity of both notions concerning empirical and causal models we shall use the following simple practical indicator in order to distinguish them: a model obtained by means of an identification procedure will be called an empirical model, while one obtained by using the physical laws will be called causal.

1.4 THE INFLUENCE OF PURPOSE OF MODELLING ON THE FINAL FORM OF THE MODEL

The choice of a physical model depends substantially on the purpose of modelling, i.e. which properties of the investigated phenomenon are we interested in. A single factor may be unimportant for the investigation of one property and very important during the investigation of another. This means that a physical model has always a limited range of applications and is useful only for investigation of a certain subset of all the properties of a phenomenon. A good example illustrating the last statement is provided by the modelling of the motion of a rigid body about a horizontal axis, i.e. the compound pendulum. Probably everyone is familiar with the compound pendulum oscillating about its equilibrium position. We may observe that free motion is nearly periodic, decaying in time, and that after a certain time the body stops moving.

Let us assume that we are interested in the following properties of the pendulum in its motion about the horizontal axis:

P1. period T,
P2. decay time to *half-life* oscillation, i.e. *time to half* T_h,
P3. time after which pendulum stops, i.e. *dead time*.

In order to obtain information about the three properties mentioned above let us consider several physical and corresponding mathematical models:

Physical model PM-1. We shall consider a rigid body of a mass m rotating about a horizontal axis x through the point 0, the axis being distant l from mass centre C (see Fig. 1.8). Let J_0 be the body moment of inertia about the axis x. Additionally, we shall simplify the description of the real object assuming that:

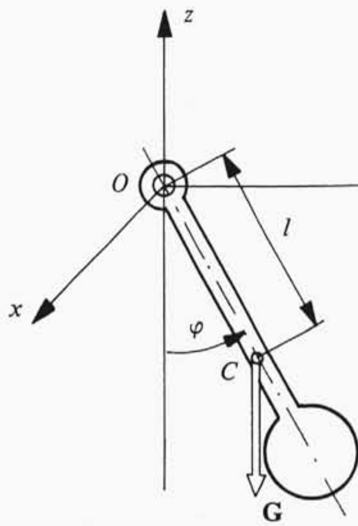


Fig. 1.8.

A1. frictional resistance in a bearing is negligible,
A2. drag is negligible also,
A3. the body moves around the equilibrium state oscillating with small angles φ.

During rotational motion of the body many physical laws are fulfilled simultaneously but two of them directly influence the shape of mathematical model. They are

(1) law of rate of change of the angular momentum,
(2) Newton's law of gravity.

Mathematical models for physical model PM-1. The law of rate of change of angular momentum for rotational motion of a body about the x-axis states that

$$\frac{\mathrm{d}H_x}{\mathrm{d}t} = M_x, \tag{1.12}$$

where H_x is a projection on the x-axis of the vector of angular momentum of the body about this axis, and M_x is a projection on the x-axis of the moment of external forces about this axis. Then, Newton's law of gravity enables us to express the force of gravity, which gives the nonzero moment about the x-axis. We have $\mathbf{G} = m\mathbf{g}$, where $\mathbf{G}$ is the force of gravity, and $\mathbf{g}$ is the acceleration due to gravity. Taking into account assumptions A1 and A2, as well as the relations $H_x = J_0\dot{\varphi}$ and $M_x = -Gl\sin\varphi = -mgl\sin\varphi$, and substituting them into (1.12), we get the first mathematical model, MM-1.1., corresponding to the physical model PM-1, i.e.

MM-1.1.: $J_0\ddot{\varphi} + mgl\sin\varphi = 0.$ (1.13)

The nonlinear equation (1.13) thus obtained may be simplified due to assumption A3 and we get

MM-1.2: $J_0\ddot{\varphi} + mgl\varphi = 0.$ (1.14)

Equation (1.14) describes the well-known simple harmonic motion, and its solution has the following form:

$$\varphi = A\cos(\omega_n t + \beta), \tag{1.15}$$

where $\omega_n = \sqrt{(mgl/J_0)}$ is the *undamped natural frequency*. The *amplitude* A and *initial phase angle* β are constants which are evaluated from initial conditions $\varphi(0)$ and $\dot{\varphi}(0)$. The solution (1.15) of the mathematical model MM-1.2 provides the answers to the three interesting properties:

P1. period $T = 2\pi/\omega_n = 2\pi\sqrt{(J_0/mgl)}$,
P2. time to half T_h is infinite, since the solution (1.15) is periodic,
P3. dead time is also infinite, for the same reason.

The solutions obtained with respect to properties P2 and P3 are evidently contradictory with the phenomenon described before. Thus we can try to change the physical model hoping that this would improve the resulting model of the phenomenon.

Physical model PM-2. All elements from physical model PM-1 remain unchanged except for the assumption A2. Now we shall assume the existence of a drag, which will damp the oscillations of the pendulum. As a consequence of this new assumption the set of data has to be supplemented with the damping coefficient denoted by B. Insofar as we shall further assume the oscillatory character of pendulum motion the damping has to be light. Hence, $B < B_c$, where $B_c = 2m\omega_n$ is '*critical damping*'.

Mathematical models for the physical model PM-2. Since the resistance moment about the x-axis caused by drag is $B\dot{\varphi}$, the mathematical model is

MM-2.1.: $J_0\ddot{\varphi} + B\dot{\varphi} + mgl\sin\varphi = 0.$ (1.16)

Due to the assumption A3 the nonlinear model (1.16) may be simplified and we get

MM-2.2.: $J_0\ddot{\varphi} + B\dot{\varphi} + mgl\varphi = 0.$ (1.17)

The solution of (1.17) is

$$\varphi = Ce^{-\zeta\omega_n t}\cos(\omega_d t + \beta), \tag{1.18}$$

where $\zeta = B/B_c$ is the damping factor, $\omega_d = \omega_n \sqrt{(1-\zeta^2)}$, C and β are constants, whose values are evaluated from the initial conditions. For the detailed explanation of the meaning and evaluation of ζ, ω_d, C and β see, for example, Walshaw (1984).

The solution (1.18) provides the following answers for interesting questions:

P1. period $T = 2\pi/\omega_d = 2\pi/\omega_n \sqrt{(1-\zeta^2)}$,
P2. time to half $T_h = \ln 2/\zeta\omega_n$,
P3. dead time is infinite (since the function $\exp(-\zeta\omega_n t)$ is always positive).

Since the solution of the model MM-2.2 does not give a proper answer concerning property P3, there is a necessity of further improvement of the physical model.

Physical model PM-3. In order to improve a physical model we shall change the model PM-2, now assuming the existence of frictional resistance in a bearing. We shall assume that in the bearing O the dry friction forces result in a resistance torque M, which is characterized by the three graphs of Fig. 1.9.

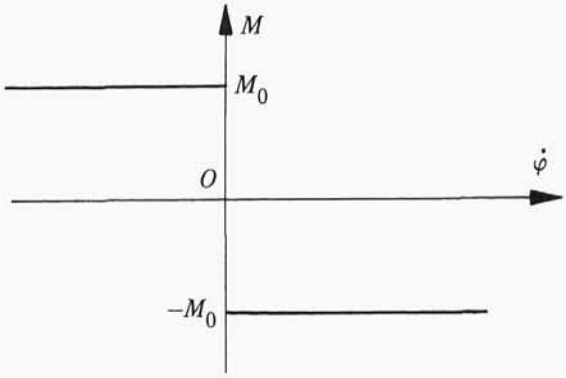
The graph in Fig. 1.9(a) describes the relation between the moment of dry friction forces and the angular velocity $\dot{\varphi}$. We see that if $\dot{\varphi} \neq 0$ the moment is uniquely determined. This is not the case when $\dot{\varphi} = 0$. The situation of $\dot{\varphi} = 0$ is described in turn by the graph of Fig. 1.9(b). In this case the moment depends on the angular acceleration $\ddot{\varphi}$ of the pendulum. When this acceleration differs from zero ($\ddot{\varphi} \neq 0$), i.e. when the velocity changes its sign, the moment is uniquely defined and has a constant value. If, however, both angular acceleration and angular velocity are simultaneously equal to zero, i.e. the body does not move, the moment of dry friction forces may assume any value from a range $(-M_0, M_0)$ and its exact value depends on the position in which the body stopped. The quantitative relation between the moment of dry friction forces and the angular position is shown in Fig. 1.9(c). As we see, the introduction of dry friction into the bearing substantially complicates the physical model. All the other elements of the physical model remain the same as in model PM-2.

Mathematical models for physical model PM-3. In the first stage of mathematical modelling we have to form the mathematical description of frictional resistance. First of all note that the moment of dry friction forces depends on the angular acceleration $\ddot{\varphi}$, and thus we may write $M = M(\varphi, \dot{\varphi}, \ddot{\varphi})$. In any case the moment M does not exceed a certain constant value M_0, so that we can write

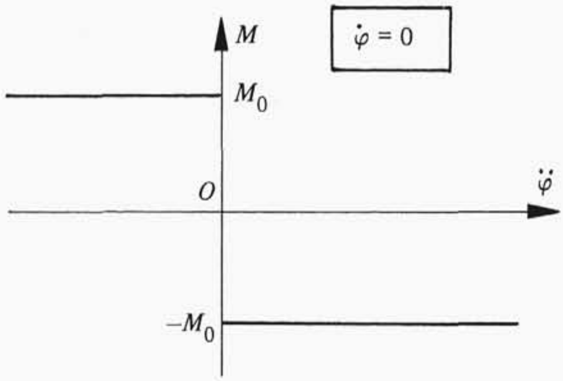
$$M = \kappa M_0, \tag{1.19}$$

where

$$\kappa = \begin{cases} -\dfrac{\dot{\varphi}}{|\dot{\varphi}|} & \text{when } \dot{\varphi} \to 0 \\ -\dfrac{\ddot{\varphi}}{|\ddot{\varphi}|} & \text{when } \dot{\varphi} = 0, \quad \text{and} \quad \ddot{\varphi} \neq 0 \\ \psi\left(\dfrac{mgl\sin\varphi}{M_0}\right) & \text{when } \dot{\varphi} = 0, \quad \text{and} \quad \ddot{\varphi} = 0. \end{cases} \tag{1.20}$$


M
M_0
O
$\dot{\varphi}$
$-M_0$

(a)


M
$\dot{\varphi} = 0$
M_0
O
$\ddot{\varphi}$
$-M_0$

(b)

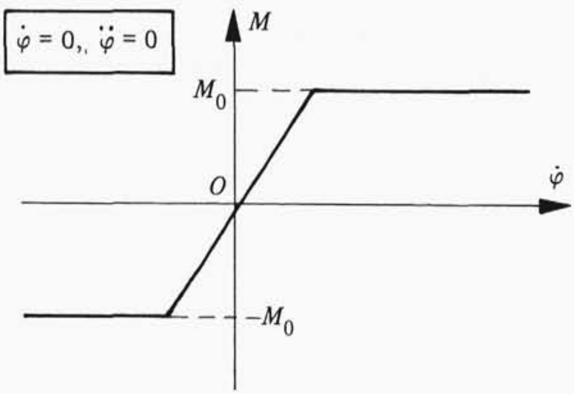

$\dot{\varphi} = 0,, \ddot{\varphi} = 0$
M
M_0
O
$\dot{\varphi}$
$-M_0$

Fig. 1.9

The function ψ appearing in (1.20) is defined as follows:

$$\psi = \begin{cases} \operatorname{sign} z & \text{for } |z| > 1 \\ z & \text{for } |z| < 1, \end{cases} \tag{1.21}$$

where, in turn,

$$\operatorname{sign} z = \begin{cases} +1 & \text{when} \quad z > 0 \\ -1 & \text{when} \quad z < 0. \end{cases} \tag{1.22}$$

Now we are in a position to formulate the equation of motion, i.e. the mathematical model

MM-3.1.: $$J_0\ddot{\varphi} + B\dot{\varphi} + mgl\sin\varphi = \kappa M_0. \tag{1.23}$$

where κ is defined by the expressions (1.20).

Equation (1.23) has no analytical solution and if we need to get the answers concerning the interesting properties P1, P2, and P3 we have to simulate a corresponding initial value problem on a computer.

The time-dependent behaviour of the system can be illustrated if three instantaneous values of displacement are plotted against time (Fig. 1.10); shown are the 'decay curves' for

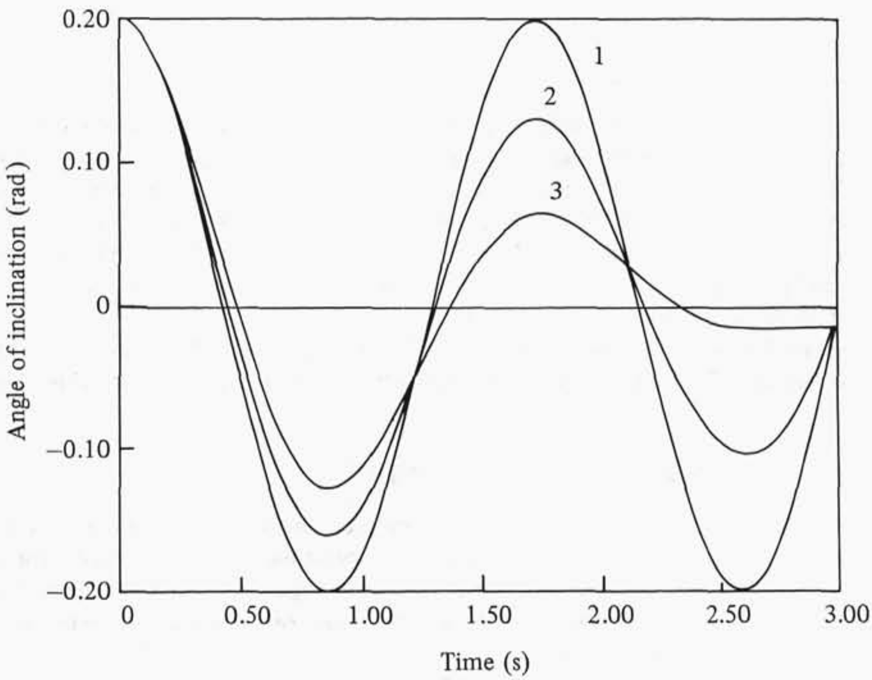


Fig. 1.10.

(1) undamped motion which is, of course, simple harmonic—equation (1.14),
(2) viscous damped motion with damping factor $\zeta = 0.1$—equation (1.17),
(3) motion influenced by viscous damping and frictional resistance, with $M_0/J_0 = 0.3\ \mathrm{s}^{-2}$.

We can see that when dry friction forces are accounted for in the model, the body stops, according to this model, after a certain finite time. This result, read out directly from computer plots, and approximately conforming with the true course of the phenomenon, allows us to think that the MM-3.1 model is adequate.

Obviously it is possible to formulate subsequent physical models which may give equally good or even better resulting models, but this is not our aim now. Here we want to focus our attention on the fact that a given modelling purpose (in the example considered it was determination of period, time to half-life, and dead time), may require a change in the physical model and consequently in the mathematical ones.

In this example the changes of the physical model were relatively small. They concerned only these assumptions, which cause introduction of two kinds of forces into the model, i.e. the viscous damping force and the dry friction force. All the other assumptions concerning the object remained unchanged. It often happens that the purpose of modelling has much greater consequences. Modelling of aircraft provides a good example of this kind of change.

An aircraft may be modelled as

(1) a particle,
(2) a system of rigid bodies,
(3) a system of deformable bodies,

and the choice depends on the problem we have to solve. If we are interested in optimization of flight trajectory with respect, for example, to fuel consumption, then a sufficient and a good model of the aircraft is a particle. When considering the problem of flight stability, i.e. when studying aircraft behaviour on a flight trajectory under small disturbances, we need to model the aircraft as a rigid-body system. Finally, when performing flutter analysis, i.e. determining the so-called critical speed of flutter, we have to model the aircraft as a system of deformable bodies (Fig. 1.11).

Thus, on the basis of the two examples considered, we see that the purpose of modelling has substantial influence on the final form of the physical and mathematical models.

1.5 DISCRETE AND CONTINUOUS MODELS

One of the most important decisions, made usually at the initial stage of the modelling process, is whether a given object should be transformed into a **discrete**, **continuous** or **mixed** (i.e. discrete–continuous) model. Let us note that the notions 'discrete' and 'continuous' concern the model, and not the object. To demonstrate that difference let us again refer to the example considered in section 1.4. A real object such as an aircraft, depending on the modelling purpose, may be modelled as a mass particle, a system of rigid bodies or a system of deformable bodies. Every aircraft possesses a structure, i.e. a unique system of element interconnections, such as wings, fuselage, stabilizer, elevator,

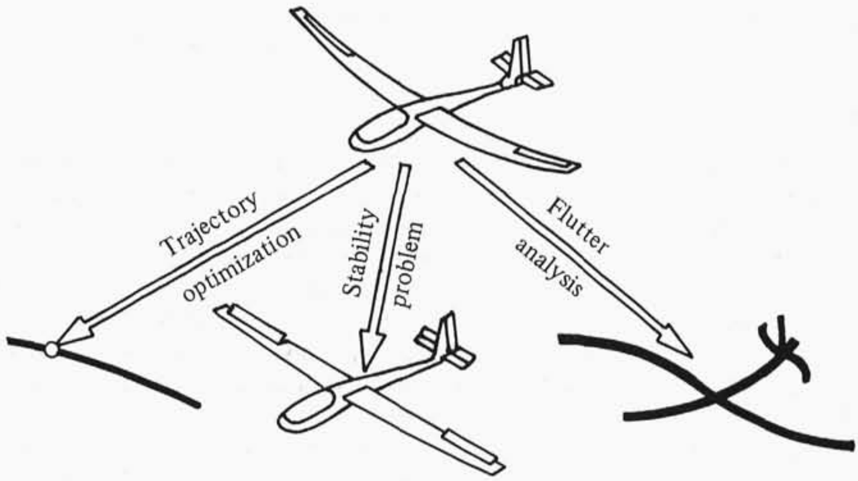


Fig. 1.11.

rudder, aileron, engine, propeller, etc. Moreover, some of the aircraft elements are massive, heavy and relatively rigid, while others are light and flexible. This complex structure of the object is transformed at the very beginning of modelling into a substantially simplified structure of the physical model. After this transformation, physical models of a different kind may be obtained, namely discrete (as for example particles or rigid bodies), continuous (like deformable bodies), or mixed, i.e. consisting of both kinds of components—discrete and continuous. Of course this means, however, that the choice of the kind of a model depends strongly upon the aim of modelling. At the same time the properties of the object are its attributes and they can be accepted or not. It is known from physics that the microstructure of matter is in fact grainy, i.e. discrete. It consists of molecules and atoms. All processes occurring on micro-level, such as collisions, absorption or emission of energy, have discontinuous character, i.e. they are of quantum nature. However, in engineering mechanics only the phenomenological effects, which appear in the macroscale, are practically important. There the individual behaviour of the molecules as well as the granulation of the matter gets blurred. The matter can then be treated as a spatially continuous medium, and all phenomena occurring in it are recognized as time-continuous. The fundamental simplification related to the objects to be considered in this book relies upon neglecting their discrete structure, i.e. abandoning the fact that from microscopic viewpoint material substance is composed of discrete particles—molecules and atoms. In other words an object will be (and already has been) treated, in the first instance, as a **continuous medium**, or **continuum**, for simplicity.

The above statements can be expressed in a more formal form as a so-called **continuum postulate**.

The continuum postulate assumes that

every element volume of a body contains a tremendous number of molecules and that the average statistical properties of the molecules contained in an elementary volume represent the macroscopic properties of the body in the region of that elementary volume.

Consequently, the continuum model is a satisfactory one only for those situations where the characteristic dimensions of the body under consideration are very large when compared with the average molecular distance between the molecules constituting the body. When considering phenomena on the macro-level for solid bodies or liquids, the continuum postulate is fulfilled, of course. However, in the case where a gas is at a low pressure (e.g. atmospheric air at high altitudes), the gas density may be so low that the applicability of the continuum postulate may be an open question. It is, therefore, essential that analytical criteria be available for determination of the limitations to the application of the continuum postulate. In order to obtain them let us introduce the notion of *mean free path*. Under normal conditions of pressure and temperature a gas molecule moves only a short linear distance, called the *molecular free path*, before it collides with another gas molecule. The average value of the free path for an assemblage of molecules is termed the mean free path, denoted by λ.

For a gas to satisfy the continuum postulate, the molecular mean free path must be small compared to a significant characteristic linear dimension L pertinent to the flow field. By definition, the ratio λ/L is termed the **Knudsen number**, and is denoted by Kn. Thus,

$$\mathrm{Kn} = \lambda/L. \tag{1.24}$$

The continuum postulate is applicable to those flows of gas for which the appropriate Knudsen number is less than approximately 0.01. Accordingly, when $\mathrm{Kn} > 0.01$, the gas should be treated as an assemblage of discrete particles.

The continuum postulate, having a purely physical nature, is, from the mathematical viewpoint, equivalent to the assumption that the functions describing the motion of this medium are continuous. This, in turn, makes it possible to apply a relatively simple apparatus of mathematical analysis.

Now a question arises: how, for a continuous object, may its different models, particularly the continuous one, be obtained? In addition which model should be applied for a proper representation of a continuous object?

Generally, the model will be referred to as **continuous** if its mathematical description requires introduction of variables depending not only on time but also on spatial coordinates. This kind of model may be mathematically represented by means of partial differential equations. Continuous models are alternatively called **distributed-parameter models**. If, however, an idealization made at the initial stage of modelling, enables the description of the object be made by means of a finite number of variables, then such a model is referred to as a **discrete** or alternatively a **lumped-parameter** model. For this kind of physical model the mathematical model consists of ordinary differential equations. The process of passing from a continuous object to a discrete model is called **discretization**. The question as to when a continuous object may be represented by a lumped-parameter model without a considerable decrease of exactness of the final results is quite serious, and the answer to this question is not unique. The choice of a physical model made at the very beginning results from the existing structure of the object considered, and from the aim of modelling. However, a proper choice depends also on the experience, sometimes called intuition, of an engineer who deals with a given modelling problem.

To develop one's skills and learn the art of modelling, let us first consider various simple examples of both the discrete and the continuous kinds.

A typical example of a discrete model is the vibratory system of Fig. 1.12. Its mathematical model has the form

$$m\ddot{x} + b\dot{x} + kx = F(t) \tag{1.25}$$

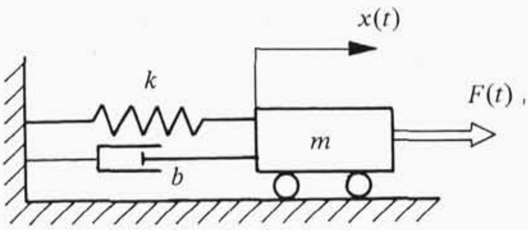


Fig. 1.12.

Another, probably more complicated system, which also may be modelled as a discrete one, is provided by the torsional vibrations of a ship propeller shaft. It can be described to within a close approximation by neglecting the mass of the shaft and replacing the propeller and the turbine by two discs, located at each end of the shaft. The scheme of such a system is shown in Fig. 1.13, while its mathematical model is given by the two following equations:

$$\begin{aligned} I_1\ddot{\theta}_1 + k(\theta_1 - \theta_2) &= 0 \\ I_2\ddot{\theta}_2 - k(\theta_1 - \theta_2) &= M(t), \end{aligned} \tag{1.26}$$

where torsional stiffness $k = GJ/L$.

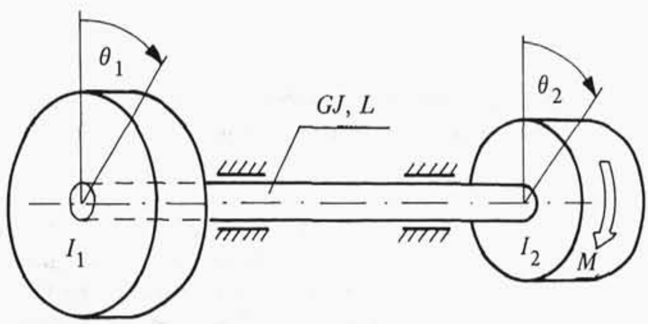


Fig. 1.13.

We would like to point out that although absolutely discrete objects do not exists in reality, investigation of their discrete models provides, in a relatively simple way, important information about the behaviour of objects. Thus, for example, the transmission of force from an unbalanced spring-mounted machine to its foundation may usually be described by the model (1.25) (the machine being considered as a concentrated mass mounted on a simple equivalent spring).

One must recognize the high degree of idealization of a discrete model, for in reality springs and dampers possess some mass, and a mass possesses some deformability and damping capabilities. To realize the influence of the continuous-parameter distribution on both the form of the mathematical model and on some object properties, let us consider a typical example of a continuous model—a cantilever bar excited in vibration by a longitudinal loading $f(x, t)$, which is assumed to vary arbitrarily with position and time (Fig. 1.14). The significant physical properties of this bar are assumed to be Young's modulus E, cross-sectional area $A(x)$, and the mass per unit length $\mu(x)$.

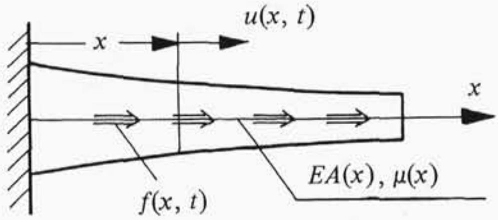


Fig. 1.14.

If damping effects can be neglected, then the equation of motion takes the form (for a derivation see Meirovitch (1967))

$$\mu(x)\frac{\partial^2 u}{\partial t^2}-\frac{\partial}{\partial x}\left[EA(x)\frac{\partial u}{\partial x}\right]=f(x,t), \tag{1.27}$$

where $u(x, t)$ denotes the displacement of the cross-section given by the coordinate x. Assuming in (1.27) $\mu = const$, $A = const$, and $f(x, t) = 0$, we get

$$\mu\frac{\partial^2 u}{\partial t^2}-EA\frac{\partial^2 u}{\partial x^2}=0, \tag{1.28}$$

i.e. the standard model of free longitudinal vibration of an elastic bar, known also as the wave equation. The velocity for longitudinal waves in a bar is thus

$$c=\sqrt{(EA/\mu)}=\sqrt{(E/\rho)}. \tag{1.29}$$

This gives velocities of the order of 5000 ms^{-1} for typical values of E and ρ for steel. This result tells us that any longitudinal disturbance travels along the bar with a finite speed. The finite speed of disturbance propagation is probably the feature in which the discrete and continuous models differ the most. The speed of disturbance propagation resulting from a discrete model is infinite, if quantifiable.

Except for purely discrete or continuous models there are often situations encountered in modelling practice, in which a mixed model provides the best results. Here are several examples:

(1) Smoke stacks with heavy, compact installations, such as a platform or an electrofilter on it. Such a smoke stack can be modelled as a beam (a continuous subsystem) with attached lumped masses (-lumped subsystem) (see Fig. 1.15). The longitudinal vibrations of a liquid fuel propulsion rocket may be investigated by means of the same physical model. Fuel in tanks is modelled as a mass–spring lumped subsystem while a rocket body is often modelled by a beam.

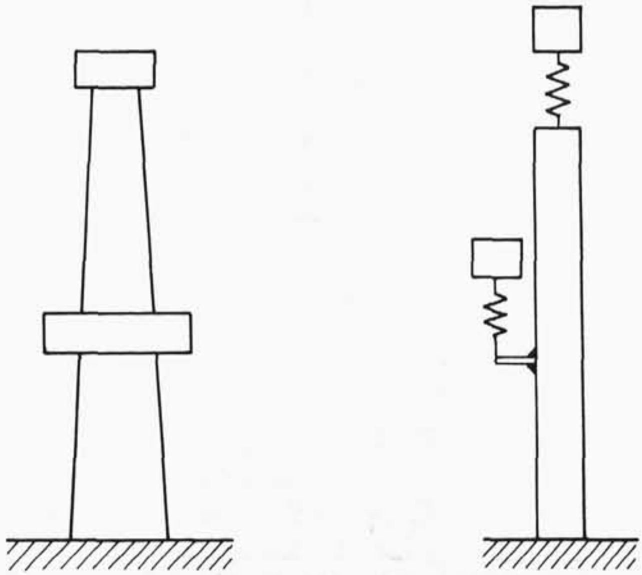


Fig. 1.15.

(2) Antiseismic buildings in regions subject to earthquakes or frequent motions of the rocks forming the Earth's crust are constructed on special elastic foundations, which may be modelled as a spring–dashpot lumped subsystem, while a multistorey building may be modelled as a continuous subsystem—a beam (see Fig. 1.16).
(3) A bridge along which a heavy vehicle moves. The vehicle is usually modelled as lumped system, while the bridge is represented again by a beam (see Fig. 1.17). A similar physical model but with an immovable lumped subsystem may be used for modelling the suspension–turboengine system of a helicopter.

The mathematical models for mixed physical models are either combined ordinary and partial differential equations or integro-differential equations. Both classes of equations present serious difficulties in solving them.

It should be added that both notions—discrete and continuous—may also concern time. This meaning is very popular among automatic engineers where, for instance, the phrase 'continuous model' means a model in which input signals may vary continuously in time. It is therefore better to speak about discrete and continuous mechanical models as models with lumped and distributed parameters, respectively.

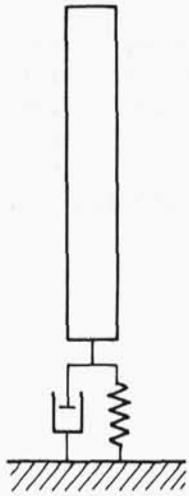


Fig. 1.16.

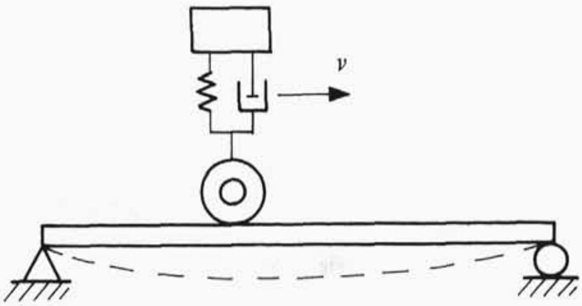


Fig. 1.17.

1.6 STOCHASTIC VERSUS DETERMINISTIC MODELS

We know already from previous considerations that if real-world observations are obviously in contradiction with the solution of a model, then we have ben employing an inadequate model. In this situation we must try to find another, better model. Examples of this type of problem were shown in section 1.4. In the practice of modelling, though, it can happen that this type of consecutive modification does not ultimately yield satisfactory results, and it might be necessary to take up an entirely new approach. This could entail application of an entirely different mathematical tool.

A very instructive example for this situation is provided by the event described by A. C. Hall in Oldenburger (1956). In 1941 MIT and Sperry Company were jointly elaborating a control system of aircraft radiolocator. Hall and his friend worked through the