The latter formula implies that for a given value of k/b the essential influence is exerted on the power transmitted from the coil to the membrane by the parameter (4.254). Thus, the properties of the loudspeaker can be improved through: (1) increase of ϕ_c, i.e. growth of magnetic induction of the permanent magnet, (2) increase of the damping coefficient b, and (3) decrease of resistance R of the coil.

4.4 MODELLING OF NONHOLONOMIC SYSTEMS

4.4.1 Introductory remarks

We introduced in section 2.2.2, after Hertz, the notion of a nonholonomic system. The theory of nonholonomic systems started to develop at the end of the nineteenth century, when it unexpectedly turned out that the wonderful and apparently universal formalism of Lagrange is useless even for simple questions of rolling without slipping of a rigid disc on a plane. As improbable as it may seem, **Joseph Louis Lagrange** (1736–1813) himself did not suspect that such constraints might exist. He set out his belief in his famous *Mécanique Analytique* (1788), in which he states that it is possible, for every mechanical system, to select independent coordinates having independent variations. No exceptions were noticed for many years, until the problem of the rolling rigid bodies without slippage was studied. Recall that Hertz introduced his classification into holonomic and nonholonomic constraints as late as 1894. The development of the theory took a circuit course, with numerous mistakes and errors committed by known exponents of mechanics and mathematics. The series of mishaps lasted until the second half of the 1960s, when the monograph of Neimark and Fufaev (1967) was published, resolving many existing doubts. The present section of this book owes much to that book and in general to the Russian school of mechanics. It is, simultaneously, worth emphasizing that many questions are still subjects of studies. We present here only a well-established apparatus for modelling mechanical systems on which imposed constraints that are linear with respect to velocities. Such constraints have, in generalized coordinates, the form (see also (2.24))

$$\sum_{\sigma=1}^{s} B_{\beta\sigma}(t, q_\sigma)\dot{q}_\sigma + B_\beta = 0, \qquad \beta = 1, \ldots, b, \tag{4.258}$$

where b denotes the number of nonholonomic constraints.

We will see later that the practical modelling of nonholonomic systems reduces mainly to obtaining equation (4.258) and to the determination of the coefficients of this equation. In order to develop certain skills which would then facilitate understanding of analytical mechanics, we will comment on two well-known examples of nonholonomic constraints and we will transform them to the form (4.258). The best-known example is probably that of a billiard ball rolling without slipping on a rough table surface (Fig. 4.17). The location of the ball will be posed by the coordinates x_C and y_C of its centre and the three Euler angles ψ, θ and φ.

The fact that the ball rolls without slipping may be expressed through the statement that $\mathbf{v}_S = \mathbf{0}$, where $\mathbf{v}_S$ is the velocity of the point of the ball in contact with the surface. Since

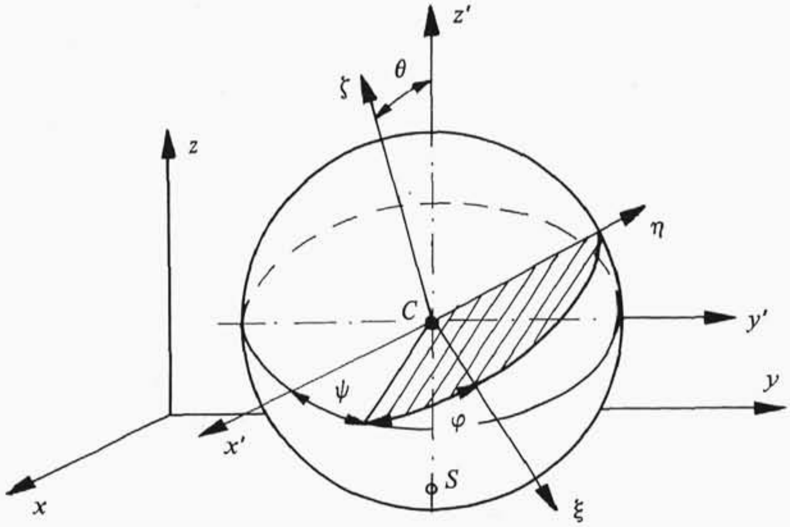

Fig. 4.17.

$$\mathbf{v}_S = \mathbf{v}_C + \boldsymbol{\omega} \times \mathbf{r}_{CS} \tag{4.259}$$

we have that

$$\mathbf{v}_C + \boldsymbol{\omega} \times \mathbf{r}_{CS} = \mathbf{0}, \tag{4.260}$$

where $\boldsymbol{\omega}$ is the instantaneous angular velocity of the ball and $\mathbf{r}_{CS}$ is the radius vector of the point of the ball with the table.

Condition (4.260) could be written as

$$\begin{aligned} \dot{x}_C - r\omega_y &= 0 \\ \dot{y}_C - r\omega_x &= 0 \\ \dot{z}_C &= 0, \end{aligned} \tag{4.261}$$

whence, having accounted for the kinematic equations of Euler and integrating the third equation of the system (4.261), we obtain

$$\begin{aligned} \dot{x}_C - r(\sin\psi)\dot{\theta} + r(\sin\theta\cos\psi)\dot{\varphi} &= 0 \\ \dot{y}_C + r(\cos\psi)\dot{\theta} + r(\sin\theta\sin\psi)\dot{\varphi} &= 0 \\ \dot{z}_C &= r. \end{aligned} \tag{4.262}$$

Hence the motion of the ball is subject to two nonholonomic constraints, expressed by the two first equations of the system (4.262). The third equation represents a holonomic constraints which means that the ball neither jumps during motion nor sinks into the table, so that its centre is always located at a constant distance r from the table.

We determine now the values necessary for presenting equations (4.262) in general notation: $s = 5$, $b = 2$. With this, we have

$$\sum_{\sigma=1}^{s} B_{\beta\sigma}\dot{q}_{\sigma} + B_{\beta} = 0, \qquad \beta = 1,2, \tag{4.263}$$

or, having broken down the summation

$$\begin{aligned} &B_{11}\dot{q}_1 + B_{12}\dot{q}_2 + B_{13}\dot{q}_3 + B_{14}\dot{q}_4 + B_{15}\dot{q}_5 + B_1 = 0, \\ &B_{21}\dot{q}_1 + B_{22}\dot{q}_2 + B_{23}\dot{q}_3 + B_{24}\dot{q}_4 + B_{25}q_5 + B_2 = 0. \end{aligned} \tag{4.264}$$

When we take account of the fact that $q_1 = x_C$, $q_2 = y_C$, $q_3 = \psi$, $q_4 = \theta$ and $q_5 = \varphi$, and compare with (4.262), we get

$$\begin{aligned} &B_{11} = 1, \quad B_{12} = 0, \quad B_{13} = 0, \quad B_{14} = -r\sin\psi, \quad B_{15} = r\sin\theta\cos\psi, \quad B_1 = 0, \\ &B_{21} = 0, \quad B_{22} = 1, \quad B_{23} = 0, \quad B_{24} = r\cos\psi, \quad B_{25} = r\sin\theta\sin\psi, \quad B_2 = 0. \end{aligned} \tag{4.265}$$

The second nonholonomic system which is popular in theoretical mechanics is the so-called *Caratheodory–Chaplygin sledge*. This name refers to a rigid body, mounted on three support points, of which two are ideally smooth and the third is constituted by a sharp straight-line runner directed perpendicularly to the line joining the other two support points (see Fig. 4.18). The existence of this runner is such that the sledge cannot move perpendicularly to it, implying that the velocity of the mass centre $C(x_C, y_C)$ of the sledge is always directed perpendicularly to the runner.

Location of the sledge is determined by three generalized coordinates: x_C, y_C, and the angle φ between the runner and an *a priori* selected direction. The manner of motion of the sledge can be expressed by

$$\frac{\dot{y}_C}{\dot{x}_C} = \tan\varphi, \tag{4.266}$$

whence we have the standard form of the constraint equation, i.e.

$$(\tan\varphi)\dot{x}_C - \dot{y}_C = 0. \tag{4.267}$$

Hence, the sledge has two degrees of freedom ($l = s - b = 3 - 1 = 2$). Equation (4.258) therefore takes the form ($\beta = 1$)

$$B_{11}\dot{q}_1 + B_{12}\dot{q}_2 + B_{13}\dot{q}_3 + B_1 = 0 \tag{4.268}$$

which, after equalization of $q_1 \equiv x_C$, $q_2 \equiv y_C$, $q_3 \equiv \varphi$, yields

$$B_{11} = \tan\varphi, \qquad B_{12} = -1, \qquad B_{13} = 0, \qquad B_1 = 0. \tag{4.269}$$

4.4.2 Lagrange equations of the first kind with multipliers

We believe that the best and clearest method of deriving these equations is based upon the d'Alembert principle in the Lagrangian form, i.e.

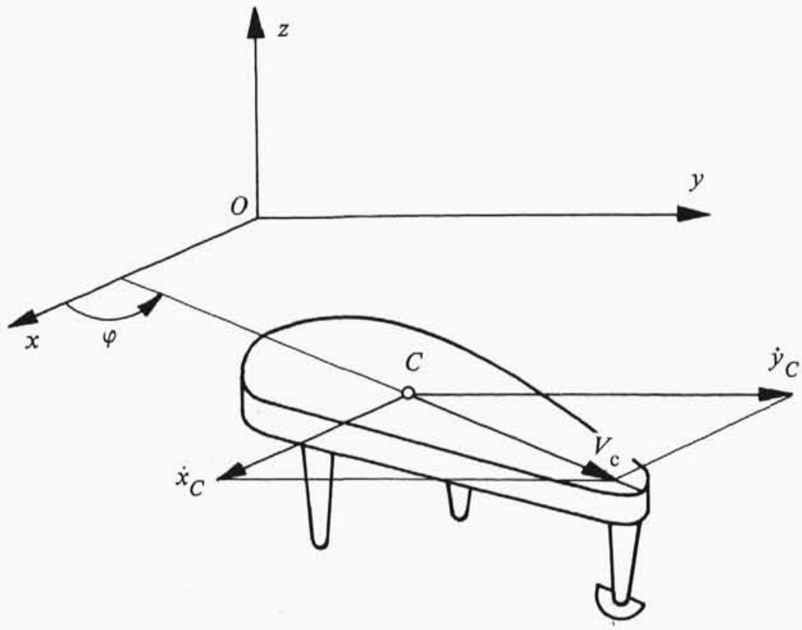

Fig. 4.18.

$$\sum_{\sigma=1}^{s}\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \dot{q}_\sigma}-\frac{\partial T}{\partial q_\sigma}-Q_\sigma\right)\delta q_\sigma=0. \tag{4.270}$$

In this case, in which only holonomic constraints apply, the variations $\delta q_\sigma\,(\sigma=1,\ldots,s)$ are independent and we have obtained the Lagrange equations of the second kind. Now these variations are mutually dependent because of relations (4.258). With the help of the mnemonic rule (see section 4.2.2.2) these relations yield additional equations

$$\sum_{\sigma=1}^{s}B_{\beta\sigma}\delta q_\sigma=0, \tag{4.271}$$

which are fulfilled by variations δq_σ. The situation is as follows: we have s variations δq_σ which are bound by two relations, (4.270) and (4.271); we know that the number of independent variations is equal to the number of degrees of freedom of the mechanical system, i.e. $l=s-b$. A popular method of choice of independent variations is the method of indeterminate Lagrange multipliers. Thus, we multiply relations (4.271) by $\lambda_\beta\,(\beta=1,\ldots,b)$ and sum from 1 to b, yielding

$$\sum_{\beta=1}^{b}\lambda_\beta\sum_{\sigma=1}^{s}B_{\beta\sigma}\delta q_\sigma=0. \tag{4.272}$$

Next, subtracting (4.272) from the sum (4.270) gives the equation

$$\sum_{\sigma=1}^{s}\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}_\sigma}\right)-\frac{\partial T}{\partial q_\sigma}-Q_\sigma-\sum_{\beta=1}^{b}\lambda_\beta B_{\beta\sigma}\right]\delta q_\sigma=0. \tag{4.273}$$

Making use of the arbitrariness of the multipliers λ_β, we choose them in such a manner as to make the contents of the square brackets (multiplying the variations δq_β ($\beta = 1,\ldots, b$)) equal to zero. Since the other variations, i.e. $\delta q_{b+1},\ldots, \delta q_s$ may be regarded as independent, the remaining square brackets corresponding to these variations are also brought to zero. We thus obtain equations for the nonholonomic system in generalized coordinates:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}_\sigma}\right)-\frac{\partial T}{\partial q_\sigma}=Q_\sigma+\sum_{\beta=1}^{b}\lambda_\chi B_{\beta\sigma}, \qquad \sigma=1,\ldots,s, \tag{4.274}$$

which are called the **Lagrange equations of the first kind with multipliers** or simply the **Lagrange equations with multipliers**.

In order to use equations (4.274) in modelling, one must therefore calculate kinetic energy in terms of generalized coordinates, determine generalized forces Q_σ, and identify coefficients $B_{\beta\sigma}$ of the nonholonomic constraints. An example of such an identification was provided in section 4.4.1. The unknowns in equations (4.274) are values of the generalized coordinates q_σ whose number is s and Lagrange multipliers, λ_β, whose number is b. Thus altogether we have $s + b$ unknowns. They can be determined through s equations (4.274) and b constraint relations (4.258). If we recall first that the unknown multipliers λ_β are related to the unknown reactions of nonholonomic constraints, and then we realize that these reactions are not always required, we conclude that the model obtained is not yet the minimal one necessary for describing the motion of a material system, and so we will be looking for a better model.

4.4.3 Maggi equations

The Maggi equations originate also from the d'Alembert principle in Lagrangian form, but they rely upon a different manner of eliminating the dependent variations. **Gian Antonio Maggi** (1856–1937) introduced independent parameters $\dot{e}_1,\ldots,\dot{e}_l$, whose number equals the number of degrees of freedom of the nonholonomic system, and with the help of these parameters he expressed all the generalized velocities, that is

$$\dot{q}_\sigma=\sum_{\lambda=1}^{l}C_{\lambda\sigma}\dot{e}_\lambda+C_\sigma, \qquad \sigma=1,\ldots,\ s, \tag{4.275}$$

where $C_{\lambda\sigma}$ and C_σ are usually functions of variables t and q_σ ($\sigma = 1,\ldots, k$). Magnitudes $\dot{e}_\lambda$ ($\lambda = 1,\ldots,\ l$) are called **kinematic characteristic** or **kinematic parameters**.

Note that relations (4.275) can be always written down if relations (4.258) are valid for quantities q_σ and $\dot{q}_\sigma$, which would mean that nonholonomic constraints exist in this case. In order to illustrate this statement consider the motion of the Caratheodory–Chaplygin sledge, introduced in section 4.4.1. We define as kinematic parameters

$$\dot{e}_1=\dot{x}_C, \qquad \dot{e}_2=\dot{\varphi}, \tag{4.276}$$

and the constraint equations (4.267) can be presented in the form

$$\dot{y}_C = \dot{x}_C \tan\varphi = \dot{e}_1 \tan\varphi. \tag{4.277}$$

Thus equations (4.275) take the form ($s = 3$, $b = 1$, $l = s - b = 2$):

$$\begin{aligned} \dot{q}_1 = \dot{x}_C &= \sum_{\lambda=1}^{2} C_{\lambda 1}\dot{e}_\lambda + C_1 = C_{11}\dot{e}_1 + C_{21}\dot{e}_2 + C_1 \\ \dot{q}_2 = \dot{y}_C &= \sum_{\lambda=1}^{2} C_{\lambda 2}\dot{e}_\lambda + C_2 = C_{12}\dot{e}_1 + C_{22}\dot{e}_2 + C_2 \\ \dot{q}_3 = \dot{\varphi}_C &= \sum_{\lambda=1}^{2} C_{\lambda 3}\dot{e}_\lambda + C_3 = C_{13}\dot{e}_1 + C_{23}\dot{e}_2 + C_3. \end{aligned} \tag{4.278}$$

Having used (4.276) and (2.77) and compared coefficients standing at $\dot{e}_1$ and $\dot{e}_2$ we obtain the following values for the coefficients in expression (4.275):

$$\begin{aligned} &C_{11} = 1, && C_{21} = 0, && C_1 = 0, \\ &C_{12} = \tan\varphi, && C_{22} = 0, && C_2 = 0, \\ &C_{13} = 0, && C_{23} = 1, && C_3 = 0. \end{aligned} \tag{4.279}$$

On the basis of (4.275) we have

$$C_{\lambda\sigma} = \frac{\partial \dot{q}_\sigma}{\partial \dot{e}_\lambda}, \qquad \lambda = 1,\ldots,l, \quad \sigma = 1,\ldots,s, \tag{4.280}$$

and, using the mnemonic rule (see section 4.2.2.2), we get

$$\delta q_\sigma = \sum_{\lambda=1}^{l} \mathrm{X}_{\lambda\sigma}\delta e_\lambda, \qquad \sigma = 1,\ldots,s, \tag{4.281}$$

which, considering (4.280), yields

$$\delta q_\sigma = \sum_{\lambda=1}^{l} \frac{\partial \dot{q}_\sigma}{\partial \dot{e}_\lambda}\delta e_\lambda. \tag{4.282}$$

Once we introduce relations (4.282) to equation (4.275) and group expressions corresponding to respective variations δe_λ, we obtain

$$\sum_{\lambda=1}^{l}\left[\sum_{\sigma=1}^{s} C_{\lambda\sigma}\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \dot{q}_\sigma} - \frac{\partial T}{\partial q_\sigma}\right)\right]\delta e_\lambda = \sum_{\lambda=1}^{l}\left(\sum_{\sigma=1}^{s} C_{\lambda\sigma}Q_\sigma\right)\delta e_\lambda, \tag{4.283}$$

whence, on the basis of independence of δe_λ, we finally get the **Maggi equations**

$$\sum_{\sigma=1}^{s} C_{\lambda\sigma}\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \dot{q}_\sigma} - \frac{\partial T}{\partial q_\sigma}\right) = \phi_\lambda, \qquad \lambda = 1,\ldots,l, \tag{4.284}$$

where

$$\phi_\lambda = \sum_{\sigma=1}^{s} C_{\lambda\sigma} Q_\sigma. \tag{4.285}$$

These linear combinations of generalized forces Q_σ will be called **modified generalized forces**.

We see, therefore, that the Maggi method provides us with the possibility of eliminating the indeterminate Lagrange multipliers λ_β ($\beta = 1,\ldots, b$), due to which the number of unknowns decreased by b, that is, by the number of nonholonomic constraints. But, while we are dealing with l Maggi equations, the number of unknowns is s, and this is the number of generalized coordinates. Hence, we must add s relations (4.275) to the Maggi equations, and these additional relations introduce l unknowns $\dot{e}_\lambda$. Together, there are thus $s + l$ equations for determining $s + l$ unknowns $q_1,\ldots, q_s$ and $\dot{e}_1,\ldots,\dot{e}_l$.

Making use of equations (4.284) requires only determination of the coefficients $C_{\lambda\sigma}$ standing at the kinematic parameters, which is easy (see example), as well as standard calculation of kinetic energy and generalized forces.

4.4.4 The Gibbs–Appell equations

The question arises of whether it is possible to further reduce the number of equations below the $s + l$ attained in the Maggi method. This turns out to be impossible. We shall, however, give yet another set of equations because of its rare advantage: it provides the most concise form of equations of motion in all mechanics. There are various methods of deriving the Gibbs–Appell equations. The shortest, and simultaneously the clearest, method is based upon Gauss's principle (see section 4.2.3.6).

Recall that the constraint has, by definition, the form

$$Z = \tfrac{1}{2}\sum_{\nu=1}^{n} m_\nu\left(\mathbf{w}_\nu - \frac{\mathbf{F}_\nu}{m_\nu}\right)^2. \tag{4.286}$$

After squaring and introducing the notation

$$S = \tfrac{1}{2}\sum_{\nu=1}^{n} m_\nu \ddot{\mathbf{r}}_\nu^2, \tag{4.287}$$

formula (4.286) can be transformed into

$$Z = S - \sum_{\nu=1}^{n} \mathbf{w}_\nu \mathbf{F}_\nu + \tfrac{1}{2}\sum_{\nu=1}^{n} \frac{1}{m_\nu}\mathbf{F}_\nu^2. \tag{4.288}$$

The magnitude defined by formula (4.287) is called the **acceleration energy** (or the **Appell function**) and it plays a key role in the Gibbs–Appell equations.

To conform with the essence of the Gauss principle, we will consider only terms containing acceleration and we will therefore take into consideration the second term in formula (4.288). Transforming this in such a way as to introduce generalized coordinates and following the requirement (2.21)

$$\mathbf{r}_\nu = \mathbf{r}_\nu(t, q_\sigma), \qquad \nu = 1,\ldots,n, \quad \sigma = 1,\ldots,s, \tag{4.289}$$

gives

$$\mathbf{v}_\nu = \frac{\partial \mathbf{r}_\nu}{\partial t} + \sum_{\sigma=1}^{s} \frac{\partial \mathbf{r}_\nu}{\partial q_\sigma} \dot{q}_\sigma. \tag{4.290}$$

Differentiating (4.290) with respect to time gives

$$\mathbf{w}_\nu = \frac{\mathrm{d}\mathbf{v}_\nu}{\mathrm{d}t} = \frac{\partial^2 \mathbf{r}_\nu}{\partial t^2} + \sum_{\sigma=1}^{s} \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathbf{r}_\nu}{\partial q_\sigma}\right)\dot{q}_\sigma + \sum_{\sigma=1}^{s} \frac{\partial \mathbf{r}_\nu}{\partial q_\sigma} \ddot{q}_\sigma, \tag{4.291}$$

which can be transformed into

$$\mathbf{w}_\nu = \sum_{\sigma=1}^{s} \frac{\partial \mathbf{r}_\nu}{\partial q_\sigma} \ddot{q}_\sigma + g, \tag{4.292}$$

where $\mathbf{g}$ denotes an expression independent of $\ddot{q}_\sigma$, i.e.

$$\mathbf{g} = \frac{\partial^2 \mathbf{r}_\nu}{\partial t^2} + \sum_{\sigma=1}^{s} \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathbf{r}_\nu}{\partial q_\sigma}\right)\dot{q}_\sigma. \tag{4.293}$$

Finally the announced transformation of the term

$$\begin{aligned}\sum_{\nu=1}^{n} \mathbf{w}_\nu \cdot \mathbf{F}_\nu &= \sum_{\nu=1}^{n}\left(\sum_{\sigma=1}^{s} \frac{\partial \mathbf{r}_\nu}{\partial q_\sigma}\ddot{q}_\sigma + \mathbf{g}\right)\cdot \mathbf{F}_\nu \\ &= \sum_{\sigma=1}^{s}\left(\sum_{\nu=1}^{n} \frac{\partial \mathbf{r}_\nu}{\partial q_\sigma}\mathbf{F}_\nu\right)\ddot{q}_\sigma + \sum_{\nu=1}^{n} \mathbf{g}\cdot\mathbf{F}_\nu\end{aligned} \tag{4.294}$$

together with the definition of generalized force (4.110) results in

$$\sum_{\nu=1}^{n} \mathbf{w}_\nu \cdot \mathbf{F}_\nu = \sum_{\sigma=1}^{s} Q_\sigma \ddot{q}_\sigma + \sum_{\nu=1}^{n} \mathbf{g}\cdot\mathbf{F}_\nu. \tag{4.295}$$

The subsequent step to be taken on the way to obtaining the final equations consists of the fact that nonholonomic constraints are still in for the form (4.258). An expression of this is relation (4.275), from which, after differentiation with regard to time, we get

$$\ddot{q}_\nu = \sum_{\lambda=1}^{l} C_{\lambda\sigma}\ddot{e}_\lambda + h, \qquad \sigma = 1,\ldots,s, \tag{4.296}$$

where h denotes the magnitude independent of $\ddot{e}_\lambda$.

Having introduced (4.296) into (4.295), we get

$$\sum_{\nu=1}^{n} \mathbf{w}_\nu \cdot \mathbf{F}_\nu = \sum_{\sigma=1}^{s} \left(Q_\sigma \sum_{\lambda=1}^{l} C_{\lambda\sigma} \ddot{e}_\lambda + h \right) + \sum_{\nu=1}^{n} \mathbf{g} \cdot \mathbf{F}_\nu \tag{4.297}$$

and, taking account of notation (4.285), we obtain

$$\sum_{\nu=1}^{n} \mathbf{w}_\nu \cdot \mathbf{F}_\nu = \sum_{\lambda=1}^{l} \phi_\lambda \ddot{e}_\lambda + \sum_{\sigma=1}^{s} h + \sum_{\nu=1}^{n} \mathbf{g} \cdot \mathbf{F}_\nu. \tag{4.298}$$

Now we are already able to present the constraint (4.298) in a form which is convenient for application of Gauss's principle. After introduction of (4.298) to (4.288) we obtain

$$Z = S - \sum_{\lambda=1}^{l} \varphi_\lambda \ddot{e}_\lambda - \sum_{\sigma=1}^{s} h + \sum_{\nu=1}^{n} \left(\frac{1}{2m_\nu} \mathbf{F}_\nu^2 - \mathbf{g} \cdot \mathbf{F}_\nu \right), \tag{4.299}$$

from which one can easily see that of all terms, only the first two depend upon accelerations (because S depends upon $\ddot{\mathbf{r}}_\nu$). If these quantities are treated as the ones which can be subject to variation then, by virtue of Gauss's principle ($\delta Z = 0$) and through application of the heuristic principle, we get

$$\delta Z = \left(\frac{\partial S}{\partial \ddot{e}_\lambda} - \phi_\lambda \right) \delta \ddot{e}_\lambda = 0, \tag{4.300}$$

which, in view of the independence of variations $\delta \ddot{e}_\lambda$, immediately gives

$$\frac{\partial S}{\partial \ddot{e}_\lambda} = \phi_\lambda, \qquad \lambda = 1, \dots, l. \tag{4.301}$$

These are the **Gibbs–Appell equations**. They contain $s + l$ (or $2s - b$, since $l = s - b$) unknowns: $q_1, \dots, q_s$ and $\ddot{e}_1, \dots, \ddot{e}_l$. Insofar as the number of Gibbs–Appell equations is l, we must complement them with s relations (4.275), which together gives the complete system of first-order differential equations.

To use equations (4.301), one would need the Appell function S. We shall therefore provide a useful formula, being an analogue of the known formula for kinetic energy (see (3.79)). Since the method of derivation is the same, we will not quote it here for the sake of brevity. The formula in question has the form

$$S = \tfrac{1}{2} M w_C^2 + S_r, \tag{4.302}$$

where

$$S_r = \tfrac{1}{2} \sum_{\nu=1}^{n} m_\nu w_\nu^2 \tag{4.303}$$

is the acceleration energy in relative rotational motion.

Note that the application of equation (4.301) requires expression of the acceleration energy through kinematic parameters $\ddot{e}_\lambda$ ($\lambda = 1,\ldots,l$). This can be done with the help of relation (4.296).

4.4.5 Case studies

4.4.5.1 Constant speed drive

In order to show the manner in which these three kinds of equations function we shall apply them, consecutively, to modelling of a mechanism which was first analysed by V. S. Novoselov (see Novoselov (1957)). This mechanism is shown in Fig. 4.19. Its purpose is to transmit the rotation of a driving shaft 1 (motor) to a drive shaft 2 (machine) by means of a disc mounted on the roller 3 so that it is free to rotate, and to have the speed of the driven shaft remain sensibly constant even though that of the driving shaft is not. The principle of functioning is as follows:

The vertical driving shaft has a rigidly attached horizontal disc. A intermediate horizontal shaft 3 has a thin disc of radius a. The disc can translate along its axis of rotation in both directions: to the left (towards the centre of the horizontal disc) owing to a centrifugal governor, and to the right owing to the spring with the stiffness k_3. The disc

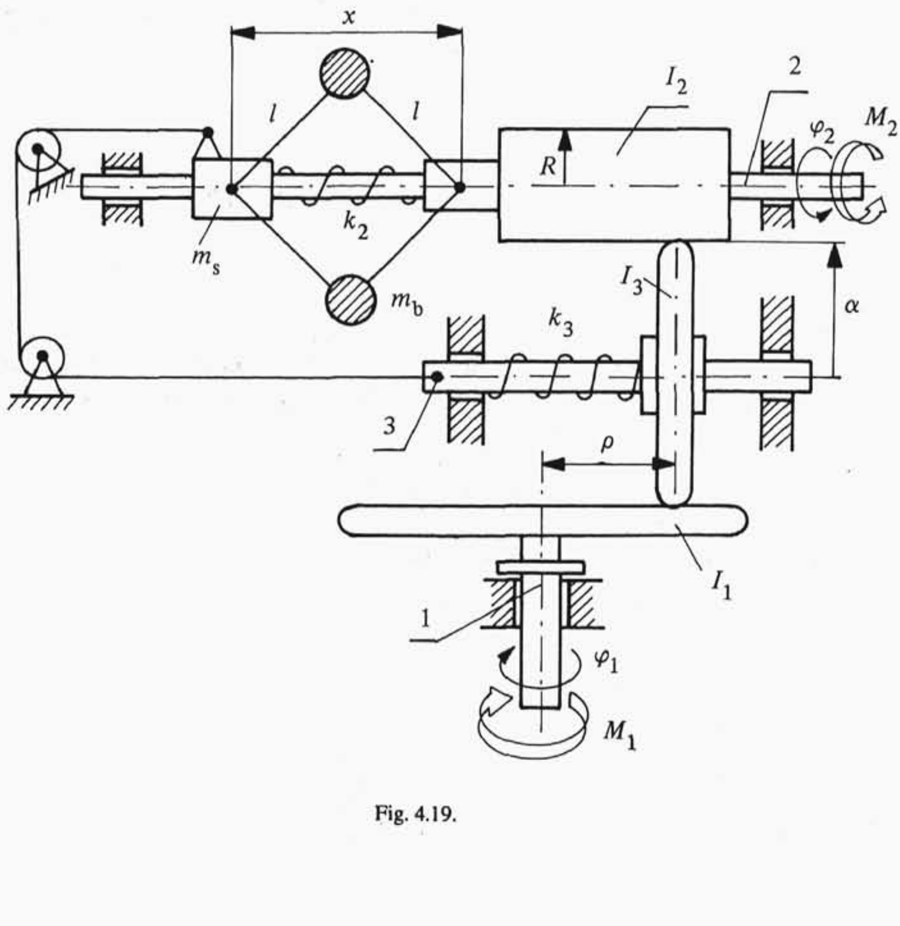

Fig. 4.19.