in time. It is therefore better to speak about discrete and continuous mechanical models as models with lumped and distributed parameters, respectively.

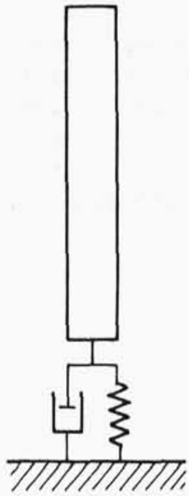

Fig. 1.16.

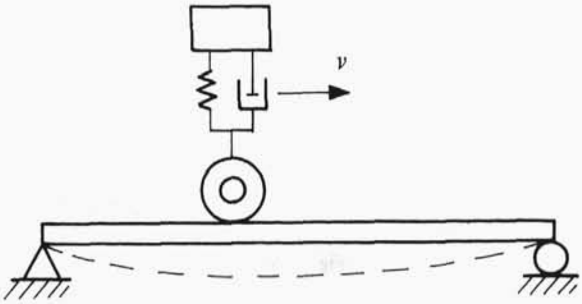

Fig. 1.17.

1.6 STOCHASTIC VERSUS DETERMINISTIC MODELS

We know already from previous considerations that if real-world observations are obviously in contradiction with the solution of a model, then we have ben employing an inadequate model. In this situation we must try to find another, better model. Examples of this type of problem were shown in section 1.4. In the practice of modelling, though, it can happen that this type of consecutive modification does not ultimately yield satisfactory results, and it might be necessary to take up an entirely new approach. This could entail application of an entirely different mathematical tool.

A very instructive example for this situation is provided by the event described by A. C. Hall in Oldenburger (1956). In 1941 MIT and Sperry Company were jointly elaborating a control system of aircraft radiolocator. Hall and his friend worked through the whole day, but they were disappointed, for although they had constructed a good study system, they completely forgot the importance of noise, so that their system was unstable and totally unsatisfactory. It is only when noise was taken into account that the system could be made stable, while ensuring that the output dispersion was smaller by an order of magnitude. It took three months to make all these improvements.

Thus, consideration of noise leads us to this new branch of mathematics, which is termed theory of random processes. It is, however, not our goal to present this theory here, and we think that a comprehensive treatment of the modelling problem for complex mechanical systems cannot neglect the effects which influence the overall behaviour of systems in such a decisive manner, as described by Hall.

In order to realize the difference between the deterministic and stochastic approaches let us consider two examples. The first one will be constituted by a compound pendulum, whose motion is described by the relation: $\varphi = A\cos(\omega_n t + \beta)$, where $\omega_n = \sqrt{(mgl/J_0)}$ (see (1.15)). Assume that pendulum was inclined from the equilibrium position by the angle φ_0, and then let go freely at the time $t = 0$. We have then

$$\varphi(t) = \varphi_0 \cos \omega_n t, \tag{1.30}$$

whence we see that the inclination of the pendulum is strictly determined for any time t in future. It should be emphasized that, now and at any specific future moment, this is true only when $\varphi_0 = const$, and $\omega_n = const$, which means that the parameters of pendulum, (m, l, J_0) are also constant. Because of the fact that the inclination of the pendulum, given the assumptions made, can be strictly described, we say that the motion of the pendulum is a deterministic phenomenon. All the models considered up till now are of this character.

Another example is provided by the so-called **Brownian motion**. In 1827, the English botanist **Robert Brown** (1773–1858) observed that particles suspended in a liquid perform particularly irregular movements. This phenomenon is now referred to as Brownian motion. The kinetic theory of matter explains the mechanism of this phenomenon in the following way: particles change their state because of impacts with particles of liquid, which are in chaotic motion. This impact chaos causes the state of the observed particle to be a random magnitude, or—as it is referred to in probabilistic theory—a random variable. Even knowing the state of a particle at the initial instant does not ensure the capacity to predict its exact state in the future. We can at most try to give the probability of the particle's being in a certain spatial area. Thus, we are saying that Brownian motion is a random phenomenon.

Now let us present some situations which necessitate the construction of a **stochastic model**.

We start with noise, since it was with noise that our considerations of stochastic effects began. Basically, there are two important types of noise: *shot noise* and *thermal noise*. Shot noise is present in both vacuum tubes and semiconductor devices. In vacuum tubes shot noise arises due to random emission of electrons from the cathode. In semiconductor devices this effect arises because of the random diffusion of majority carriers and because of the random generation and recombination of the hole–electron pairs. The second type of noise—thermal noise—arises due to the random motion of free electrons in a conducting medium such as a resistor. Each free electron inside a resistor is in motion due to its thermal energy. The motion of an electron is random and it zigzags due to collisions with the lattice structure. We thus see that both types of noise are very similar in nature to Brownian motion.

Although noise is, as it appears to us, a good example for a random phenomenon, a mechanical engineer is dealing with such phenomena usually in situations of measurement. Let us now then give some instances in which the nature of phenomena is random.

(1) *Missile flight.* It is not possible to strictly determine the actual location where a missile hits, since many factors influence the missile trajectory: imprecision of aiming, dispersion of powder mass in the missile, speed and direction of wind, humidity and temperature of air. In some cases these factors are treated as deterministic, as for example in classical external ballistics, but this is just a concession caused by the difficulties resulting from randomness.

(2) *Driving or taxiing of a vehicle.* Unevenness of the surface entails continuously changing accelerations in various elements of the vehicle (a car on the road, an aircraft on the runway). If the surface irregularities could be described in a deterministic manner (e.g. as a kind of harmonic excitation), then we would be able to calculate relatively easily the resulting acceleration (using traditional methods of theory of vibrations). But real-life surface profiles are by their very nature random.

(3) *Earthquake.* Recurring tragedies caused by this terrible phenomenon indicate clearly the inability to predict its occurrence. Thus, the very occurrence of an earthquake is a random event of energy release in a certain point of the Earth's crust. Secondly, movements of the ground, on which buildings are founded, constitute a random phenomenon because of multiple diffraction and interference of seismic waves. This, in turn, results from the extreme variability of the structure of the Earth's crust. In spite of the obviously random character of this phenomenon there have been efforts at a deterministic treatment of its appearance and course.

Many more examples could be given, but it may be more useful to provide a short comment. There could be an interest in, or even a doubt about, the proper basis on which certain phenomena are considered random. The question is even more valid than initially surmised, since in the three examples cited, deterministic approaches parallel the stochastic ones. In our opinion, the fundamental significance in distinguishing the deterministic and probabilistic relevance lies in experiment. Thus, the decision on the selection of attributes for a given phenomenon is made depending on whether or not the phenomenon may be repeated under experimental conditions. Indeed, the question is how to proceed when experiment is yet far ahead and its details unknown (for instance, we are only designing a shock absorber for a car to be driven on uneven roads, or for an aircraft which will be taxiing over irregular runways). In such cases, typical for modelling, one can only use the following advice: it is necessary *a priori* to already have a broad knowledge of the phenomena which we would like to study with the help of models. Such knowledge is of course provided by practice, but experiment confirms and deepens it. Experiment often also indicates new effects which have to be considered in description, in order for the results obtained to have practical significance. This is the inspiring role of experiment, a very important role, sometimes underestimated by the theoreticians.

When starting to model a phenomenon one uses to a large extent the results of experimental studies.

The purpose of the above comments is to note that the overlapping of theoretical and experimental studies and to emphasize the need for mutual interaction of both types of studies. Let us now present a list of phenomena whose modelling accounted for the above remarks and involved random effects:

— aircraft flight in a turbulent atmosphere;
— rolling of a ship on irregular waves;
— dynamic load of engineering constructions (towers, masts, bridges, smoke stacks) due to gusty winds;
— laminar flow of liquids in tubes with rough walls;
— variable load of aircraft constructions in conditions of fluctuating pressure in reaction engines;
— river flow accounting for precipitation and evaporation;
— destruction due to fatigue in deformable bodies.

This list is by no means complete, but it is sufficiently rich to persuade doubters to treat seriously the alternative approach—in similar cases the deterministic approach may not be adequate and acceptance of the stochastic approach should be considered a real possibility. Transition from one approach to the other requires the breaking of some habits related to the traditional manner of formulating the problems of mechanics. This might even turn out more important than the need to grasp the concepts of the theory of random processes.

Consequently, mathematical description of physical phenomena in terms of suitable equations for appropriate functions must be modified in order to account for stochastic effects. This can be achieved by regarding the values of these functions not as deterministic variables, but rather as random variables with associated probabilistic, i.e. stochastic properties. Such a change of interpretation corresponds to the change of the point of view from the **deterministic** to the stochastic mathematical model.

Because of the introductory nature of the present section we shall not be dealing here directly with stochastic modelling in phenomena. As in deterministic modelling, causal and empirical models can be distinguished, as well as discrete and continuous ones. Besides that, in order to better understand the final conclusions, let us state that it is convenient to distinguish three basic types of stochastic differential equations according to the form in which random elements enter the equations:

(1) random initial conditions,
(2) random forcing functions,
(3) random coefficients.

These three types are not mutually exclusive, and most of the technical situations involve equations of mixed type (see the exemplary situations described, notably 'missile flight').

Finally, we shall consider in more detail a situation in which a deterministic model is wrong (see section 1.2.5). For this purpose consider the simplest stochastic model of the third type, i.e.

$$\frac{dx}{dt} = [a + b(t)]x(t), \tag{1.31}$$

where $x(t)$ is a random magnitude of zero mean, while $b(t)$ is so-called white noise (see Bendant and Piersol (1971)).

Note that if $b(t) \equiv 0$, we are dealing with the usual deterministic model. This corresponds to the situation when we are disregarding small fluctuations around the mean value given by the expression

$$\hat{x} = x_0 \exp(at). \tag{1.32}$$

It can be shown that variance of x is expressed as

$$D^2(x) = x_0^2 \exp(2at)[\exp(\sigma^2 t) - 1], \tag{1.33}$$

where σ^2 is variance of the white noise $b(t)$. On the basis of (1.32) and (1.33) we get

$$\frac{\sqrt{D^2(x)}}{\hat{x}} = \sqrt{[\exp(\sigma^2 t) - 1]} \tag{1.34}$$

which means that deviations of x from the mean value increase with time. In other words the deterministic model

$$\frac{dx}{dt} = ax \tag{1.35}$$

is not stable. It seems that this is the way to explain the error depicted earlier by Hall; thus it can be concluded that in some situations random factors must of necessity be taken into account.

Finally it would be appropriate to note the question of the nature of the laws of physics, but without taking up philosophical arguments. The direct question is: are laws of physics deterministic or stochastic? The followers of the concept of random nature of phenomena tend to agree with **Richard Bellman**, who said that 'deterministic models are very useful, but stochastic ones are more real'. On the other hand the adversaries of the randomness concept repeat after **Albert Einstein** that 'der Alte würfelt nicht' (the Old One does not toss dice). We, as the authors of a book on modelling, think that resolution of this argument, very important in the cognitive aspect, is interesting for a mechanical engineer insofar as it facilitates reasonable modelling for the purposes of providing a proper solution to the practical problem he faces. Thus, for instance, an engineer is rather indifferent as to whether reliability of an aircraft over a given period of use is random or deterministic, provided he is able to establish precisely the states of stress in the craft, and thus ensure the avoidance of faulty functioning during use. Yet another example: when analysing wave phenomena a situation is considered in which a wave encounters an obstacle. The most important difference between smooth and rough surfaces is that a smooth surface reflects the wave, while a rough one disperses it. However, this depends also upon the wavelength, so that a non-polished metal surface is smooth for radio waves, but is rough for light. In the latter case a stochastic description is necessary. The nature of a given phenomenon is decided upon on the basis of the purpose of modelling, as we have already emphasized many times. Generally speaking, we recommend that problems in which knowledge of the phenomena occurring is scanty or imprecise should be treated by an engineer with special care and analysed against the background of wider knowledge.

1.7 MODELS RELATED TO THE DIFFERENTIAL MODEL

On the basis of examples given hitherto it might seem that mathematical models of dynamical phenomena consist of ordinary or partial differential equations. Since this impression might be reinforced by later chapters of the book, in which we shall be obtaining just this type of model, we would like to present now some examples of formulation of models other than the **differential** ones.

Equations other than differential ones may arise from two causes. First, they can be obtained through expression of the essence of the nature of the phenomenon described (see Examples 1.1. and 1.2). Second, they can appear through transformations of the original differential equation into other forms (see Example 1.3). In other words, one could say that in the first case we have to abandon the differential form of the model, while in the second case, we want, for some reason, to substitute some other form, usually more convenient for qualitative analysis or numerical solving, for the differential one.

Example 1.1. Self-induced oscillations in machining with a lathe. Let us first consider an example of a problem leading to the formulation of a differential equation with a delayed argument. In the processes of machining with machine tools (polishing, milling, turning) unwanted oscillations are often observed, due to which the surface of the object of machining becomes undulating. Dynamic phenomena which take place during cutting are very complicated. It is, however, sufficient to employ a relatively simple physical model in order to obtain a rough explanation for the oscillations. In order to focus attention consider cutting on a turning machine; assume that the object subject to machining is revolving on a rigid bearing, while the cutting edge is moving with respect to this object with a constant speed and is perfectly rigid in this direction. The cutter can undergo deformations and oscillate only in the direction perpendicular to the longitudinal axis of the object subject to machining (see Fig. 1.18).

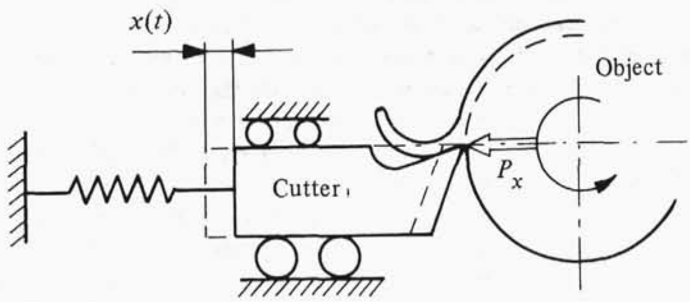

Fig. 1.18.

The analysis of the process of cutting leads to the conclusion that the forces of cutting depend not only upon the mutual motion of cutter and object in a given instant, but also upon their mutual position one revolution before. It can therefore be said that there is a feedback between the cutter position x and cutting force P_x. This feedback is brought about via the delay element constituted just by the surface being machined, which 'remembers', through the shape, for the period of one revolution of the object, its displacement, and after this period introduces this magnitude back to the process in the form of the variable thickness of the cutting layer, and hence also the cutting force P_x. This type of turning is defined in the literature as *regenerative chatter*. When this image of the machining process is accepted, the thickness of the layer being currently cut is expressed as

$$a(t) = a_0 + \Delta a = a_0 + x(t) - \lambda x(t - \tau), \tag{1.36}$$

where a_0 is the thickness of the layer taken off when the cutter does not oscillate, $\lambda \in [0, 1]$ is the coefficient of overlapping of machining traces, $x(t)$ and $x(t - \tau)$ are the positions of the cutter at current time t and at the time earlier by one revolution, $t - \tau$.

The cutter is loaded by three forces: the cutting force P_x, the elasticity force exerted by the cutter–carriage system, which will be modelled as kx, and the sum of various resistance and damping forces depending on cutter velocity in the x-direction, which can be expressed as $b\dot{x}$. Because cutting force depends, first of all, upon the thickness of the cutting layer, the equation representing the cutter oscillations can be represented as

$$m\ddot{x} + b\dot{x} + kx = P_x[a(t)], \tag{1.37}$$

where m is the reduced mass of the cutter–carriage system.

After linearization of the force $P_x[a(t)]$ with respect to x, accounting for (1.36) and substituting into (1.37), this equation takes the form

$$m\ddot{x} + b\dot{x} + (k - P_x')x + \lambda P_x' x(t - \tau) = 0, \tag{1.38}$$

where $P_x' = \mathrm{d}P_x \big/ \mathrm{d}x\big|_{a_0}$. The force referred to as $P_x[a(t)]$ is produced due to the revolving motion of the object (or, more precisely, the drive of the lathe). In the linear form presented above, this force was broken down into two components: one $P_x' x(t)$, analogous to the 'ordinary' elasticity force $kx(t)$, and the other is $\lambda P_x' x(t - \tau)$, which can be referred to as *action with delay*. It is just this component of the cutting force that can increase the energy of oscillations of the cutter–carriage system. And this energy inflow appears through instability. The energy is provided to the system by the revolving object. Oscillations of this type are called *self-induced oscillations*.

Equations of the form of (1.38), which are characterized by the fact that the unknown magnitude depends not only upon t but also upon $t - \tau$ are called **differential equations with delayed argument**. They appear everywhere where we do not neglect the delays. Typical delays are the ones arising from transport phenomena (such as in the mixing of liquids with various concentrations), as well as information delays (e.g. transmission of signals from the control column to the helm in the automatic stabilization of a ship's course).

Example 1.2. Fundamental equation of finite-wing theory. Let us consider now a problem, which leads to an **integrodifferential equation**. The main problem of finite-wing theory is determination of the spanwise distribution of airloads on a wing of given characteristics (e.g. geometry) that is flying with a given speed and orientation in space. This airload can be determined on the basis of the known distribution of circulation $\Gamma = \Gamma(y)$, where y is the axis along the wing (see Fig. 1.19). The circulation of a velocity is a linear integral of the velocity vector **v** along an arbitrary closed curve.

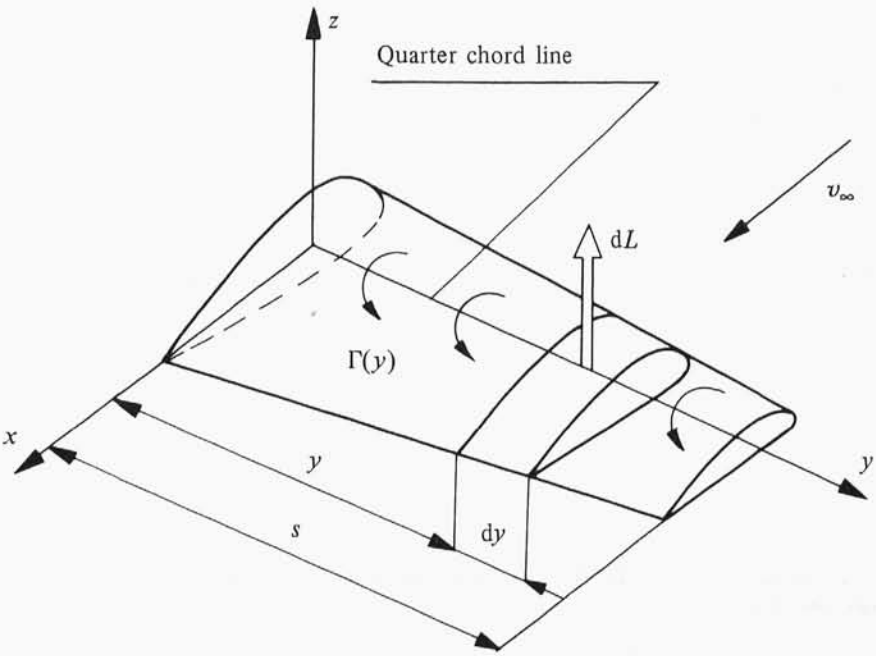

Fig. 1.19.

The first rational attempt at predicting circulation distribution on subsonic, three-dimensional wings was a method due to **Prandtl**, which was especially adapted, in an approximate sense, to the large aspect-ratio, unswept platforms prevalent during the early twentieth century. From the standpoint of the aerodynamicist, practically all important characteristics of these wings are predicted by the so-called *thin airfoil* theory. This means that the section of the finite wing behaves just as though it were a section of an infinite wing. Thus the flow about each unit span of the finite wing is two-dimensional and the force experienced by every differential element dy of the wing is given by

$$\mathrm{d}L = \rho_\infty \Gamma(y) v_\infty \mathrm{d}y \tag{1.39}$$

or

$$\mathrm{d}L = \tfrac{1}{2}\rho_\infty v_\infty^2 \mathrm{c}(\mathrm{y}) \mathrm{C_L}(\mathrm{y})\, \mathrm{d}y, \tag{1.40}$$

where ρ_∞ and v_∞ are the air density and the velocity of undisturbed flow, respectively, $c(y)$ is the local chord of the wing, and $C_L(y)$ is the local lift coefficient.

By comparing (1.39) and (1.40) we get

$$\Gamma(y) = \tfrac{1}{2}c(y)v_\infty C_L(y), \tag{1.41}$$

i.e. the relation between circulation of a velocity and the lift coefficient at the section y of the wing. According to the two-dimensional flow model we have

$$C_L(y) = \frac{dC_L}{d\alpha}\alpha_e, \tag{1.42}$$

where α_e is the effective incidence of the wing at the y-section. Let the geometric incidence at the same section be $\alpha(y)$, which may vary across the span. Both these angles be measure, not from the free stream velocity vector, but from the local zero-lift angle. Then

$$\alpha_e(y) = \alpha(y) - \alpha_i(y), \tag{1.43}$$

where $\alpha_i(y)$ is the induced incidence, caused by the induced velocity $\mathbf{v}_i$. A simple application of the **Biot–Savart law** (Milne-Thomson (1948)) yields

$$v_i(y) = \frac{1}{4\pi v_\infty}\int_{-s}^{s}\frac{d\Gamma/d\eta}{\eta - y}\,d\eta. \tag{1.44}$$

We assume that v_i is small compared with v_∞, so that

$$\alpha_i = \frac{v_i}{v_\infty}. \tag{1.45}$$

If we now insert (1.42)–(1.45) into (1.41) we obtain the following integrodifferential equation for $\Gamma(y)$:

$$\Gamma(y) = \tfrac{1}{2}c(y)v_\infty\frac{dC_L}{d\alpha}\left(\alpha(y) - \frac{1}{4\pi v_\infty}\int_{-s}^{s}\frac{d\Gamma/d\eta}{\eta - y}\,d\eta\right). \tag{1.46}$$

This equation enables determination of circulation distribution along the wing, whose form is known, if we know both the function $dC_L(y)/d\alpha$, i.e. airfoil distribution along the wing, and the function $\alpha(y)$.

Example 1.3. Transformation of the standard model of control theory into an integral model. The standard model of control theory is a differential model of the form

$$\dot{\mathsf{x}} = \mathsf{A}(t)\mathsf{x} + \mathsf{B}(t)\mathsf{u} \tag{1.47}$$

where x and u are vectors of state and control, respectively, while A and B are matrices of state and control, respectively. The problem is to determine the control vector which ensures a satisfactory course of the process as described by model. There are several methods of solving this problem. One of them consists in obtaining the integral form, to which the differential form (1.47) can be transformed.

Assume that the initial state of the system is given:

$$\mathrm{x}(t_0)=\mathrm{x}_0 \tag{1.48}$$

together with the terminal state

$$\mathrm{x}(t_f)=\mathrm{x}_f. \tag{1.49}$$

The general solution to equation (1.47) has the form (see, e.g., Hartman (1964)):

$$\mathrm{x}(t)=\mathrm{X}(t)\mathrm{x}_0+\mathrm{X}(t)\int_{t_0}^{t}\mathrm{X}^{-1}(\tau)\mathrm{B}(\tau)\mathrm{u}(\tau)\,\mathrm{d}\tau, \tag{1.50}$$

where $\mathrm{X}(t)$ is the normed fundamental matrix of the uncontrolled system, i.e.

$$\dot{\mathrm{X}}=\mathrm{A}(t)\mathrm{X}. \tag{1.51}$$

After premultiplication of (1.50) by $\mathrm{X}^{-1}(t)$ we obtain

$$\mathrm{X}^{-1}(t)\mathrm{x}(t)=\mathrm{x}_0+\int_{t_0}^{t}\mathrm{X}^{-1}(\tau)\mathrm{B}(\tau)\mathrm{u}(\tau)\,\mathrm{d}\tau, \tag{1.52}$$

whence, for $t=t_f$, using condition (1.4), we get

$$\mathrm{X}^{-1}(t)\mathrm{x}_f=\mathrm{x}_0+\int_{t_0}^{t_f}\mathrm{X}^{-1}(\tau)\mathrm{B}(\tau)\mathrm{u}(\tau)\,\mathrm{d}\tau. \tag{1.53}$$

Equation (1.53) can be transformed into

$$\int_{t_0}^{t_f}\mathrm{H}(\tau)\mathrm{u}(\tau)\,\mathrm{d}\tau=\alpha(t_f), \tag{1.54}$$

where

$$\boldsymbol{\alpha}(t_f)\equiv\mathrm{X}^{-1}(t_f)\mathrm{x}_f-\mathrm{x}_0 \tag{1.55}$$

and

$$\mathrm{H}(t)\equiv\mathrm{X}^{-1}(t)\mathrm{B}(t). \tag{1.56}$$

Since $\boldsymbol{\alpha}$ given by formula (1.55) is known (as x_0 and x_f are assumed known and $\mathrm{X}(t)$ can be calculated for a given uncontrolled system), equation (1.54) is an integral equation with regard to the unknown function $\mathrm{u}(t)$.

For a better understanding let us illustrate the above considerations with a simple example; consider the simple pendulum described in section 1.5, but with the following viewpoint; what external force should be applied to the ball in order to bring the pendulum to equilibrium in a predefined time with minimum expense of work. The most effective method of solving this problem consists simply in the transformation of the differential equation into an integral equation.

Equation of the controlled pendulum (see Fig. 1.20) has the form

$$ml^2\ddot{\varphi}+mgl\varphi=F(t)l, \tag{1.57}$$

whence we obtain

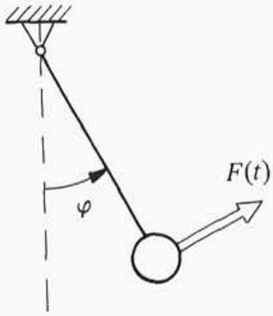

Fig. 1.20.

$$\ddot{\varphi} + \omega_0^2 \varphi = u(t), \tag{1.58}$$

where $\omega_0^2 = g/l$ is the natural frequency of free oscillation of the pendulum, and $u(t) = F(t)/ml$ is an unknown control function. Thus, a control needs to be determined which in the predefined time T will bring the pendulum from the initial state

$$\varphi(0) = \varphi_0, \qquad \dot{\varphi}(0) = \dot{\varphi}_0 \tag{1.59}$$

to the terminal state

$$\varphi(T) = 0, \qquad \dot{\varphi}(T) = 0. \tag{1.60}$$

Let us first present the model according to the general format. Then equation (1.58) takes the form (1.47), in which

$$\mathsf{x} = \begin{bmatrix} \varphi \\ \dot{\varphi} \end{bmatrix}, \qquad \mathsf{A} = \begin{bmatrix} 0 & 1 \\ -\omega_0^2 & 0 \end{bmatrix}, \qquad \mathsf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{1.61}$$

Conditions (1.48) and (1.49) take, with due account of (1.59) and (1.60), the form of

$$\mathsf{x}_0(t_0) = [\varphi_0, \ \dot{\varphi}_0]^{\mathrm{T}}, \tag{1.62}$$

$$\mathsf{x}_{\mathrm{f}}(T) = [0, \ 0]^{\mathrm{T}}. \tag{1.63}$$

The fundamental matrix for the uncontrolled oscillator has the form

$$\mathsf{X}(t) = \begin{bmatrix} \cos\omega_0 t & \dfrac{1}{\omega_0}\sin\omega_0 t \\ -\omega_0 \sin\omega_0 t & \cos\omega_0 t \end{bmatrix}, \tag{1.64}$$

whence

$$\mathsf{X}^{-1}(t) = \begin{bmatrix} \cos\omega_0 t & -\dfrac{1}{\omega_0}\sin\omega_0 t \\ \omega_0 \sin\omega_0 t & \cos\omega_0 t \end{bmatrix}. \tag{1.65}$$

On the basis of formula (1.56) we obtain functions $h_j(t)$, $j = 1, 2$:

$$h_1 = -\frac{1}{\omega_0}\sin\omega_0 t, \qquad h_2 = \cos\omega_0 t, \tag{1.66}$$

and on the basis of formula (1.55)—numbers α_j, $j = 1, 2$:

$$\alpha_1 = -\varphi_0, \qquad \alpha_2 = -\dot{\varphi}_0. \tag{1.67}$$

When (1.66) and (1.67) are accounted for, the general equation (1.54) takes the form

$$\int_0^T (\sin\omega_0 t)u(t)\,\mathrm{d}t = \omega_0\varphi_0, \qquad \int_0^T (\cos\omega_0 t)u(t)\,\mathrm{d}t = -\dot{\varphi}_0. \tag{1.68}$$

Now, let us finally compare the initial differential model (1.58) with the final integral model (1.68). This shows clearly the advantages of the integral model—initial conditions (1.62) and terminal conditions (1.63) are already contained in equation (1.68).

It should perhaps be noted that we mentioned in section 1.5, when commenting on the discrete–continuous model, that mathematical representations of such a physical model are integrodifferential equations. These equations are not obtained out of 'physics', but after transformations of the differential model.