$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_\rho} - \frac{\partial L}{\partial q_\rho} = Q_\rho^{\mathrm{nc}}, \qquad \rho = 1, \ldots, r, \tag{4.202}$$

where

$$L = (T_{\mathrm{mech}} + T_{\mathrm{el}}) - (V_{\mathrm{mech}} + V_{\mathrm{el}}). \tag{4.203}$$

Equation (4.202) and function (4.203) will be called, respectively, the **Lagrange–Maxwell equation** and **function**.

Since we have not presented a decent derivation of the Lagrange–Maxwell equations, we present three assumptions whose fulfilment is necessary for application of these equations:

(1) we assume that the behaviour of the mechanical part of an electromechanical system can be described by means of a discrete model having s 'mechanical' degrees of freedom;

(2) we assume that in every system electrical circuits are closed, meaning that conductors do not touch each other; the possibility of contact, for instance, via a commutator gives rise to nonholonomic constraints and would require separate treatment;

(3) a continuous electrical part of the electro-mechanical system can be described with the help of a finite number of 'electrical' generalized coordinates, if the condition of quasi-stationarity is satisfied; that is, changes over time in the intensity of an electromagnetic field do not influence the value of magnetic induction.

4.3.4 Case studies

4.3.4.1 Does a bell always ring?

This problem has a certain historical interest due to experiments performed with the giant bell *Kaiserglocke* of the famous Cologne Cathedral in Germany. In some situations a bell does not ring owing to the failure of the clapper to strike the side of the bell; we will establish the condition under which a bell fails to ring. A bell, together with its clapper, is modelled as a mechanical system composed of two compound pendulums. The pendulum which represents the shell rotates about the fixed, horizontal axis through O, called the axis of suspension, while the pendulum representing the clapper rotates about the axis A, connected to the first pendulum at the hinge axis (Fig. 4.14a). We assume that the bell and its clapper move in one plane. Then, the system has two degrees of freedom. The coordinates are the angular displacement α of the first pendulum, and the angular displacement β of the second one relative to the vertical direction (4.14b). Both α and β are assumed to be small. We use the following notations:

m_{b}, m_{c} are the masses of the bell and the clapper, respectively;
I_0 is the moment of intertia of the bell about its axis of suspension;
I_C is the moment of intertia of the clapper with respect to its centre of gravity C;
a is the distance between the axis of suspension and the hinge axis;
b is the distance of the centre of gravity, B, of bell from the axis of suspension;
c is the distance of the centre of gravity, C, of clapper from the hinge axis.

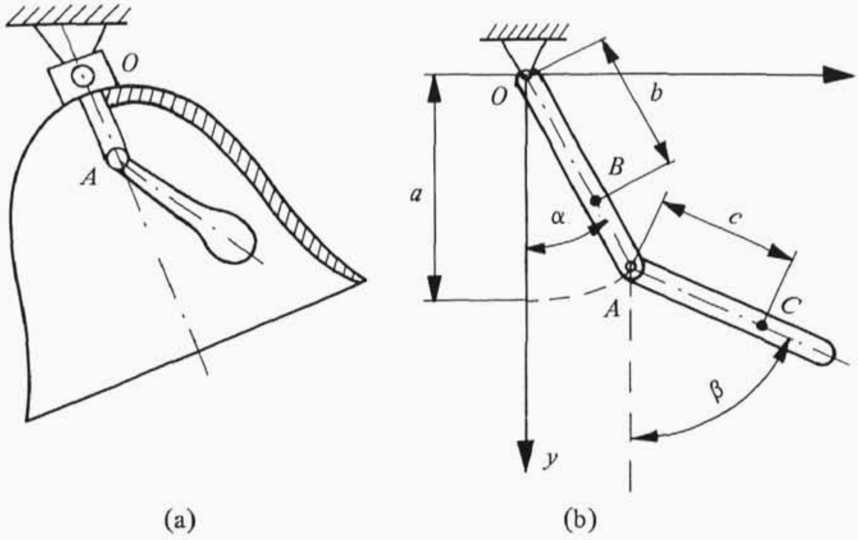

Fig. 4.14.

The kinetic energy of the entire system is

$$T = T_b + T_c, \tag{4.204}$$

where

$$T_b = \frac{1}{2} I_0 \dot{\alpha}^2 \tag{4.205}$$

is the kinetic energy of the bell, and T_c, the kinetic energy of the clapper, is given as follows:

$$T_c = \frac{1}{2} m_c v_c^2 + I_c \dot{\beta}^2, \tag{4.206}$$

where v_c denotes the velocity of the centre C of gravity of the clapper. To find an expression for v_c we make use of Fig. 3.13b).

$$x_c = a \sin\alpha + c \sin\beta, \qquad y_c = a\cos\alpha + c\cos\beta \tag{4.207}$$

and then

$$\dot{x}_c = a\dot{\alpha}\cos\alpha + c\dot{\beta}\cos\beta, \qquad \dot{y}_c = -a\dot{\alpha}\sin\alpha - c\dot{\beta}\sin\beta.$$

Thus

$$v_c^2 = \dot{x}_c^2 + \dot{y}_c^2 = a^2\dot{\alpha}^2 + 2ac\dot{\alpha}\dot{\beta}\cos(\beta - \alpha) + c^2\dot{\beta}^2. \tag{4.208}$$

On the basis of (4.104)–(4.206) as well as (4.208), the expression for the total kinetic energy is

$$T = \frac{1}{2}(I_0 + m_c a^2)\dot{\alpha}^2 + \frac{1}{2}(I_c + m_c c^2)\dot{\beta}^2 + m_c ac\dot{\alpha}\dot{\beta}\cos(\beta - \alpha). \tag{4.209}$$

Since the only forces acting on the system are gravitation forces, the potential of our system is

$$V = V_b + V_c, \tag{4.210}$$

where

$$\begin{aligned} V_b &= -m_b g y_b = -m_b g b \cos\alpha, \\ V_c &= -m_c g y_c = -m_c g(a\cos\alpha + c\cos\beta). \end{aligned} \tag{4.211}$$

The Lagrange equations (4.169) are

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \dot\alpha} - \frac{\partial T}{\partial \alpha} + \frac{\partial V}{\partial \alpha} &= 0, \\ \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \dot\beta} - \frac{\partial T}{\partial \beta} + \frac{\partial V}{\partial \beta} &= 0. \end{aligned} \tag{4.212}$$

First we will now obtain the equation for α. The derivatives in this case are

$$\begin{aligned} \frac{\partial T}{\partial \dot\alpha} &= (I_0 + m_c a^2)\dot\alpha + m_c ac\dot\beta\cos(\beta - \alpha), \\ \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \dot\alpha} &= (I_0 + m_c a^2)\ddot\alpha + m_c ac\ddot\beta\cos(\beta-\alpha) - m_c ac\dot\beta(\dot\beta - \dot\alpha)\sin(\beta-\alpha). \\ \frac{\partial T}{\partial \alpha} &= m_c ac\dot\alpha\dot\beta\sin(\beta-\alpha), \\ \frac{\partial V}{\partial \alpha} &= (m_b b + m_c a) g\sin\alpha. \end{aligned}$$

The equation for α becomes

$$(I_0 + m_c a^2)\ddot\alpha + m_c ac\ddot\beta\cos\varphi - m_c ac\dot\beta^2\sin\varphi + (m_b b + m_c a)g\sin\alpha = 0, \tag{4.213}$$

where $\varphi = \beta - \alpha$ for the sake of simplification.

Similarly, for the coordinate β we have the derivatives

$$\begin{aligned} \frac{\partial T}{\partial \dot\beta} &= (I_c + m_c c^2)\dot\beta + m_c ac\dot\alpha\cos\varphi, \\ \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \dot\beta} &= (I_c + m_c c^2)\ddot\beta + m_c ac\ddot\alpha\cos\varphi - m_c ac\dot\alpha\dot\varphi\sin\varphi, \\ \frac{\partial T}{\partial \beta} &= -m_c ac\dot\alpha\dot\beta\sin\varphi, \\ \frac{\partial V}{\partial \beta} &= m_c gc\sin\beta \end{aligned}$$

and the equation for β becomes therefore

$$(I_c + m_c c^2)\ddot\beta + m_c ac\ddot\alpha\cos\varphi + m_c ac\dot\alpha^2\sin\varphi + m_c gc\sin\beta = 0. \tag{4.214}$$

It is perhaps now a good opportunity for turning attention to the physical sense of the quantities obtained. Thus, if we make use of definition (4.159) we obtain the following:

$$Q_\alpha = (m_b b + m_c a) g \sin\alpha \qquad \text{and} \qquad Q_\beta = m_c g c \sin\beta, \tag{4.215}$$

i.e. the generalized forces are moments.

In order to be able to use equations (4.2130 and (4.214) to explain the curious phenomenon that some church bells, when set in motion, do not ring, we shall take advantage of the assumption of small angles α and β, and additionally neglecting the higher-order nonlinear terms, obtaining thereby

$$(I_0 + m_c a^2)\ddot{\alpha} + m_c a c \ddot{\beta} + (m_b b + m_c a) g \alpha = 0, \tag{4.216}$$

$$(I_c + m_c c^2)\ddot{\beta} + m_c a c \ddot{\alpha} + m_c g c \beta = 0. \tag{4.217}$$

The silence of a bell may occur when one of the principal oscillation modes is not accompanied by the relative motion of the clapper with respect to the shell, i.e. when

$$\alpha - \beta = 0. \tag{4.218}$$

Let us investigate under what conditions the relation (4.218) can represent the principal mode of oscillation.

Substituting

$$\alpha = \beta = \psi \tag{4.219}$$

into equations (4.216) and (4.217), we obtain

$$\begin{aligned} (I_0 + m_c a^2 + m_c a c)\ddot{\psi} &= -(m_b b + m_c a) g \psi, \\ (I_c + m_c c^2 + m_c a c)\ddot{\psi} &= -m_c g c \psi. \end{aligned} \tag{4.220}$$

The two equations (4.220) are compatible if

$$\frac{I_0 + m_c a^2 + m_c a c}{I_c + m_c c^2 + m_c a c} = \frac{m_b b + m_c a}{m_c c}. \tag{4.221}$$

Since the moment of inertia of the clapper is much smaller than that of the bell, i.e. $I_c \ll I_0$, we can simplify equation (4.221) assuming that

$$I_c = 0, \tag{4.222}$$

which corresponds to a modification of the physical model. Now the clapper is assumed to be a simple pendulum and not a compound pendulum as before. Thus, introducing (4.222), we get from (4.221), after simple transformation, the relation

$$l = a + c \tag{4.223}$$

in which

$$l = \frac{I_0}{m_b b}. \tag{4.224}$$

The value given by (4.224) is the so-called length of the equivalent simple pendulum.

There is still an important property of a compound pendulum: every centre of suspension O has a corresponding point O', called the centre of oscillation, such that if the pendulum is suspended at this latter point, it will oscillate with the same period as when suspended at point O.

Hence, equations (4.223) and (4.224) state that the bell may be silent if the centre of gravity of the clapper coincides with the centre of oscillation of the bell. This conclusion is the answer to the question why Kaiserglocke could not ring.

4.3.4.2 Longitudinal motion of an aircraft

Probably all readers are aware of the necessity of high lift for flying. However, this is not sufficient for ensuring the safe flight of an aircraft or of some other type of airframe. An airframe must be able in all circumstances to preserve equilibrium and it should return of its own accord to a predetermined position when it is disturbed from this position by some external agents, for example a violet gust of wind. Thus the airframe must be *stable*, and this is necessary for the ability to fly.

The purpose of modelling could be the establishment of the conditions for stable flight. For this, however, an adequate mathematical model must be built. To do this we will apply the Boltzmann–Hamel equations. In order to avoid undue calculational complications caused by computing the Boltzmann symbols (4.187), we will not consider any arbitrary motion, but will concentrate on plane (two-dimensional) motion.

This situation is shown in Fig. 4.15, in which two coordinate axes are shown: the inertial axis $Ox_gy_gz_g$ relative to the Earth and the non-inertial axis $Cxyz$, rigidly connected with the aircraft and having its origin at the centre of mass of the aircraft. Within the $Ox_gy_gz_g$ system we introduce the following generalized coordinates: x_C, z_C are the coordinates of the mass centre and θ is the rotation about the y-axis. On the other hand, in the $Cxyz$ system we introduce quasi-velocities: U in the direction of the axis Cx, W in the direction of z-axis, and Q, the angular quasi-velocity about the axis Cy. The only quasi-coordinate which in this case is identical with the usual generalized coordinate is the rotation angle θ (see Fig. 4.15). In order to simplify the algebra we introduce notations as in the table below, remembering that $\dot{q}_\sigma = \mathrm{d}q_\sigma/\mathrm{d}t$ and $\overset{\circ}{\vartheta}_\sigma = \mathrm{d}\vartheta_\sigma/\mathrm{d}t$.

Table 4.1.

Generalized coordinates	q_1 x_C	q_2 z_C	q_3 θ
Generalized velocities	$\dot{q}_1$ $\dot{x}_C$	$\dot{q}_2$ $\dot{z}_C$	$\dot{q}_3$ Q
Quasi-coordinates	ϑ_1 ϑ_U	ϑ_2 ϑ_W	ϑ_3 θ
Quasi-velocities	$\overset{\circ}{\vartheta}_1$ U	$\overset{\circ}{\vartheta}_2$ W	$\overset{\circ}{\vartheta}_3$ Q

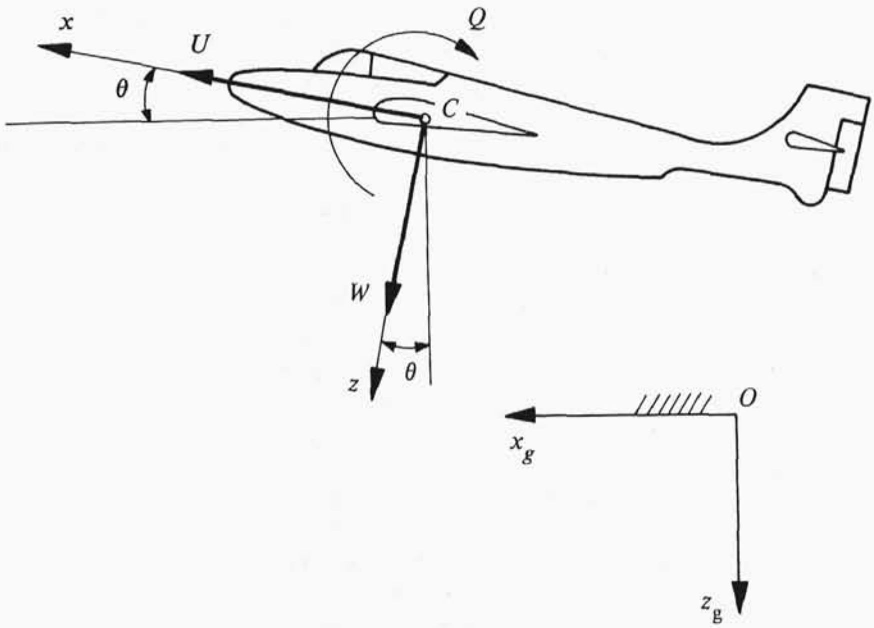

Fig. 4.15.

For the establishment of the physical model we shall assume the following:

(1) the aircraft is a rigid body;
(2) the mass and the mass distribution of the airplane are constant;
(3) the xCz plane is a plane of symmetry;
(4) the rotor gyroscopic effects are considered negligible;
(5) control surfaces are locked;
(6) the Earth can be regarded as ideally flat and nonrotating;
(7) there is no wind (the flight takes place in calm atmosphere).

For notations from Table 4.1 the Boltzmann–Hamel equations, (4.185), take the form

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T^*}{\partial U}-\frac{\partial T^*}{\partial \vartheta_U}+\sum_{\sigma,\tau=1}^{3}\gamma_{1\sigma\tau}\frac{\partial T^*}{\partial \overset{\circ}{\vartheta}_\sigma}\overset{\circ}{\vartheta}_\tau = P_1,\\
&\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T^*}{\partial W}-\frac{\partial T^*}{\partial \vartheta_W}+\sum_{\sigma,\tau=1}^{3}\gamma_{2\sigma\tau}\frac{\partial T^*}{\partial \overset{\circ}{\vartheta}_\sigma}\overset{\circ}{\vartheta}_\tau = P_2,\\
&\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T^*}{\partial Q}-\frac{\partial T^*}{\partial \theta}+\sum_{\sigma,\tau=1}^{3}\gamma_{3\sigma\tau}\frac{\partial T^*}{\partial \overset{\circ}{\vartheta}_\sigma}\overset{\circ}{\vartheta}_\tau = P_3,
\end{aligned} \tag{4.225}$$

where due to (4.180)

$$P_\rho = \sum_{\sigma=1}^{3} b_{\rho\sigma} Q_\sigma, \qquad \rho = 1,2,3. \tag{4.226}$$

Thus, we should calculate: (1) kinetic energy and express it in quasi-coordinates, (2) Boltzmann symbols (4.187) and (3) quasi-forces (4.226).

(1) On the basis of formula (3.79) we obtain kinetic energy expressed in generalized velocities

$$T = \tfrac{1}{2} m\left(\dot{x}_C^2 + \dot{z}_C^2\right) + \tfrac{1}{2} I_y \dot{\theta}^2. \tag{4.227}$$

In order to replace generalized velocities by quasi-velocities we must find a relation between them. On the basis of Fig. 4.15 we can write

$$\begin{aligned} \dot{x}_C &= U\cos\theta + W\sin\theta, \\ \dot{z}_C &= -U\sin\theta + W\cos\theta. \end{aligned} \tag{4.228}$$

Now, having introduced (4.228) to (4.227) and using the fact that

$$Q = \dot{\theta}, \tag{4.229}$$

we obtain

$$T^* = \tfrac{1}{2} m U^2 + \tfrac{1}{2} m W^2 + \tfrac{1}{2} I_y Q^2 \tag{4.230}$$

and

$$\begin{aligned} \frac{\partial T^*}{\partial U} &= mU; \qquad \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T^*}{\partial U} = m\dot{U} \\ \frac{\partial T^*}{\partial W} &= mW; \qquad \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T^*}{\partial W} = m\dot{W} \\ \frac{\partial T^*}{\partial Q} &= I_y Q; \qquad \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T^*}{\partial Q} = I_y\dot{Q} \end{aligned} \tag{4.231}$$

$$\frac{\partial T^*}{\partial \vartheta_U} = 0, \qquad \frac{\partial T^*}{\partial \vartheta_W} = 0, \qquad \frac{\partial T^*}{\partial \theta} = 0. \tag{4.232}$$

(2) It can be seen from the structure of formula (4.187) that coefficients $a_{\sigma i}$ and $b_{i\rho}$ ($\sigma = 1, 2$; $i, \rho = 1, 2, 3$) will be needed and to get them we use the relation (4.175), which takes the form (see the notation—Table 4.1)

$$\begin{aligned} \dot{q}_1 &= b_{11}U + b_{12}W + b_{13}Q = \dot{x}_C \\ \dot{q}_2 &= b_{21}U + b_{22}W + b_{23}Q = \dot{z}_C \\ \dot{q}_3 &= b_{31}U + b_{32}W + b_{33}Q = Q. \end{aligned} \tag{4.233}$$

By comparing (4.232) with (4.228) and (4.229) we conclude that

$$\begin{aligned} &b_{11} = \cos\theta, && b_{12} = \sin\theta, && b_{13} = 0, \\ &b_{21} = -\sin\theta, && b_{22} = \cos\theta, && b_{23} = 0, \\ &b_{31} = 0, && b_{32} = 0 && b_{33} = 1. \end{aligned} \tag{4.234}$$

Coefficients $a_{\rho i}$ can be calculated from the relation (4.174) since matrix B is nonsingular and its inverse exists. The inversion of B gives

$$\begin{aligned} &a_{11} = \cos\theta, && a_{12} = -\sin\theta, && a_{13} = 0, \\ &a_{21} = \sin\theta, && a_{22} = \cos\theta, && a_{23} = 0, \\ &a_{31} = 0, && a_{32} = 0, && a_{33} = 1, \end{aligned} \tag{4.235}$$

Now, on the basis of formula (4.187) we have

$$\gamma_{\rho\sigma\tau} = \sum_{i=1}^{3}\sum_{j=1}^{3} b_{i\rho} b_{j\tau}\left(\frac{\partial a_{\sigma i}}{\partial q_j} - \frac{\partial a_{\sigma j}}{\partial q_i}\right), \qquad \sigma, \rho, \tau = 1, 2, 3 \tag{4.236}$$

Since the coefficients $a_{\sigma i}$, $a_{\sigma j}$ depend only upon the coordinate $q_3 = \theta$, then the only derivatives in formula (4.236) which do not vanish are

$$\begin{aligned} &\frac{\partial a_{11}}{\partial\theta} = -\sin\theta, && \frac{\partial a_{12}}{\partial\theta} = -\cos\theta, \\ &\frac{\partial a_{21}}{\partial\theta} = \cos\theta, && \frac{\partial a_{22}}{\partial\theta} = -\sin\theta. \end{aligned} \tag{4.237}$$

After deploying the formula (4.236) we get

$$\gamma_{\rho\sigma\tau} = \frac{\partial a_{\sigma 1}}{\partial\theta} b_{1\rho} b_{3\tau} + \frac{\partial a_{\sigma 2}}{\partial\theta} b_{2\rho} b_{3\tau} - \frac{\partial a_{\sigma_1}}{\partial\theta} b_{3\rho} b_{1\tau} - \frac{\partial a_{\sigma 2}}{\partial\theta} b_{3\rho} b_{2\tau}, \tag{4.238}$$

whence for $\sigma = 1$, using (4.237), we obtain

$$\gamma_{\rho 1\tau} = -b_{1\rho} b_{3\tau}\sin\theta - b_{2\rho} b_{3\tau}\cos\theta + b_{3\rho} b_{2\tau}\sin\theta + b_{3\rho} b_{2\tau}\sin\theta, \tag{4.239}$$

while for $\sigma = 2$

$$\gamma_{\rho 2\tau} = -b_{1\rho} b_{3\tau}\cos\theta - b_{2\rho} b_{3\tau}\sin\theta + b_{3\rho} b_{1\tau}\cos\theta + b_{3\rho} b_{2\tau}\sin\theta. \tag{4.240}$$

What remains to be done is an exercise in patience in writing down indices and observation of formulae (4.234). The results are presented in Tables 4.2 and 4.3.

Knowledge of Boltzmann symbols makes it possible already now to write down the terms with sums in equations (4.225):

Table 4.2

$\sigma=1$	$\rho=1$	$\tau=1$	$\gamma_{111}=0$
		$\tau=2$	$\gamma_{112}=0$
		$\tau=3$	$\gamma_{113}=0$
	$\rho=2$	$\tau=1$	$\gamma_{211}=0$
		$\tau=2$	$\gamma_{212}=0$
		$\tau=3$	$\gamma_{213}=-1$
	$\rho=3$	$\tau=1$	$\gamma_{311}=0$
		$\tau=2$	$\gamma_{312}=1$
		$\tau=3$	$\gamma_{313}=0$

Table 4.3

$\sigma=2$	$\rho=1$	$\tau=1$	$\gamma_{121}=0$
		$\tau=2$	$\gamma_{122}=0$
		$\tau=3$	$\gamma_{123}=1$
	$\rho=2$	$\tau=1$	$\gamma_{221}=0$
		$\tau=2$	$\gamma_{222}=0$
		$\tau=3$	$\gamma_{223}=0$
	$\rho=3$	$\tau=1$	$\gamma_{321}=-1$
		$\tau=2$	$\gamma_{322}=0$
		$\tau=3$	$\gamma_{323}=0$

$$
\begin{aligned}
\sum_{\sigma,\tau=1}^{3}\gamma_{1\sigma\tau}\frac{\partial T}{\partial\overset{\circ}{\vartheta}_\sigma}\overset{\circ}{\vartheta}_\tau &= \gamma_{123}\frac{\partial T}{\partial\overset{\circ}{\vartheta}_2}\overset{\circ}{\vartheta}_3 = \frac{\partial T}{\partial W}Q \\
\sum_{\sigma,\tau=1}^{3}\gamma_{2\sigma\tau}\frac{\partial T}{\partial\overset{\circ}{\vartheta}_\sigma}\overset{\circ}{\vartheta}_\tau &= \gamma_{213}\frac{\partial T}{\partial\overset{\circ}{\vartheta}_1}\overset{\circ}{\vartheta}_3 = -\frac{\partial T}{\partial U}Q \\
\sum_{\sigma,\tau=1}^{3}\gamma_{3\sigma\tau}\frac{\partial T}{\partial\overset{\circ}{\vartheta}_\sigma}\overset{\circ}{\vartheta}_\tau &= \gamma_{321}\frac{\partial T}{\partial\overset{\circ}{\vartheta}_2}\overset{\circ}{\vartheta}_1+\gamma_{312}\frac{\partial Y}{\partial\overset{\circ}{\vartheta}_1}\overset{\circ}{\vartheta}_2 = -\frac{\partial T}{\partial W}U+\frac{\partial T}{\partial U}W.
\end{aligned}
\tag{4.241}
$$

(3) On the basis of (4.226), using (4.234) we obtain

$$
P_1 = Q_1\cos\theta - Q_2\sin\theta, \qquad P_2 = Q_1\sin\theta + Q_2\cos\theta \qquad \text{and} \qquad P_3 = Q_3. \tag{4.242}
$$

If the derivatives (4.231) and (4.232) are taken into consideration as well as expressions (4.241), equations (4.225) finally take the form

$$
\begin{aligned}
m(\dot{U}+QW) &= P_1, \\
m(\dot{W}-QU) &= P_2, \\
I_C\dot{Q} &= P_3.
\end{aligned}
\tag{4.243}
$$

After the specification of generalized forces $Q_\sigma\,(\sigma=1,2,3)$ acting on the aircraft one can already investigate stability of the uncontrolled longitudinal motion of the aircraft. It is a long and rather complicated procedure, so we will not do it.

4.3.4.3 When does a loudspeaker work well?

The loudspeaker (see Fig. 4.16) is composed of an iron body (1), with a core sticking out, on which, quite loosely, a movable coil (2) of induction H can move. The body of this coil is connected with a conical membrane (3), whose outside edge is attached to the immobile support. An electromotive force $E(t)$ is applied to the movable coil in

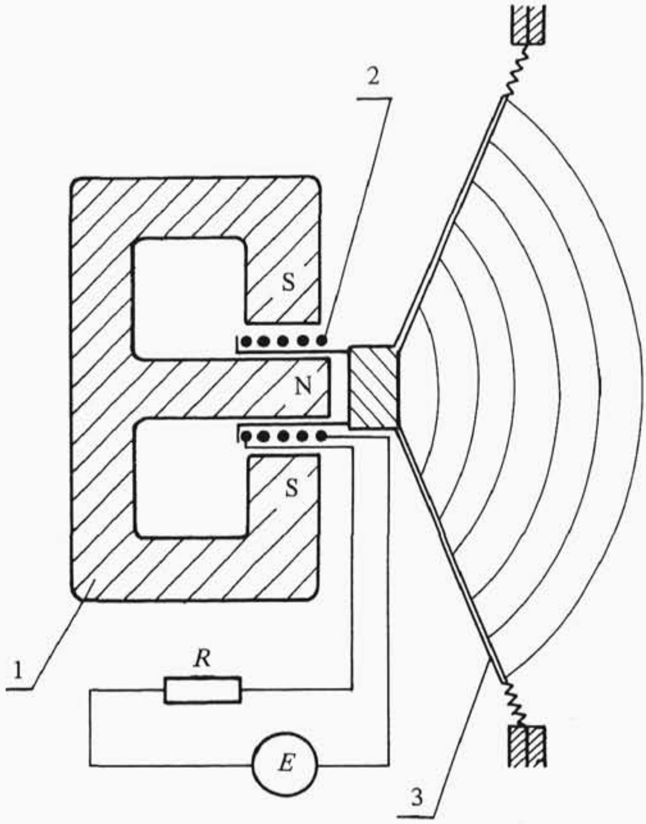

Fig. 4.16.

accordance with a signal appearing at the input. For the conventional loudspeaker, the signal comes, for example, from a radio tuner. By means of an amplifier (not shown in the figure) the current varies over time in the coil, due to which a varying magnetic force acts upon the coil. Under the influence of this force the movable coil vibrates, setting the membrane into motion. Vibrations of the membrane then create in the air acoustic pressure waves and in this way the loudspeaker generates sound. It may also act as a generator of an acoustic force. Such a force can be used to benefit people. An example of such a use of a loudspeaker is provided by the famous experiment performed in Cambridge in 1985, in which a loudspeaker was used to generate the acoustic force. This force, in 'cooperation' with the aerodynamic force, causes flutter suppression.

The first goal of modelling of the loudspeaker is establishment of conditions of its stable functioning. The second one is determination of which parameters to change to improve the quality of the loudspeaker.

In order to derive equations of motion of the system described we shall first of all model the membrane, which is a continuous object, as a harmonical oscillator of mass m

and stiffness k, and of the viscous damping coefficient b. With regard to mass m, we assume that it is the combined mass of all the movable parts.

The system considered can be described in section 4.3.3, with two coordinates—x, displacement of the movable coil, and e, electric charge flowing through the circuit. The only difference consists in the fact that we shall be calculating magnetic energy T_{el}, somewhat differently. Now it is stored in the magnetic field of the permanent magnet coupled with the field of the movable coil. Thus

$$T_{\mathrm{el}} = \tfrac{1}{2} H\dot{e}^2 + \tfrac{1}{2}\left(\phi_0 + \frac{\phi_{\mathrm{c}}}{d}x\right)\dot{e}, \tag{4.244}$$

where H is induction of the coil, ϕ_0 is the permanent magnetic flux of the magnet, ϕ_{c} denotes the coupled flux of coil with the magnet, and d is the diameter of the coil. The form of (4.244) results from the general formula which may be found in Feynman *et al.* (1965), Vol. 2.

Thus, the Lagrange–Maxwell function (4.203) is given by the formula

$$L = \tfrac{1}{2}\left[m\dot{x}^2 + H\dot{e}^2 + \left(\phi_0 + \frac{\phi_{\mathrm{c}}}{d}x\right)\dot{e}\right] - \tfrac{1}{2}kx^2 \tag{4.245}$$

and the Lagrange–Maxwell equations take precisely the form (4.196), and therefore we will not quote them here. We can also use formulae (4.197), somewhat simplified because here we have no external mechanical force, and this yields

$$Q_x = -b\dot{x}, \qquad Q_e = E - R\dot{e}. \tag{4.246}$$

Having introduced the function (4.245) to equations (4.202) and made use of formulae (4.246) we obtain the equations

$$\begin{aligned} m\ddot{x} + b\dot{x} - \tfrac{1}{2}\frac{\phi_{\mathrm{c}}}{d}\dot{e} + kx &= 0 \\ H\ddot{e} + R\dot{e} + \tfrac{1}{2}\frac{\phi_{\mathrm{c}}}{d}\dot{x} &= E(t), \end{aligned} \tag{4.247}$$

which describe dynamic behaviour of the loudspeaker.

Consider first the autonomous system by taking $E(t) \equiv 0$. Equilibrium position of the system is described by

$$x = 0, \qquad \dot{e} = 0. \tag{4.248}$$

The characteristic equation of (4.247) takes the form

$$\lambda^3 + \left(\frac{b}{m} + \frac{R}{H}\right)\lambda^2 + \left(\frac{k}{m} + \frac{b}{m}\frac{R}{H}\right)\lambda + \frac{k}{m}\frac{R}{H} + \frac{1}{4}\frac{\phi_{\mathrm{c}}^2}{d^2 mH} = 0, \tag{4.249}$$

and, on the basis of the *Routh criterion*, in view of positiveness of all coefficients, this implies that the equilibrium (4.248) is always stable.

When evaluating the dynamic properties of a loudspeaker one is interested in the frequency characteristic of mechanical power yielded with constant amplitude of excitation voltage. Mechanical power transmitted from the coil to the membrane is given by the formula

$$P = |i(s)|^2 \operatorname{Re}\{Z(s)\}, \tag{4.250}$$

where, for a greater clarity, we have taken account of relation (2.37). The notations $|\cdot|$ and Re denote the modulus and the real part of the complex number, respectively, $Z(s)$ is the operator impedance of the system, and s is the *Laplace variable*.

Applying Laplace transforms to the system of equations (with assumption of zero initial conditions) we obtain the impedance in the form of

$$Z(s) = Hs + R + \frac{\phi_c^2 s}{4d^2(ms^2 + bs + k)}. \tag{4.251}$$

Now, substituting (4.251) into (4.250), taking account of the fact that $s = j\omega$ (j in the imaginary unit) and neglecting terms with H and $\phi_c^2/4d^2$ (since they are a rather small influence on the final result) we obtain

$$P = \frac{b\phi_c^2 E^2 \omega_n^2 \Omega^2}{4d^2k^2R^2\left[\left(1-\Omega^2\right)^2 + \omega_n^2\Omega^2(b/k)^2\right]}, \tag{4.252}$$

where

$$\Omega = \omega/\omega_n, \qquad \omega_n^2 = k/m. \tag{4.253}$$

Since the quantities d and ϕ_c characterizing the coil, and also the quantities b, k and R, are assumed constant, so we can introduce the parameter

$$p = \frac{b\phi_c^2}{(2dkR)^2} \tag{4.254}$$

and then the equation (4.252) takes the form

$$P = \frac{E^2\omega_n^2\Omega^2 p}{(1-\Omega^2)^2 + \left(b/\sqrt{(mk)}\right)^2\Omega^2}. \tag{4.255}$$

From this, we can obtain interesting results. We shall define the system's transmittance to be

$$K = \frac{P}{E^2}, \tag{4.256}$$

and then we will calculate its extremum with regard to Ω. Then assuming light damping and using notation (4.254), we obtain

$$K_{\max} = \left(\frac{k}{b}\right)^2 p. \tag{4.257}$$

The latter formula implies that for a given value of k/b the essential influence is exerted on the power transmitted from the coil to the membrane by the parameter (4.254). Thus, the properties of the loudspeaker can be improved through: (1) increase of ϕ_c, i.e. growth of magnetic induction of the permanent magnet, (2) increase of the damping coefficient b, and (3) decrease of resistance R of the coil.

4.4 MODELLING OF NONHOLONOMIC SYSTEMS

4.4.1 Introductory remarks

We introduced in section 2.2.2, after Hertz, the notion of a nonholonomic system. The theory of nonholonomic systems started to develop at the end of the nineteenth century, when it unexpectedly turned out that the wonderful and apparently universal formalism of Lagrange is useless even for simple questions of rolling without slipping of a rigid disc on a plane. As improbable as it may seem, **Joseph Louis Lagrange** (1736–1813) himself did not suspect that such constraints might exist. He set out his belief in his famous *Mécanique Analytique* (1788), in which he states that it is possible, for every mechanical system, to select independent coordinates having independent variations. No exceptions were noticed for many years, until the problem of the rolling rigid bodies without slippage was studied. Recall that Hertz introduced his classification into holonomic and nonholonomic constraints as late as 1894. The development of the theory took a circuit course, with numerous mistakes and errors committed by known exponents of mechanics and mathematics. The series of mishaps lasted until the second half of the 1960s, when the monograph of Neimark and Fufaev (1967) was published, resolving many existing doubts. The present section of this book owes much to that book and in general to the Russian school of mechanics. It is, simultaneously, worth emphasizing that many questions are still subjects of studies. We present here only a well-established apparatus for modelling mechanical systems on which imposed constraints that are linear with respect to velocities. Such constraints have, in generalized coordinates, the form (see also (2.24))

$$\sum_{\sigma=1}^{s} B_{\beta\sigma}(t, q_\sigma)\dot{q}_\sigma + B_\beta = 0, \qquad \beta = 1, \ldots, b, \tag{4.258}$$

where b denotes the number of nonholonomic constraints.

We will see later that the practical modelling of nonholonomic systems reduces mainly to obtaining equation (4.258) and to the determination of the coefficients of this equation. In order to develop certain skills which would then facilitate understanding of analytical mechanics, we will comment on two well-known examples of nonholonomic constraints and we will transform them to the form (4.258). The best-known example is probably that of a billiard ball rolling without slipping on a rough table surface (Fig. 4.17). The location of the ball will be posed by the coordinates x_C and y_C of its centre and the three Euler angles ψ, θ and φ.

The fact that the ball rolls without slipping may be expressed through the statement that $\mathbf{v}_S = \mathbf{0}$, where $\mathbf{v}_S$ is the velocity of the point of the ball in contact with the surface. Since