Equation (4.135) now becomes

$$\int_{t_1}^{t_2} \delta(T - V)\,\mathrm{d}t + \int_{t_1}^{t_2} \delta W^{\mathrm{nc}}\,\mathrm{d}t = 0 \tag{4.161}$$

or, if use is made of the definition of the Lagrangian function $L = T - V$ and of the property (4.98),

$$\delta\int_{t_1}^{t_2} L\,\mathrm{d}t + \int_{t_1}^{t_2} \delta W^{\mathrm{nc}}\,\mathrm{d}t = 0. \tag{4.162}$$

Thus, it has the same form as the engineering form of Hamilton's principle (4.145), since δW now denotes the virtual work of only the nonconservative forces. One could therefore assign to (4.162) the label 'D', as it arises by derivation of the engineering form of Hamilton's principle from d'Alembert's principle, and to (4.145) the label 'P', indicating that it is the result of postulation.

Hamilton's principle could be considered not as a consequence of Newton's second law (although, of course, it is in agreement with the latter) but as an equivalent postulate of mechanics. There is nothing new in the idea that it is very useful in cases in which direct application of Newton's second law is cumbersome. We maintain, namely an engineer may find himself in a practical situation in which application of the method of Newton is not only cumbersome but impossible. This usually occurs when we are dealing not with a purely mechanical system but with coupled systems. Recalling section 4.2.4.3, and through the example, we would like to encourage beginners in modelling to be bold, leave the footprints of the past and go where the familiar track—that is, the relation between the Newton's second law and Hamilton's principle—is no longer visible. There, only the 'compass of analogy' can be used if for some reasons the variational principle is chosen. More simply, a well-founded tool in classical mechanics is a general instrument for developing the equations in integrated mechanics.

4.3 MODELLING OF HOLONOMIC SYSTEMS

Quite numerous complex mechanical systems encountered in engineering practice may be presented in the form of models with holonomic constraints. The equations of motion of such systems can be obtained with the help of *Lagrange's equations of the second kind*. These equations constitute the most important instrument for modelling complex holonomic systems and that is why the present action begins with various methods of derivation of Lagrange's equations. However, in some specific cases, it may be more suitable or even necessary to use other types of equations. That is why the *Boltzmann–Hamel* and *Lagrange–Maxwell* equations are presented in sections 4.3.2 and 4.3.3.

4.3.1 Lagrange equations of the second kind

We shall show two methods of derivation: one originating from d'Alembert's principle in the Lagrange form, which is also called the fundamental equation, and the second from Hamilton's principle in the standard form. The fundamental equation in generalized coordinates (4.130) has been found to have the form

$$\sum_{\sigma=1}^{s}\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \dot{q}_\sigma}-\frac{\partial T}{\partial q_\sigma}-Q_\sigma\right)\delta q_\sigma=0, \tag{4.163}$$

where the kinetic energy, T, and the generalized forces, Q_σ, are generally functions of all the generalized coordinates q_σ and the generalized velocities $\dot{q}_\sigma$ ($\sigma=1,\ldots,s$), and of time t.

Because the system is holonomic and the coordinates q_σ are independent, the δq_σ are completely arbitrary. Thus, one may deduce directly from (4.163) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \dot{q}_\sigma}-\frac{\partial T}{\partial q_\sigma}=Q_\sigma, \qquad \sigma=1,\ldots,s. \tag{4.164}$$

Equations (4.164) are called **Lagrange's equations of the second kind**. We now consider the action in Hamilton's sense defined by the formula

$$S=\int_{t_0}^{t_e} L(t,q_\sigma,\dot{q}_\sigma)\,\mathrm{d}t, \tag{4.165}$$

where

$$L=T(t,q_\sigma,\dot{q}_\sigma)-V(t,q_\sigma) \tag{4.166}$$

denotes the Lagrangian function. According to Hamilton's principle

$$\delta S=0, \tag{4.167}$$

and applying the Euler–Lagrange equations (4.103), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_\sigma}-\frac{\partial L}{\partial q_\sigma}=0, \qquad \sigma=1,\ldots,s. \tag{4.168}$$

Equations (4.168) are also called Lagrange's equations of the second kind.

Although equations (4.164) and (4.168) bear the same name, they are not equivalent. Having substituted (4.166) into (4.168) we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \dot{q}_\sigma}-\frac{\partial T}{\partial q_\sigma}+\frac{\partial V}{\partial q_\sigma}=0. \tag{4.169}$$

Now, taking account of definition (4.159) we can transform equation (4.169) to

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \dot{q}_\sigma}-\frac{\partial T}{\partial q_\sigma}=Q_\sigma^{\mathrm{pot}}. \tag{4.170}$$

The essential difference between the equations is constituted by the fact that the left-hand side of (4.164) is more general, since relation (4.158) holds. Similarly in the opposite direction, i.e. introducing (4.158) into (4.164) and accounting for (4.159) and (4.166), we get the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_\sigma}-\frac{\partial L}{\partial q_\sigma}=Q_\sigma^{\mathrm{nc}}, \tag{4.171}$$

which we prefer, because it is short and also more general than (4.168). This, however, by no means proves that the d'Alembert principle is more general, at least not in terms of the equations of motion. In the case considered this means only that Hamilton's principle was not taken in the general form (4.145). For purposes of modelling there is no need, however, to repeat the derivation of equations (4.171) in a different manner. It is much more important to be able to calculate the generalized nonconservative forces Q_σ^{nc}.

There exist two methods of calculations of Q_σ^{nc}. The first method is apparently more direct, as it is based upon the definition (4.110) adopted for the present goal. Thus, we have

$$Q_\sigma^{\mathrm{nc}} = \sum_{\nu=1}^{n} \mathbf{F}_\nu^{\mathrm{nc}} \cdot \frac{\partial \mathbf{r}_\nu}{\partial q_\sigma}, \qquad \sigma = 1,\ldots,s. \tag{4.172}$$

The second method is based upon the definition of virtual work (4.111), i.e.

$$\delta W = \sum_{\sigma=1}^{s} Q_\sigma \delta q_\sigma.$$

Thus, we calculate the differential of work, apply the mnemonic rule (see section 4.2.2.2) and the generalized forces are obtained as coefficients at variations of generalized coordinates. Although the second method seems to be in a way roundabout, we advise application of this method, since, in fact, it is much quicker than the first one.

4.3.2 The Boltzmann–Hamel equations

These equations differ from the well-known Lagrange equations of the second kind in that quasi-coordinates are used in them in place of generalized coordinates. They were obtained by **Ludwig Boltzmann** (in 1902) and **George Hamel** (in 1904). Since Hamel published them under the name of the Lagrange–Euler equation, one still encounters this name in the literature. Because of the formal similarity with Lagrange's equations of the second king these equations are also referred to as Lagrange's equations in quasi-coordinates.

We have commented already in section 2.2.6 on the advantages resulting from the use of quasi-coordinates in classical mechanics. We shall try to use them here to illustrate the modelling of a holonomic system consisting of an aircraft, although the principle purpose of the Boltzmann–Hamel equations, according to the intention of their authors, was to model nonholonomic systems.

For derivation of the Boltzmann–Hamel equations we shall use the fundamental equation (4.130), which we shall write down as

$$\sum_{\sigma=1}^{s}\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \dot{q}_\sigma} - \frac{\partial T}{\partial q_\sigma}\right)\delta q_\sigma = \sum_{\sigma=1}^{s} Q_\sigma \delta q_\sigma. \tag{4.173}$$

The method of derivation consists in the presentation of equation (4.173) with the help of quasi-coordinates using relations (see section 2.2.6)

$$\overset{\circ}{\vartheta}_\rho = \sum_{\sigma=1}^{s} a_{\rho\sigma}\dot{q}_\sigma, \qquad \rho = 1,\dots,s, \tag{4.174}$$

$$\dot{q}_\sigma = \sum_{\rho=1}^{s} b_{\sigma\rho}\overset{\circ}{\vartheta}_\rho, \qquad \sigma = 1,\dots,s. \tag{4.175}$$

On the basis of (4.174) we can obtain relations

$$\frac{\partial \overset{\circ}{\vartheta}_\rho}{\partial \dot{q}_\sigma} = a_{\rho\sigma} \tag{4.176}$$

and

$$\frac{\partial \overset{\circ}{\vartheta}_\rho}{\partial q_\sigma} = \sum_{\tau=1}^{s} \frac{\partial a_{\rho\tau}}{\partial q_\sigma}\dot{q}_\tau. \tag{4.177}$$

From (4.175), using the mnemonic rule, we obtain the formula

$$\delta q_\sigma = \sum_{\rho=1}^{s} b_{\sigma\rho}\delta\vartheta_\rho. \tag{4.178}$$

Introducing (4.178) into (4.173) and using the property of independence of variation of quasi-coordinates $\delta\vartheta_\rho$ $(\rho = 1,\dots,s)$ we get

$$\sum_{\sigma=1}^{s} b_{\sigma\rho}\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \dot{q}_\sigma} - \sum_{\sigma=1}^{s} b_{\sigma\rho}\frac{\partial T}{\partial q_\sigma} = P_\rho, \tag{4.179}$$

where the notation

$$P_\rho = \sum_{\sigma=1}^{s} b_{\sigma\rho}Q_\sigma, \qquad \rho = 1,\dots,s \tag{4.180}$$

is used. Since Q_σ are generalized forces, we shall refer to the quantities P_ρ as *quasi-generalized forces*.

Denote by T^* the function obtained from function T by substitution of the generalized velocities $\dot{q}_\sigma$ $(\sigma = 1,\dots,s)$ with quasi-velocities $\overset{\circ}{\vartheta}_\rho$ $(\rho = 1,\dots,s)$ using relation (4.175), that is

$$T(q_\sigma, \dot{q}_\sigma) = T\left(q_\sigma, \sum_{\rho=1}^{s} b_{\sigma\rho}\overset{\circ}{\vartheta}_\rho \right) \equiv T^*\left(q_\sigma, \overset{\circ}{\vartheta}_\sigma \right). \qquad \sigma = 1,\dots,s. \tag{4.181}$$

We can now calculate expressions appearing on the left-hand side of equation (4.179) with the help of this function T^*. Using (4.176) and (4.177) we obtain

$$\frac{\partial T}{\partial \dot{q}_\sigma} = \sum_{i=1}^{s} \frac{\partial T^*}{\partial \mathring{\vartheta}_i} \frac{\partial \mathring{\vartheta}_i}{\partial \dot{q}_\sigma} = \sum_{i=1}^{s} \frac{\partial T^*}{\partial \mathring{\vartheta}_i} a_{i\sigma} \tag{4.182}$$

and

$$\frac{\partial T}{\partial q_\sigma} = \frac{\partial T^*}{\partial q_\sigma} + \sum_{i=1}^{s} \frac{\partial T^*}{\partial \mathring{\vartheta}_i} \frac{\partial \mathring{\vartheta}_i}{\partial q_\sigma} = \frac{\partial T^*}{\partial q_\sigma} + \sum_{i=1}^{s} \frac{\partial T^*}{\partial \mathring{\vartheta}_i} \sum_{\tau=1}^{s} \frac{\partial a_{i\tau}}{\partial q_\sigma} \dot{q}_\tau . \tag{4.183}$$

Having introduced expressions (4.182) and (4.183) to equation (4.179) we obtain

$$\begin{aligned} &\sum_{\sigma=1}^{s} b_{\sigma\rho} \left[\frac{\mathrm{d}}{\mathrm{d}t} \left(\sum_{i=1}^{s} \frac{\partial T^*}{\partial \mathring{\vartheta}_i} a_{i\sigma} \right) \right] - \sum_{\sigma=1}^{s} b_{\sigma\rho} \frac{\partial T^*}{\partial q_\sigma} \\ &\qquad - \sum_{\sigma=1}^{s} \sum_{\rho=1}^{s} b_{\sigma\rho} \frac{\partial T^*}{\partial \mathring{\vartheta}_\rho} \sum_{\tau=1}^{s} \frac{\partial a_{\rho\tau}}{\partial q_\sigma} \dot{q}_\tau = P_\rho, \qquad \sigma = 1, \ldots, s. \end{aligned} \tag{4.184}$$

After differentiation with respect to time the sum in the first term (accounting for the fact that $a_{i\sigma} = a_{i\sigma}(q_\sigma)$ as well as for relation (2.65)) equation (4.184) can be expressed in the form

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial T^*}{\partial \mathring{\vartheta}_\rho} + \sum_{\sigma=1}^{s} \sum_{\tau=1}^{s} \gamma_{\rho\sigma\tau} \frac{\partial T^*}{\partial \mathring{\vartheta}_\sigma} \mathring{\vartheta}_\tau - \frac{\partial T^*}{\partial \vartheta_\rho} = P_\rho, \qquad \rho = 1, \ldots, s, \tag{4.185}$$

where the purely conventional notation

$$\frac{\partial T^*}{\partial \vartheta_\rho} = \sum_{\sigma=1}^{s} b_{\sigma\rho} \frac{\partial T^*}{\partial q_\sigma} \tag{4.186}$$

was used. The quantities $\gamma_{\rho\sigma\tau}$ $(\sigma, \rho, \tau = 1, \ldots, s)$ denote the so-called **Boltzmann symbols** given by the formula

$$\gamma_{\rho\sigma\tau} = \sum_{i=1}^{s} \sum_{j=1}^{s} b_{i\rho} b_{j\tau} \left(\frac{\partial a_{\sigma i}}{\partial q_j} - \frac{\partial a_{\sigma j}}{\partial q_i} \right). \quad \vartheta_\rho \ (\rho = 1, \ldots, s) \tag{4.187}$$

Equations (4.185) are called the **Boltzmann–Hamel equations**. They describe the motion of holonomic systems in quasi-coordinates. If ϑ_ρ $(\rho = 1, \ldots, s)$ are the usual generalized coordinates, implying that relations (4.174) are integrable, then the symbols $\gamma_{\rho\sigma\tau}$ given with formula (4.187) vanish and equations (4.185) take the form of the usual Lagrange equations of the second kind.

In practical applications the antisymmetric property of the Boltzmann symbols is very useful. From the definition (4.187) we have

$$\gamma_{\rho\sigma\tau} = \sum_{i=1}^{s} \sum_{j=1}^{s} b_{i\rho} b_{j\tau} \left(\frac{\partial a_{\sigma i}}{\partial q_j} - \frac{\partial a_{\sigma j}}{\partial q_i} \right)$$

$$= \sum_{j=1}^{s} \sum_{i=1}^{s} b_{j\rho} b_{i\tau} \left(\frac{\partial a_{\sigma j}}{\partial q_i} - \frac{\partial a_{\sigma i}}{\partial q_j} \right), \tag{4.188}$$

and one can conclude that

$$\gamma_{\rho\sigma\tau} = -\gamma_{\rho\tau\sigma}, \qquad \sigma, \rho, \tau = 1, \ldots, s, \tag{4.189}$$

where, in particular,

$$\gamma_{\rho\sigma\sigma} = 0, \qquad \sigma, \rho = 1, \ldots, s. \tag{4.190}$$

Finally it is worth emphasizing that values of the Boltzmann symbols depend only upon the definition of quasi-velocity $\overset{\circ}{\vartheta}_\rho$ through generalized velocities $\dot{q}_\sigma$ (see (4.174) and do not depend upon the motion of the modelled system.

4.3.3 The Lagrange–Maxwell equations

Electromechanical analogies presented in section 2.2.3 suggest the idea of using the method of Lagrange in modelling the dynamics of electromechanical systems. A first step was to admit that electrical charge has the nature of a generalized coordinate.

The second step requires recalling the elementary information from electricity and magnetism (see e.g. Rogers (1960)). We learned there that in certain circumstances an electrodynamic force arises. This force can set the conductor in motion, and that is what in fact happens and is the basis for the functioning of an electric motor. We may also remember an experiment in which the appearance of electromotive force in a coil due to motion of a magnet was demonstrated. These facts together mean that we cannot treat the electromechanical system as a straightforward combination of electrical and mechanical parts. Speaking of this we mention that the electrical part alone could well be described with the help of Kirchhoff's laws, and the mechanical part alone by means of linear and angular momentum balance laws. In view of the mutual influence of both parts, neither of these kinds of laws is applicable separately. In such situations it seems that the ideal issue is to use the energy approach, since energy is an additive quantity. An energy approach of this type suggests the use of the Lagrangian method. We must only define two energies components, viz.

$$T_{\text{el}} = \tfrac{1}{2} H \dot{e}^2, \tag{4.191}$$

where the notation for induction of the coil has been changed from L to H, in order to avoid confusion with the Lagrangian, and

$$V_{\text{el}} = \frac{1}{2C} e^2. \tag{4.192}$$

In order to illustrate the above, consider a simple electromechanical system composed of an oscillator and an electric circuit (see Fig. 4.13). The current state of the system can

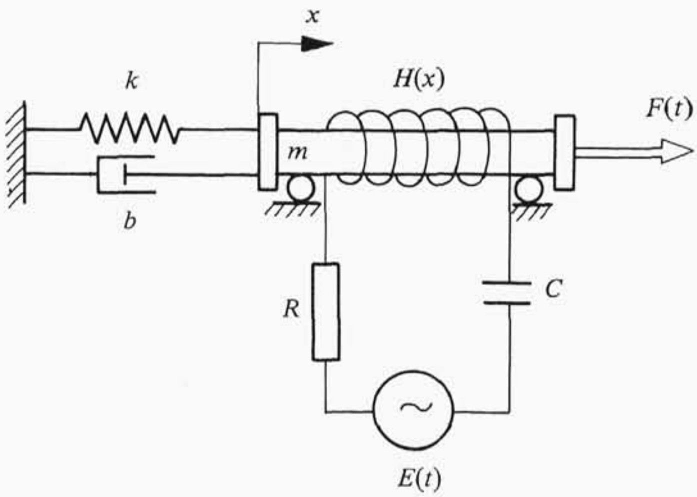

Fig. 4.13

be described by means of two coordinates: displacement x of the mass from the equilibrium position and the electric charge e carried by the current flowing in the circuit.

Now comes the most important moment—and not only for the system currently considered—the combination of energy of both subsystems, i.e.

$$L = L_{\mathrm{mech}} + L_{\mathrm{el}}, \tag{4.193}$$

where, in our example,

$$L_{\mathrm{mech}} = T_{\mathrm{mech}} - V_{\mathrm{mech}} = \tfrac{1}{2} m\dot{x}^2 - \tfrac{1}{2} kx^2 \tag{4.194}$$

and on the basis of (4.191) and (4.192)

$$L_{\mathrm{el}} = T_{\mathrm{el}} - V_{\mathrm{el}} = \tfrac{1}{2} H(x)\dot{e}^2 - \frac{1}{2C} e^2. \tag{4.195}$$

The Lagrange equations (4.171) take the form

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{dt}}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = Q_x, \\ &\frac{\mathrm{d}}{\mathrm{dt}}\frac{\partial L}{\partial \dot{e}} - \frac{\partial L}{\partial e} = Q_e, \end{aligned} \tag{4.196}$$

where Q_x and Q_e are nonconservative forces corresponding to coordinates x and e, respectively. We shall determine these forces in the same way as for a 'usual' mechanical system (see section 4.2.3.2). From (4.111) we have

$$\delta W = (F - b\dot{x})\delta x + (F - R\dot{e})\delta e,$$

whence

$$Q_x = F - b\dot{x} \qquad \text{and} \qquad Q_e = E - R\dot{e}. \tag{4.197}$$

Having used (4.193)–(4.195) and (4.197) in equations (4.196) we obtain

$$\begin{aligned} m\ddot{x} + b\dot{x} + kx &= F(t) + \frac{1}{2}\frac{\mathrm{d}H(x)}{\mathrm{d}x}\dot{e}^2, \\ H(x)\ddot{e} + R\dot{e} + \frac{1}{C}e &= E(t) - \frac{\mathrm{d}H(x)}{\mathrm{d}x}\dot{e}\dot{x}. \end{aligned} \tag{4.198}$$

Comparing (4.198) with equations (2.39) and (2.40) we observe now the appearance of new expressions: in the first equation of the system (4.198) the quantity

$$F_{\mathrm{el}} = \frac{1}{2}\frac{\mathrm{d}H(x)}{\mathrm{d}x}i^2(t) \tag{4.199}$$

appears, while in the second equation of the system, (4.198), the quantity

$$U = -\frac{\mathrm{d}H(x)}{\mathrm{d}x}i(t)\dot{x}(t) \tag{4.200}$$

appears, where relation (2.37) has been made use of.

Quantity (4.199) is an electrically induced force acting upon the oscillator, while quantity (4.200) is voltage induced in the circuit due to a change of magnetic resistance of the coil circuit; note that this voltage depends upon the velocity of the coil motion. It is that term which is responsible for the effects observed during classroom experiments in electromagnetic induction.

One can state therefore that application of the method of Lagrange to the modelling of a simple electromechanical system has been successful. Now, the question arises as how universally applicable this method is. It turns out that this has so far remained unknown, which is surprising, since this natural manner of treating the electromechanical system was formulated by **James C. Maxwell** (1831–1879) some 120 years ago in the form of a hypothesis. Thus, Maxwell formulated the hypothesis that the equation describing the behaviour of a complex electro-acoustic-mechanical system can be represented in the following form:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \dot{q}_\rho} - \frac{\partial T}{\partial q_\rho} + \frac{\partial J}{\partial \dot{q}_\rho} + \frac{\partial V}{\partial q_\rho} = Q_\rho^{\mathrm{nc}}, \qquad \rho = 1,\dots,r, \tag{4.201}$$

where $T = T_{\mathrm{mech}} + T_{\mathrm{el}}$, $V = V_{\mathrm{mech}} + V_{\mathrm{el}}$, J is a dissipation function of both parts of the system, Q_ρ^{nc} is the nonconservative external generalized force corresponding to the generalized coordinate q_ρ, and $r = s + f$ with s denoting the number of degrees of freedom of the mechano-acoustic part of the complex system, and f denoting the number of degrees of freedom of its magneto-electric part.

In order to obtain a more concise formulation used for the electro-mechanical systems, we refer to the definition of the Lagrangian function (4.193), while dissipation forces will be included in the nonconservative forces. We then obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_\rho}-\frac{\partial L}{\partial q_\rho}=Q_\rho^{\mathrm{nc}}, \qquad \rho=1,\ldots,r, \tag{4.202}$$

where

$$L=(T_{\mathrm{mech}}+T_{\mathrm{el}})-(V_{\mathrm{mech}}+V_{\mathrm{el}}). \tag{4.203}$$

Equation (4.202) and function (4.203) will be called, respectively, the **Lagrange–Maxwell equation** and **function**.

Since we have not presented a decent derivation of the Lagrange–Maxwell equations, we present three assumptions whose fulfilment is necessary for application of these equations:

(1) we assume that the behaviour of the mechanical part of an electromechanical system can be described by means of a discrete model having s 'mechanical' degrees of freedom;

(2) we assume that in every system electrical circuits are closed, meaning that conductors do not touch each other; the possibility of contact, for instance, via a commutator gives rise to nonholonomic constraints and would require separate treatment;

(3) a continuous electrical part of the electro-mechanical system can be described with the help of a finite number of 'electrical' generalized coordinates, if the condition of quasi-stationarity is satisfied; that is, changes over time in the intensity of an electromagnetic field do not influence the value of magnetic induction.

4.3.4 Case studies

4.3.4.1 Does a bell always ring?

This problem has a certain historical interest due to experiments performed with the giant bell *Kaiserglocke* of the famous Cologne Cathedral in Germany. In some situations a bell does not ring owing to the failure of the clapper to strike the side of the bell; we will establish the condition under which a bell fails to ring. A bell, together with its clapper, is modelled as a mechanical system composed of two compound pendulums. The pendulum which represents the shell rotates about the fixed, horizontal axis through O, called the axis of suspension, while the pendulum representing the clapper rotates about the axis A, connected to the first pendulum at the hinge axis (Fig. 4.14a). We assume that the bell and its clapper move in one plane. Then, the system has two degrees of freedom. The coordinates are the angular displacement α of the first pendulum, and the angular displacement β of the second one relative to the vertical direction (4.14b). Both α and β are assumed to be small. We use the following notations:

m_b, m_c are the masses of the bell and the clapper, respectively;
I_0 is the moment of intertia of the bell about its axis of suspension;
I_C is the moment of intertia of the clapper with respect to its centre of gravity C;
a is the distance between the axis of suspension and the hinge axis;
b is the distance of the centre of gravity, B, of bell from the axis of suspension;
c is the distance of the centre of gravity, C, of clapper from the hinge axis.