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APPROXIMATE SOLUTION OF THE PROBLEM OF BENDING OF PLATES WITH MIXED BOUNDARY CONDITIONS

by

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SUMMARY

In the paper approximate solutions of the problem of bending of a plate are given; to derive these solutions the corresponding differential equation has been written in the finite-difference form. Sec.2 gives the solution in a form of ordinary finite series, in Sec.3 the equation is considered as a differential-difference one. In Sec.4 double finite series are applied.

The problem of discontinuous boundary conditions can be reduced to a system of equations involving finite sums; these equations correspond to the Fredholm integral equations of the 1st kind which are obtained from the differential equation of the problem.

SOMMAIRE

Le travail est consacré à la recherche de la solution de l'équation différentielle de flexion d'une plaque, en remplaçant les déri-

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vées par les quotients des différences finies.

Au par. 2 une solution de flexion d'une plaque est donnée;on y applique des séries unitaires finies.Au par.3 l'équation est considérée comme étant différentielle et aux différences finies. Au par 4. on donne une solution basée sur l'emploi des séries finies doubles.

Le problème des conditions mixtes aux limites dans la théorie des plaques peut être réduit au système d'équations de sommation.Ces equations présentent une analogie avec les équatinos intégrales de Fredholm du premier type que l'on peut obtenir en résolvant l'équation différentielle de flexion d'une plaque.

INTRODUCTION

The calculus of finite differences found a broad application in the theory of plates owing to the works by N.J.NIELSEN, /1/, H.MARCUS, /2/, P.M. WARWAK, /3/. By replacing the derivatives in the differential equation of the theory of plates by difference quotients, a differential equation is replaced by a partial difference equation, the continuous deflection surface thus being represented - in an approximate manner by means of a polyhedron. Treating the partial difference equation as a system of linear algebraic equations, and solving these equations by known methods /the Gaussian elimination method or various iteration methods/ approximate values of plate deflection are obtained. The partial difference equation may also be treated as a matrix equation /4/, /5/ and solved by means of matrix methods in a manner shown by E. EGERVÁRY, /6/.

Another way, which will be followed in the present paper, is to solve the partial difference equation of plate deflection by means of methods of finite differences successfully applied to plane gridworks by H. BLEICH and E. MELAN, /7/. In this way full analogy between the solution of the differential equation and that of a difference equation of plate deflection is obtained. By letting the net become more and more dense we can always pass to the results obtained in the domain of differential equations.

In this paper the solution of the problem of plate bending is given in the form of ordinary and double finite series. The applied procedure has been constructed from the standpoint of application of digital computers.

The differential equation of bending of a plate has the form

/1.1/ $$N \nabla^4 w(x,y) = q(x,y)$$

In this equation w denotes the deflection, q - the load, N - the bending rigidity of the plate.

Let us remplace the derivaties in /1.1/ by difference quotients. Dividing the edge a of the rectangle into n equal segments Δx, the edge b into n equel segments Δy, we reduce the $Eq.$ /1.1/ to the form

/1.2/ $$L_{xy}(w_{xy}) = \varkappa q_{xy}, \quad \varkappa = \Delta x^4/N \qquad (x=0,1,2,\dots,n; \quad y=0,1,2\dots,m),$$

where

/1.3/ $$L_{xy}(w_{xy}) = (\Delta_x^4 + 2\varepsilon^2 \Delta_x^2 \Delta_y{}^2 + \varepsilon^4 \Delta_y^4) w_{xy}, \qquad \varepsilon = \frac{\Delta x}{\Delta y} = \frac{am}{bn},$$

and Δ_x^2, Δ_x^4 denote the second and the fourth difference in the x-direction and Δ_y^2, Δ_y^4 the second and the fourth difference in the y-direction, respectivly, where

/1.4/ $$\begin{cases} \Delta_x^2 (w_{xy}) = w_{x-1,y} - 2 w_{xy} + w_{x+1,y} \\ \Delta_x^4 (w_{xy}) = w_{x-2,y} - 4 w_{x-1,y} + 6 w_{xy} - 4 w_{x+1,y} + w_{x+2,y} \end{cases}$$

APPLICATION OF SIMPLE FINITE SERIES TO THE SOLUTION OF PLATE PROBLEMS

Let us consider a rectangular plate simply supported on two opposite edges, and supported in any way on the remaining edges. Our considerations will be confined to the static case, although there is no obstacle to generalize them to the problem of forced vibration and simultaneous bending and compression. The solution of the difference equation of the plate

/2.1/ $$L_{xy}(w_{xy}) = \varkappa q_{xy} \qquad (x=0,1,\dots,n; \quad y=0,1,2,\dots,m)$$

will be sought-for /assuming that the edges $y=0$, $y=m$ are simply supported/ in the form of the finite simple series

/2.2/ $$w_{xy} = \sum_{\mu=1}^{m-1} Y_y^\mu X_x^\mu ,$$

where

/2.3/ $$Y_y^\mu = \sqrt{\frac{2}{m}} \sin \beta_\mu y, \qquad \beta_\mu = \frac{\mu\pi}{m} \quad (\mu=1,2,\dots, m-1).$$

The series /2.2/ will constitute the accurate solution of the Eq./2.1/ if the functions $L_{xy}(w_{xy}) - q_{xy}$ are orthogonal to each of the functions Y_y^μ.

The following conditions must be satisfied

/2.4/ $$\sum_{y=0}^{m-1} \left[L_{xy}\left(\sum_{\mu=1}^{m-1} Y_y^k X_x^k \right) - q_{xy} \varkappa \right] Y_y^\mu = 0 \qquad (\mu=1,2,\dots, m-1).$$

The Eq. /2.4/ is reduced to the form

/2.5/ $$\sum_{\mu=1}^{m-1} \sum_{y=1}^{m-1} Y_y^\mu L_{xy}(Y_y^k X_x^k) = \varkappa q_x^\mu , \qquad q_x^\mu = \sum_{y=1}^{m-1} q_{xy} Y_y^\mu$$

Bearing in mind that

/2,6/ $$L_{xy}(Y_y^k X_x^k) = (\Delta_x^4 + 2\varepsilon^2 b_k \Delta_x^2 + \varepsilon^4 b_k^2) X_x^k Y_y^k, \qquad b_k = 2(\cos\beta_k - 1)$$

we obtain from the Eq. /2.5/ the following ordinary difference equation

/2.7/ $$(\Delta_x^4 + 2\varepsilon^2 b_\mu \Delta_x^2 + \varepsilon^4 b_\mu^2) X_x^\mu = \varkappa q_x^\mu \quad (x=0,1,2,\dots,n; \;\; y=0,1,2,\dots, m-1).$$

This equations may also be represented in the form

/2.8/ $$X_{x-2}^\mu - c_\mu X_{x-1}^\mu + d_\mu X_x^\mu - c_\mu X_{x+1}^\mu + X_{x+2}^\mu = \varkappa q_x^\mu ,$$

where $$c_\mu = 4 - 2\varepsilon^2 b_\mu , \qquad d_\mu = 6 - 2\varepsilon^2 b_\mu + \varepsilon^4 b_\mu^2 .$$

The solution of the Eq. /2.8/ is composed of a particular solution and the general solution of the homogeneous equation

/2.9/ $$X_x^\mu = \bar{X}_x^\mu + (C_1^\mu + x C_2^\mu)\lambda_\mu^x + (C_3^\mu + C_4^\mu x)\lambda_\mu^{-x} ,$$

where $$\lambda_y = (1+\varepsilon^2\varrho_\mu^2) - \varepsilon\varrho_\mu\sqrt{2+\varepsilon^2\varrho_\mu^2}\,, \qquad \varrho_\mu = \left|\sin\frac{\beta_\mu}{2}\right|,$$

/2.10/ $$X_x^\mu = \bar{X}_x^\mu + A_1^\mu \operatorname{ch}\vartheta_\mu x + A_2^\mu x \operatorname{sh}\vartheta_\mu x + A_3^\mu x \operatorname{ch}\vartheta_\mu x + A_4^\mu \operatorname{sh}\vartheta_\mu x,$$

where $$\vartheta_\mu = \ln\varrho_\mu .$$

The function $\bar{X}_x^\mu$ is a particular solution of the Eq. /2.7/. From the boundary conditions /two for each edge/ we find the constants A_1^μ, ..., A_4^μ and X_x^μ . The knowledge of this function enables the deflection w_{xy} to be obtained from the Eq. /2.2/. This procedure is of importance for plates, of which one edge is very long or infinite /an infinite strip/.

If the number of segments is small $/m < 10/$ it is more convenient to treat the Eq. /2.8/ as a system of algebraic equations.

Let us consider the case of a plate strip simply supported on the edges $y=0$, $y=m$ and loaded by a concentrated force P at the point $(0,\eta)$. The solution of the Eq. /2.8/ will be sought-for by making use of the Fourier integral transformation proposed by I. BABUŠKA,/8/, for difference equations.

The Fourier transformation in an infinite interval is defined by the infinite series

/2.11/ $$\mathcal{F}(X_x) = X^*(\alpha) = \sum_{x=-\infty}^{x=\infty} X_x e^{i\alpha x},$$

where the sum $\sum_{x=-\infty}^{x=\infty} |X_x|$ should be bounded. The inverse Fourier transformation is defined by the equation

/2.12/ $$X_x = \mathcal{F}^{-1}(X_x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X^*(\alpha) e^{-i\alpha x} d\alpha .$$

From the definition /2.12/ it follows, for instance, that the transformation of the Dirac function $\delta(x-\xi)$ is

/2.13/ $$\mathcal{F}[\delta(x-\xi)] = \sum_{x=-\infty}^{x=\infty} \delta(x-\xi) e^{i\alpha x} = e^{i\alpha\xi} .$$

In our consideration use will be made of the following rule:

/2.14/ $$\mathcal{F}(X_{x\pm p}) = \mathcal{F}(X_x) e^{\mp i p \alpha} .$$

Let us express the Eq. /2.8/ in the form

/2.15/ $$X^{\mu}_{x-2} - c_{\mu} X^{\mu}_{x-1} + d_{\mu} X^{\mu}_{x} - c_{\mu} X^{\mu}_{x+1} + X^{\mu}_{x+2} = \varkappa Y^{\mu}_{\eta} P \delta(x)$$

in view of the relation

$$q^{\mu}_{x} = P \sum_{y=1}^{m-1} \delta(y-\eta)\,\delta(x)\, Y^{\mu}_{y} = P Y^{\mu}_{\eta}\, \delta(x).$$

Performing on the Eq. /2.15/ the Fourier transformation, we obtain, bearing in mind /2.11/, /2.13/ and /2.14/

$$\left[e^{2i\alpha} + e^{-2i\alpha} - c_{\mu}(e^{-i\alpha} + e^{i\alpha}) + d_{\mu}\right] X^{\mu *}(\alpha) = \varkappa Y^{\mu}_{\eta} P,$$

or

$$\left[a(\alpha) + \varepsilon^2 b_{\mu}\right]^2 X^{\mu *}(\alpha) = \varkappa Y^{\mu}_{\eta} P, \quad a(\alpha) = 2(\cos\alpha - 1).$$

Therefore

/2.16/ $$X^{\mu}_{x} = \frac{\varkappa P Y^{\mu}_{\eta}}{2\pi} \int_{-\pi}^{\pi} \frac{\cos\alpha x\, d\alpha}{\left[a(\alpha) + \varepsilon^2 b_{\mu}\right]^2}.$$

Bearing in mind /2.2/ the deflection of the plate can be represented thus

/2.17/ $$W_{xy} = \frac{\varkappa P}{2\pi} \sum_{\mu=1}^{m-1} Y^{\mu}_{\eta} Y^{\mu}_{y} \int_{-\pi}^{\pi} \frac{\cos\alpha x\, d\alpha}{\left[a(\alpha) + \varepsilon^2 b_{\mu}\right]^2}.$$

Making use of the result /5.17/ a number of other problems may be solved. Thus, for a semi-strip acted on by a concentrated force at a point (ξ, η) we obtain, by super position of two forces.

Let now the plate strip be acted on by a load $q_{xy} = q_y$ /independent of y only/ and by concentrated forces P_{η} along the segment $k\Delta y$, $(k < m)$ of the line $x = 0$.

The deflection thus produced is

/2.18/ $$W_{xy} = \varkappa \sum_{\mu=1}^{m-1} \frac{q_{\mu}}{b^2_{\mu}} Y^{\mu}_{y} + \frac{\varkappa}{\pi} \sum_{\eta=1}^{k} P_{\eta} \sum_{\mu=1}^{m-1} Y^{\mu}_{\eta} Y^{\mu}_{y} \int_{-\pi}^{\pi} \frac{\cos\alpha x\, d\alpha}{\left[a(\alpha) + \varepsilon^2 b_{\mu}\right]^2}.$$

We require now that the deflection be equal to zero for $y = 1, 2, \ldots, \eta, \ldots, k$.

From the condition $w_{0y} = 0$, we obtain

$$/2.19/ \quad \begin{cases} \sum_{\mu=1}^{m-1} \frac{q_\mu}{b_\mu^2} Y_y^\mu + \frac{\varkappa}{\pi} \sum_{\eta=1}^{k} P_\eta \sum_{\mu=1}^{m-1} Y_\eta^\mu Y_y^\mu e_\mu = 0 & (y=1,2,\dots,k), \\ e_\mu = \int_{-\pi}^{\pi} \frac{d\alpha}{[a(\alpha)+\varepsilon^2 b_\mu]^2} . \end{cases}$$

Thus, a system of k equations for the determination of the unknown quantities P_η has been obtained.

The Eq. /2.19/ may be given still another form. Let us expand P_η in a series

$$/2.20/ \qquad P_\eta = \sum_{j=1}^{m-1} A_j Y_\eta^j .$$

and insert in /2.19/. We obtain

$$/2.21/ \qquad \sum_{j=1}^{m-1} A_j a_{j\mu} + \frac{q_\mu}{b_\mu^2 e_\mu} = 0, \qquad a_{j\mu} = \sum_{\eta=1}^{k} Y_\eta^j Y_\eta^k .$$

If the plate is supported over the entire width $b = m\Delta y$, we have $a_{j\mu} = \delta_{j\mu}$, and the Eq. /2.21/ becomes

$$/2.22/ \qquad A_\mu = -\frac{q_\mu}{e_\mu b_\mu^2} \qquad (\mu = 1,2,\dots, m-1).$$

The Eq./2.21/ constitutes the solution with mixed boundary conditions along the line $x=0$. As the net becomes more dense along the line $x=0$, the approximation to the reality becomes better and better. In the limit case for $m \to \infty$, the Eq./2.19/ becomes a Fredholm integral equation of the first kind, /9/.

APPLICATION OF DIFFERENCE-DIFFERENTIAL EQUATIONS TO THE THEORY OF PLATES

In a number of plate problems especially with mixed boundary conditions, it may be very useful to describe the deflection of the plate by means of a difference-differential equation. Let us divide the plate into m equal strips of width Δy in the direction of the y-axis. Let us denote the function, expressing the deflection along the lines $y = 0, 1, 2, \dots, m$ by $w_y(x)$. Then

$$/3.1/ \qquad \begin{cases} \dfrac{\partial^4 w_y(x)}{\partial x^4} + 2\varkappa^2 \dfrac{\partial^2}{\partial x^2}\left[\Delta_y^2\big(w_y(x)\big)\right] + \varkappa^4 \Delta_y^4\big(w_y(x)\big) = \dfrac{q_y(x)}{N}, \\ 0 \leqslant x \leqslant a, \quad y = 0,1,2,\dots,m, \quad \varkappa^2 = \dfrac{1}{\Delta y^2}, \quad \Delta y = \dfrac{b}{m} \end{cases}$$

is the difference-differential equation of deflection. Let us assume now that the edges $y=0$, $y=m$ are simply supported.

Assuming the solution of /3.1/ in the form

$$/3.2/ \qquad w_y(x) = \sum_{\mu=1}^{m-1} Y_y^\mu X_\mu(x), \qquad Y_y^\mu = \sqrt{\frac{2}{m}} \sin \beta_\mu y, \quad \beta_\mu = \frac{\mu\pi}{m},$$

expanding the load $q_y(x)$ in a series of functions Y_y^μ

$$/3.3/ \qquad q_y(x) = \sum_{\mu=1}^{m-1} q_\mu(x) Y_y^\mu, \qquad q_\mu(x) = \sum_{y=1}^{m-1} q_y(x) Y_y^\mu,$$

and applying the orthogonalization method, we obtain the following ordinary linear differential equation

$$/3.4/ \qquad \begin{cases} \left[\dfrac{d^4}{dx^4} - 2\varkappa^2 c_\mu^2 \dfrac{d^2}{dx^2} + \varkappa^4 c_\mu^4\right] X_\mu(x) = \dfrac{q_\mu(x)}{N}, \\ c_\mu^2 = 2(1-\cos\beta_\mu) = 4\sin^2 \dfrac{\beta_\mu}{2}. \end{cases}$$

The solution of this equation has the form

$$/3.5/ \qquad X_\mu(x) = \bar{X}_\mu(x) + C_{1,\mu} \operatorname{ch} \tau_\mu x + C_{2,\mu} x \operatorname{sh} \tau_\mu x + C_{3,\mu} \operatorname{sh} \tau_\mu x + C_{4,\mu} x \operatorname{ch} \tau_\mu x,$$

where $\bar{X}_\mu(x)$ is a particular integral of the Eq. /3.4/ and

$$/3.6/ \qquad \tau_\mu = \varkappa c_\mu = 2\varkappa \left|\sin \frac{\beta_\mu}{2}\right|$$

Let us consider an infinite plate strip acted on by a concentrated force P at the point $(0,\eta)$. Performing on the Eq. /3.4/ the Fourier transformation, and bearing in mind the fact that $q_\mu(x) = P Y_\eta^\mu \delta(x)$, we obtain

$$/3.7/ \qquad X_\mu(x) = \frac{P Y_\eta^\mu}{\pi N} \int_0^\infty \frac{\cos \alpha x \, d\alpha}{(\alpha^2 + \tau_\mu^2)^2},$$

or

$$/3.8/ \qquad w_y(x) = \frac{P}{4N} \sum_{\mu=1}^{m-1} \frac{Y_\eta^\mu Y_y^\mu}{\tau_\mu^3} (1 + x\tau_\mu) e^{-\tau_\mu x} \qquad x > 0.$$

Introducing an auxiliary function

$$\Phi_y(x) = -\frac{P}{2N}\sum_{\mu=1}^{m-1}\frac{Y_y^{\mu}Y_q^{\mu}}{\tau_\mu}e^{-\tau_\mu x} \qquad x>0, \tag{3.9}$$

we can express the quantities $\partial^2 w_y/\partial x^2$ and $\Delta_y^2(w_y)$ by the following simple equations

$$\frac{\partial^2 w}{\partial x^2} = \frac{1}{2}\left(\Phi_y + x\frac{\partial \Phi_y}{\partial x}\right), \qquad \varkappa^2\Delta_y^2(w_y) = \frac{1}{2}\left(\Phi_y - x\frac{\partial \Phi_y}{\partial x}\right). \tag{3.10}$$

Let us consider two auxiliary problems. Let a concentrated moment act at the point $(0,\eta)$ of the semi-strip simply supported on all edges - /Fig.1/. Bearing in mind the boundary conditions

$$w_y(0)=0, \quad m_y^x(0) = M\delta_{y\eta} \tag{3.11}$$

the deflection of the strip is obtained in the form

$$w_y^M(x) = \frac{xM}{2N}\sum_{\mu=1}^{m-1}\frac{Y_\eta^{\mu}Y_y^{\mu}}{\tau_\mu}e^{-\tau_\mu x} \qquad x>0. \tag{3.12}$$

If the semi-strip is acted on by the load $q_y(x)=q=$ const, then

$$w_y^q(x) = \frac{1}{N}\sum_{\mu=1}^{m-1}\frac{q_\mu}{\tau_\mu^4}Y_y^{\mu}\left[1-\left(1+\frac{x\tau_\mu}{2}\right)e^{-\tau_\mu x}\right] \qquad x>0. \tag{3.13}$$

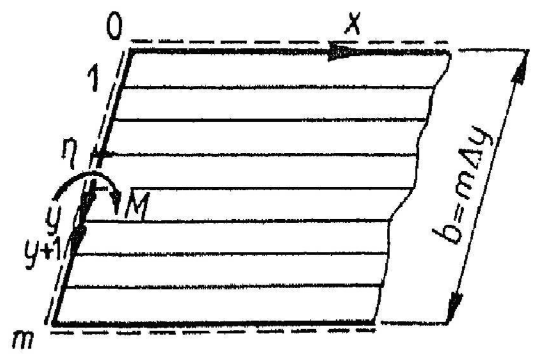

Fig.1

Let us consider a semi-strip simply supported along the edges $y=0$, $y=m$ and clamped along the segment $k\Delta y$ of the edge $x=0$, the remaining part of that edge being also simply supported. Let the plate be acted on by a load $q_y(x)=q=$ const /Fig.2/ We are concerned with a problem of mixed boundary conditions. Denoting the unknown clamping moments by $M_\eta, 1\leqslant\eta\leqslant k$, the deflection of the plate is obtained in the form

/3.14/ $$w_y(x) = w_y^q(x) + \sum_{\eta=1}^{k} M_\eta w_\eta^M(x).$$

From the condition $[\partial w/\partial x]_{x=0}$ on the segment $k\Delta y$ of the edge $x=0$ the following system of equations is obtained

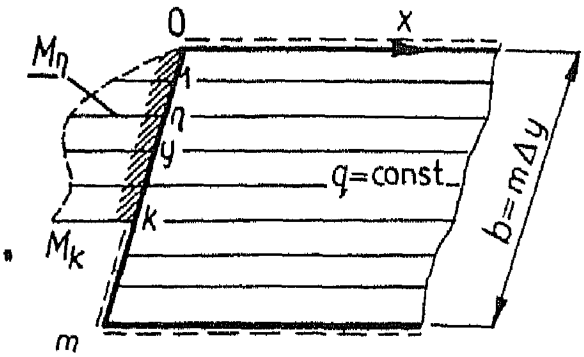

Fig.2

/3.15/ $$\sum_{\mu=1}^{m-1} \frac{q_\mu Y_y^\mu}{\tau_\mu^3} + \sum_{\eta=1}^{k} M_\eta \sum_{\mu=1}^{m-1} \frac{Y_y^\mu Y_\eta^\mu}{\tau_\mu} = 0 \qquad (y=1,2,\dots,k).$$

Solving this system of equations we find M_η, and from /3.14/ the deflection $w_y(x)$.

Another solution procedure is such. Let us expand M_η in a series of functions Y_η^μ

/3.16/ $$M_\eta = \sum_{j=1}^{m-1} A_j Y_\eta^j, \qquad A_j = \sum_{\eta=1}^{m-1} M_\eta Y_\eta^j.$$

Inserting /3.16/ in /3.15/ we obtain the system of equations

/3.17/ $$\sum_{j=1}^{m-1} A_j a_{j\mu} + \frac{q_\mu}{\tau_\mu^2} = 0, \qquad a_{j\mu} = \sum_{\eta=1}^{k} Y_\eta^\mu Y_\eta^j.$$

If the plate is supported over the entire width, then, bearing in mind that $a_{j\mu} = \delta_{j\mu}$ for $k=m-1$, the Eq. /3.17/ is reduced to the form

/3.18/ $$A_\mu + \frac{q_\mu}{\tau_\mu^2} = 0.$$

APPLICATION OF DOUBLE FINITE SERIES TO THE SOLUTION OF PLATE PROBLEMS

Let us consider the equation of bending of plate /2.1/. The solu -

tion of Eq. /2.1/ will by sought in the form of double finite series

$$w_{xy} = \sum_{\nu,\mu}^{n,m} A_{\nu\mu}\, \varphi_{xy}^{\nu\mu} \tag{4.1}$$

where the quantities $A_{\nu\mu}$ are the unknown coefficients and $\varphi_{xy}^{\nu\mu}$ are the eigenfunctions of the difference equations

$$L_{xy}\left(\varphi_{xy}^{\nu\mu}\right) = \sigma_{\nu\mu}\, \varphi_{xy}^{\nu\mu} \tag{4.2}$$

assuming that the functions $\varphi_{xy}^{\nu\mu}$ satisfy the same boundary conditions as the function w_{xy} .

The quantities $\sigma_{\nu\mu}$ are the eigenvalues $(\nu = 0,1,2,\ldots,n;\quad \mu = 0,1,2,\ldots,m)$ corresponding to the eigenfunctions $\varphi_{xy}^{\nu\mu}$. The latter constitute a complete set of orthonormal functions, therefore they satisfy the conditions

$$\sum_{x,y}^{n,m} \varphi_{xy}^{\nu\mu}\, \varphi_{xy}^{ik} = \delta_{\nu i}\, \delta_{\mu k}\,, \tag{4.3}$$

where $\delta_{i\nu}$, $\delta_{\mu k}$ are Kronecker's deltas, or

$$\delta_{i\nu} = \begin{cases} 1 & \text{if} \quad i = \nu\,, \\ 0 & \text{if} \quad i \neq \nu\,; \end{cases} \qquad \delta_{\mu k} = \begin{cases} 1 & \text{if} \quad \mu = k\,, \\ 0 & \text{if} \quad \mu \neq k\,. \end{cases} \tag{4.4}$$

If the series /4.1/ is to constitute an accurate solution of the differential equation /2.1/, the functions $L_{xy}(w_{xy}) - q_{xy}\varkappa$ should be orthogonal to every function $\varphi_{xy}^{\nu\mu}$. Therefore

$$\sum_{x,y}^{n,m} \left[L_{xy} \sum_{\nu,\mu}^{n,m} A_{\nu\mu}\, \varphi_{xy}^{\nu\mu} - \varkappa q_{xy} \right] \varphi_{xy}^{ik} = 0. \tag{4.5}$$

Changing the summation order and bearing in mind /4.2/ we obtain

$$\sum_{\nu,\mu}^{n,m} A_{\nu\mu}\, \sigma_{\nu\mu} \sum_{x,y}^{n,m} \varphi_{xy}^{\nu\mu}\, \varphi_{xy}^{ik} = \varkappa q_{ik}\,, \qquad q_{ik} = \sum_{x,y}^{n,m} q_{xy}\, \varphi_{xy}^{ik}. \tag{4.6}$$

Making use of the orthogonality condition /4.3/ we obtain finally

$$A_{ik}\, \sigma_{ik} = \varkappa q_{ik} \qquad (i = 1,2,\ldots,n; \quad k = 1,2,\ldots,m). \tag{4.7}$$

Introducing A_{ik} from the last equation in the Eq. /4.1/ we find

/4.8/ $$w_{xy} = \varkappa \sum_{\nu,\mu}^{n,m} \frac{q_{\nu\mu}}{\sigma_{\nu\mu}} \varphi_{xy}^{\nu\mu} .$$

The solutions represented here for bending of the plate are valid assuming that the functions $\varphi_{xy}^{\nu\mu}$ can be expressed in the form of a product $X_x^\nu Y_y^\mu$ or $X_x^\nu Y_y^{\nu\mu}$ or $X_x^{\nu\mu} Y_y^\mu$. It will be found that functions of the type $\varphi_{xy}^{\nu\mu} = X_x^\nu Y_y^\mu$ appear in the case of a rectangular plate simply supported on the entire contour, and functions of the type $\varphi_{xy}^{\nu\mu} = X_x^\nu Y_y^{\mu}$–in the case of a plate simply supported on the edges $x=0$, $x=n$, and supported in an arbitrary manner or free along the remaining edges.

Consider a number of particular cases of application of the double series method.

Let the plate be acted on by a load q_{xy} , and let it have an additional immovable support at the point (ξ,η) . The deflection of the plate will be composed of a deflection due to the load q_{xy} and R at the point (ξ,η) the value of R being selected in such a manner that $w_{\xi\eta}=0$. Therefore

/4.9/ $$w_{xy} = \varkappa \sum_{\nu,\mu}^{n,m} \frac{\varphi_{xy}^{\nu\mu}}{\sigma_{\nu\mu}} \left[q_{\nu\mu} + q_{\nu\mu}^{*} \right] ,$$

where

/4.10/ $$q_{\nu\mu}^{*} = R \sum_{x,y}^{n,m} \delta_{x\xi}\, \delta_{y\eta}\, \varphi_{xy}^{\nu\mu} = R \varphi_{\xi\eta}^{\nu\mu} .$$

Inserting $q_{\nu\mu}^{*}$ in /4.9/ and requiring that $w_{\xi\eta}$ 0 , we obtain the equation

/4.11/ $$R \sum_{\nu,\mu}^{n,m} \frac{(\varphi_{\xi\eta}^{\nu\mu})^2}{\sigma_{\nu\mu}} + \sum_{\nu,\mu}^{n,m} \frac{q_{\nu\mu}}{\sigma_{\nu\mu}} = 0 ,$$

from which the amplitude of the support reaction R can be found.

Let now the plate be acted on by, in addition to the load q_{xy} , a load $R_y \delta_{x\xi}$ along the line $x=\xi$. The deflection amplitude of the plate is given by /4.9/ where

/4.12/
$$q^*_{\nu\mu} = \sum_{x,y}^{n,m} R_y \, \delta_{x\xi} \, \varphi^{\nu\mu}_{xy}$$

For a plate simply supported on the entire contour we have

$$\varphi^{\nu\mu}_{xy} = X^\nu_x \, Y^\mu_y$$

Therefore, in this case, we have

/4.13/
$$q^*_{\nu\mu} = X^\nu_\xi \sum_y^m R_y \, Y^\mu_y = X^\nu_\xi b_\mu , \qquad b_\mu = \sum_y^m Y^\mu_y R_y .$$

Substituting /4.13/ in /4.9/ and requiring that the deflection of the plate along the line $x=\xi$ be zero, we obtain the following equation for the coefficients b_μ:

/4.14/
$$b_\mu \sum_\nu^n \frac{(X^\nu_\xi)^2}{\sigma_{\nu\mu}} + \sum_\nu^n \frac{q_{\nu\mu} X^\nu_\xi}{\sigma_{\nu\mu}} = 0 .$$

Knowing b_μ, the support reaction R_y can easily be found

$$R_y = \sum_\mu^m b_\mu Y^\mu_y .$$

Let us consider the case of bending of a plate simply supported on the entire contour and having an additional support along the segment $c_1 = k\Delta y$, $k<m$ of the line $x=\xi$

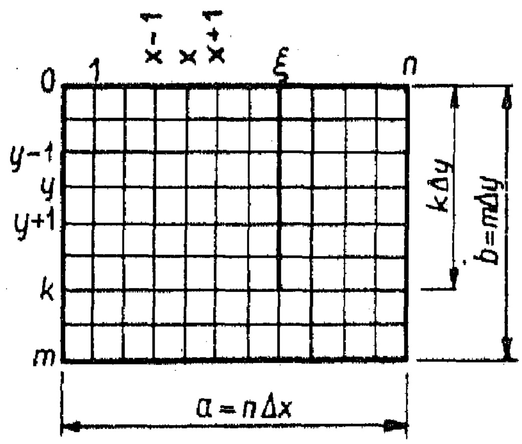

Fig.3

The deflection is espressed thus

/4.15/
$$v_{xy} = \varkappa \sum_{\nu,\mu} \frac{\varphi^{\nu\mu}_{xy}}{\sigma_{\nu\mu}} \left[q_{\nu\mu} + \sum_{\eta=1}^{k} P_\eta \, \varphi^{\nu\mu}_{\xi\eta} \right]$$

where P_η denoted the reactions forces at the points $y=1,2,..k$ along the line $x=\xi$.

Requering that the deflection at the points $y=1,2,...k$ along the line

$x=\xi$ be zero, we obtain, making of /4.9/ the equation

$$\sum_{\nu,\mu} \frac{Y_y^{\mu} X_\xi^{\nu}}{\sigma_{\nu\mu}} \left[q_{\nu\mu} + \sum_{\eta=1}^{k} P_\eta X_\xi^\nu Y_\eta^\mu \right] = 0 \qquad y=1,2,\dots k \tag{4.16}$$

from which the quantities P_η $(\eta=1,2,\dots k)$ can be found.
In the particular case of $k=m$ the bending of plate assumes the form of Eq. /4.14/.
Indeed, on expanding P_η in a series of eigenfunctions

$$P_\eta = \sum_{j=1}^{m} b_j Y_\eta^i \tag{4.17}$$

and substituting in /4.16/ and making use of the relations

$$\sum_{\eta=1}^{m} Y_\eta^j Y_\eta^\mu = \delta_{j\mu}, \quad \sum_{j}^{m} b_j \delta_{j\mu} = b\mu \tag{4.18}$$

the equation /4.14/ ist obtained from /4.16/.

Another solution method may be devised in the case of a plate with two adjacent edges clamped. This method will be discussed by means of the example of a plate clamped along the entire contour.
Let us take a complete set of orthonormal functions $[\eta_x^i]$ $(i=0,1,\dots,m)$ and a set of functions $[\xi_y^k]$ $(k=0,1,2,\dots,m)$ satisfying the difference equation

$$\Delta_x^4(\eta_x^i) = \gamma_i \eta_x^i, \qquad \Delta_y^4(\xi_y^k) = \nu_k \xi_y^k, \tag{4.19}$$

and the clamping conditions. The functions η_x^i may be treated as the natural vibration models of a bar with its ends clamped, the derivatives in the deflection equation being replaced with difference quotients.
Let us expand the deflection W_{xy} and the load q_{xy} in series of functions η_x^i, ξ_y^k

$$W_{xy} = \sum_{i,k} A_{ik} \eta_x^i \xi_y^k, \qquad q_{xy} = \sum_{i,k} q_{ik} \eta_x^i \xi_y^k, \qquad q_{ik} = \sum_{x,y} q_{xy} \eta_x^i \xi_y^k. \tag{4.20}$$

Let us insert the deflection and the load thus expressed in the difference equations of the plate /2.1/

We find

/4.21/ $$\sum_{i,k} A_{ik}\left[(\gamma_i+\vartheta_k\varepsilon^4)\eta_x^i\xi_y^k+2\varepsilon^2\Delta_x^2(\eta_x^i)\Delta_y^2(\xi_y^k)\right]=\sum_{i,k} q_{ik}\eta_x^i\xi_y^k.$$

Let us expand the expression $\Delta_x^2(\eta_x^i)\Delta_y^2(\xi_y^k)$ in a series of eigenfunctions

$$\Delta_x^2(\eta_x^i)\Delta_y^2(\xi_y^k)=\sum_{\gamma,\mu} c_{ik\gamma\mu}\eta_x^i\,\xi_y^k\ ,$$

$$c_{ik\gamma\mu}=\sum_{x,y}\Delta_x^2(\eta_x^i)\Delta_y^2(\xi_y^k)\eta_x^i\xi_y^k\ ,$$

and substitute it in /4.21/. As a result, the following system of equations is obtained

/4.22/ $$A_{ik}(\gamma_i+\vartheta_k\varepsilon^4)+2\varepsilon^2\sum_{\gamma,\mu} c_{\gamma\mu ik}A_{\gamma\mu}=q_{ik},$$

from which the quantities A_{ik} , can be found to be used later for the determination of the function W_{xy} in Eq. /4.22/.

In the case of free vibration, $q_{ik}=0$ should be assumed in equation /4.22/. Thus, a system of equations homogeneous in A_{ik} is obtained. Setting its determinant equal to zero, we obtain the free vibration condition from which the successive frequencies can be found.

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