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The Bending of a Compressed Continuous Plate.

A/ We set ourselves the problem of determining the state of strains and deflections of a plate simply supported along the edges $x = 0$; $x = a$ under the action of uniformly disposed forces q on the edges $x = 0$; $x = a$ by a simultaneous action of a load $p = \text{const.}$, directed perpendicularly to the plane of the plate (Fig.1).

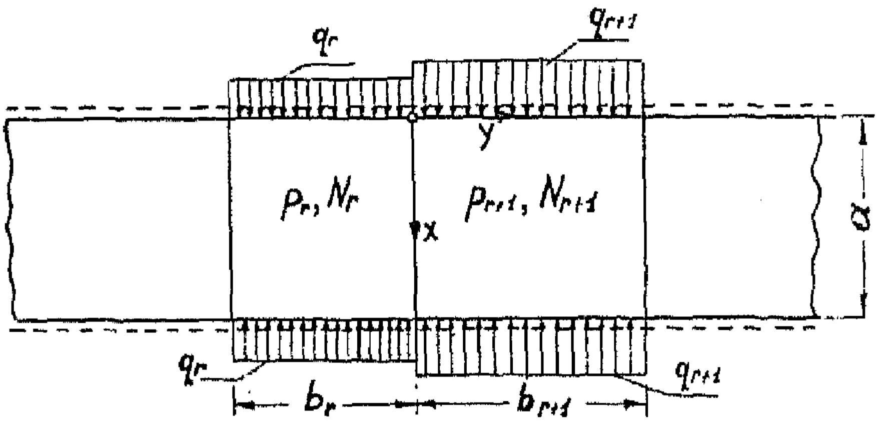


Fig.1

The bending moments $M_y(x,0)$ on the supporting lines $y = 0$; $y = b$ we assume to be redundant quantities of the problem and determine them from the conditions of continuity on these supports.
Before however fixing the three-membered equations of the problem, conditions of the simply supported plate must be defined.

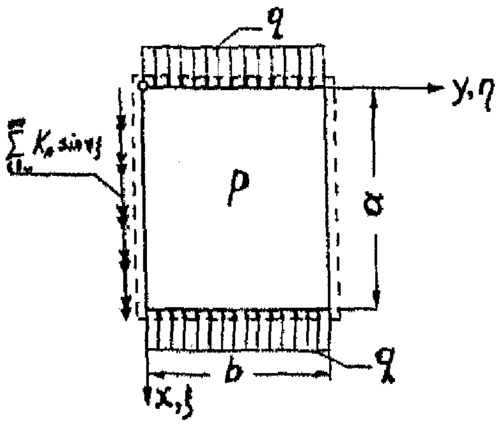


Fig.2

Denoting:

$$y = \eta b \qquad x = \xi b \qquad b = \alpha a \qquad \varphi = \frac{q \cdot b^2}{N\pi^2}$$

the differential equation reads:

$$\Delta\Delta\, \omega(\xi,\eta) + \omega_{\xi\xi}(\xi,\eta)\,\varphi\,\pi^2 = \frac{p b^4}{N} \qquad (1)$$

As the particular integral of the equation (1) we assume the deflection $\omega_0(\xi)$ of an infinitely long plate in the direction η.

$$w_0(\xi) = \frac{4pb^4}{N\pi^5\alpha^4} \sum_{1,3...}^{\infty} \frac{1}{n^5(1-\delta)} \sin \gamma\xi \qquad (2)$$

where

$$\gamma = n\pi\alpha \; ; \qquad \delta = \frac{\varphi\pi^2}{\gamma^2}$$

We shall write the complete integral of the equation as follows:

$$w(\xi,\eta) = \frac{4pb^4}{N\pi^5\alpha^4} \sum_{1,3...}^{\infty} \left[\frac{1}{n^5(1-\delta)} + Y_n \right] \sin \gamma\xi$$

where

$$Y_n(\eta) = \mathcal{U}_1 \cosh \lambda\eta + \mathcal{U}_2 \sinh \lambda\eta + \mathcal{U}_3 \cosh \beta\eta + \mathcal{U}_4 \sinh \beta\eta$$

$$\lambda, \beta = \gamma\sqrt{1 \pm \sqrt{\delta}} \qquad 0 < \delta < 1$$

From the edge conditions of the problem:

$$w(\xi,0) = 0 \; ; \quad w(\xi,1) = 0 ; \qquad w_{\eta\eta}(\xi,0) = -\frac{b^2}{N} \sum_{1,3...}^{\infty} K_n \sin \gamma\xi \qquad w_{\eta\eta}(\xi,1) = 0$$

we get the integration constants, and further on the quantities:

$$\left.\frac{\partial w}{\partial y}\right|_{y=0} = \frac{4pb^3}{N} \sum_{1,3...}^{\infty} \Theta_n \sin \gamma\xi + \frac{b}{N} \sum_{1,3...}^{\infty} K_n \Phi_n \sin \gamma\xi$$

$$\left.\frac{\partial w}{\partial y}\right|_{y=b} = -\frac{4pb^3}{N} \sum_{1,3...}^{\infty} \Theta_n \sin \gamma\xi - \frac{b}{N} \sum_{1,3...}^{\infty} K_n \Psi_n \sin \gamma\xi \qquad (3)$$

where

$$\Theta_n = \frac{\alpha}{\gamma^5} \frac{1}{1-\delta} \frac{\lambda\beta}{\lambda^2-\beta^2} \left(\lambda \tanh \frac{\beta}{2} - \beta \tanh \frac{\lambda}{2} \right) \quad \text{by} \quad 0 < \delta < 1$$

$$\Theta_n = \frac{\alpha}{\gamma^5} \frac{1}{1-\delta} \frac{\lambda\psi}{\lambda^2+\psi^2} \left(\psi \tanh \frac{\lambda}{2} - \lambda \tan \frac{\psi}{2} \right) \quad \text{by} \quad \delta > 1 \quad \psi = i\beta$$

$$\Theta_n = \frac{\alpha}{2\gamma^4}\left(\frac{\gamma}{2} - \frac{1}{\sqrt{2}}\tanh\frac{\gamma}{\sqrt{2}}\right) \qquad \text{by} \quad \delta = 1$$

and further on

$$\Phi_n = \frac{1}{\lambda^2-\beta^2}\left(\lambda\,\mathrm{ctanh}\,\lambda - \beta\,\mathrm{ctanh}\,\beta\right); \quad \Psi_n = \frac{1}{\lambda^2-\beta^2}\left(\frac{\beta}{\sinh\beta} - \frac{\lambda}{\sinh\lambda}\right) \quad \text{by} \quad 0<\delta<1 \tag{5a}$$

$$\Phi_n = \frac{1}{\lambda^2+\psi^2}\left(\lambda\,\mathrm{ctanh}\,\lambda - \psi\,\mathrm{ctan}\,\psi\right) \quad \Psi_n = \frac{1}{\lambda^2+\psi^2}\left(\frac{\psi}{\sin\psi} - \frac{\lambda}{\sinh\lambda}\right) \quad \text{by} \quad \delta>1 \tag{5b}$$

$$\Phi_n = \frac{1}{2\gamma}\left(\sqrt{2}\,\mathrm{ctanh}\,\gamma\sqrt{2} - \frac{1}{\gamma}\right); \qquad \Psi_n = \frac{1}{2\gamma}\left(\frac{1}{\gamma} - \frac{\sqrt{2}}{\sinh\gamma\sqrt{2}}\right) \quad \text{by} \quad \delta = 1 \tag{5c}$$

at last for $q = 0$:

$$\Theta_n = \frac{\alpha}{2\gamma^4}\,\frac{\sinh\gamma - \gamma}{1+\cosh\gamma}; \qquad \Phi_n = \frac{\cosh\gamma\,\sinh\gamma - \gamma}{2\gamma\,\sinh^2\gamma} \tag{5d}$$

$$\Psi_n = \frac{\gamma\cosh\gamma - \sinh\gamma}{2\gamma\,\sinh^2\gamma}$$

Let us consider two neighbouring bays of the continuous plate; the bay (r-1)-r with the side lengths a and b_r loaded with p_r, q_r and bay r-(r+1) with side lengths a, b_{r+1} loaded with q, p. We denote the bending moments along the straight lines r-1, r, r+1 by M^{r+1}, M^r, M^{r+1}.

The continuity of the plate on the support line requires the fulfliment of the following condition:

$$\frac{\partial \omega^r}{\partial y}\Big|_{y=b_r} + \frac{\partial \omega^{r+1}}{\partial y}\Big|_{y=0} = 0 \qquad (6)$$

Using the formula (3) we bring the equation (6) to the form of a system of three-membered equations:

$$K_n^{r-1}\Psi_n^r\mu_r + K_n^r\left(\Phi_n^r\mu_r + \Phi_n^{r+1}\mu_{r+1}\right) + K_n^{r+1}\Psi_n^{r+1}\mu_{r+1} +$$

$$+ 4\left(p_r b_r^2\mu_r \Theta_n^r + p_{r+1} b_{r+1}^2\mu_{r+1}\Theta_n^{r+1}\right) = 0$$

$$(n = 1,3,5\ldots\infty) \qquad (r = 1,2\ldots z-1)$$

We denoted here $\mu_r = \frac{b_r}{b_o}\frac{N_o}{N_r}$ where b_o is the comparative length of the bay and N_o the comparative rigidity of the plate in bending. For the consecutive values of n by (z+1) transverse supports we will get (z-1) three-membered equations, which solution will determine the values of k_n and therefore the bending moments:

$$M^r = \sum_{1,3\ldots}^{\infty} K_n^r \sin\nu\xi$$

It will be sufficient however on account of a strong convergency of the series to limit the numerical examples to the solution of two systems of equations by n = 1 and n = 3.

We ascertain from the equations (7), that in the direction of ξ there can appear only a symmetrical form of plate deflection, quite independently from the value of q.

The case of the determinant of the equations system (12) approaching zero with the increase of the parameter q (δ) tending to $w(\xi,\eta) \to \infty$ is to be rejected, being contrary to the assumptions of the plate theory; the equation (1) is valid only for very small deflections. The load q will never therefore attain the critical value of buckling load for the plate q_k - the plate will be destroyed in consequence of the increase of the bending stresses; it must be observed however, that the parameters q will be the nearer the value q_k, the smaller will be the load p.

We must observe at last, that in the case p = o, the homogenous system of equations (7) will not be contradictory, as the determinant of the equations system will equal zero.

Equating the determinant to zero gives the criterion of the buckling of the continuous plate.

Out of the many particular cases, included in equation (7) we give here two especially simple ones.

a) single bay plate with built in edge y = o and with simply supported edge y = b.

$$K_n = -4pb^2 \frac{\Theta_n}{\Phi_n} \qquad n = 1, 3, 5 \ldots.. \tag{8}$$

b) single bay plate with built in edges y = o, y = b

$$K_n = -4pb^2 \frac{\Theta_n}{\Phi_n + \Psi_n} \qquad n = 1, 3, 5 \ldots. \tag{9}$$

B/ Let us consider a continuous plate (Fig.3) under combined action of lateral loads uniformly disposed on edges y = o, y = b and uniformly distributed loads p perpendicular to the plane of the plate.

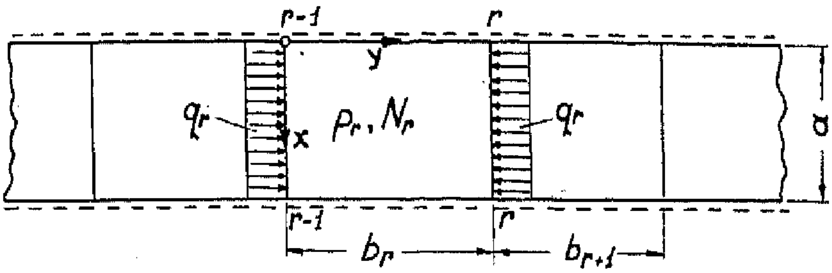


Fig.3

Similarly to the foregoing problem we assume the moments M_y (x,o) on the transverse lines of support as redundant quantities of the problem.

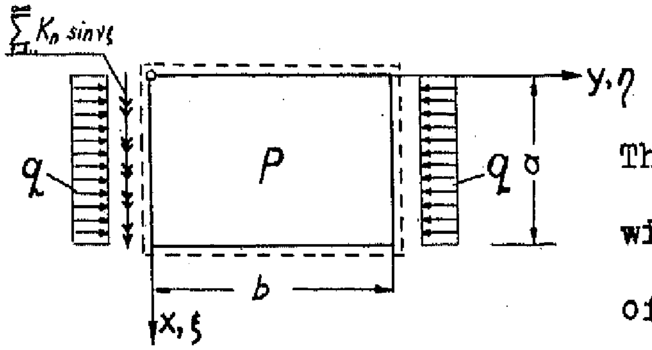


Fig.4

$$\Delta\Delta\, w(\xi,\eta) + \varphi\pi^2 w_{\eta\eta}(\xi,\eta) = \frac{pb^4}{N} \qquad (1')$$

The solution for the continuous plate will be preceded by the determination of the middle plane deflection surface shown in Fig.4.

We will assume as the particular integral of the equation (1')

$$w_0(\xi) = \frac{4pb^4}{N\pi^5\alpha^4} \sum_{1,3...}^{\infty} \frac{1}{n^5} \sin\nu\xi \qquad (1o)$$

The general solution of the equation (1') will be:

$$\omega(\xi,\eta) = \frac{4pb^4}{N\pi^5\alpha^4} \sum\left[\frac{1}{n^5} + Y_n(\eta)\right]\sin\nu\xi$$

where $Y_n(\eta)$ fulfils the common differencial equation:

$$Y_n^{IV} - \nu^2(2-\delta_0)Y_n'' + \nu^4 Y_n = 0 \qquad \delta_0 = \frac{q\alpha^2}{N\pi^2 n^2} \tag{11}$$

With $\delta > 4$ denoting $m = \frac{\nu}{\sqrt{2}}\sqrt{(\delta_0 - 2) + \sqrt{\delta_0(\delta_0 - 4)}}$

we find:

$$Y_n = A_1 \cos m\eta + A_2 \sin m\eta + A_3 \cos\frac{\nu^2}{m}\eta + A_4 \sin\frac{\nu^2}{m}\eta$$

From the edge conditions:

$$\omega(\xi,0) = 0; \quad \omega(\xi,1) = 0; \quad \omega_{\eta\eta}(\xi,0) = -\frac{b^2}{N}\sum_{1,3,\dots}^{\infty} K_n \sin\nu\xi; \quad \omega_{\eta\eta}(\xi,1) = 0$$

we define the integration constants $A_1 \dots A_4$ and futher on the quantities:

$$\left.\frac{\partial\omega}{\partial y}\right|_{y=0} = \frac{4pb^3}{N}\sum_{1,3,\dots}^{\infty}\bar{\Theta}_n \sin\nu\xi + \frac{b}{N}\sum_{1,3,\dots}^{\infty} K_n \bar{\Phi}_n \sin\nu\xi \tag{12}$$

$$\left.\frac{\partial\omega}{\partial y}\right|_{y=b} = -\frac{4pb^3}{N}\sum_{1,3,\dots}^{\infty}\bar{\Theta}_n \sin\nu\xi - \frac{b}{N}\sum_{1,3,\dots}^{\infty} K_n \bar{\Psi}_n \sin\nu\xi$$

where:

$$\bar{\Theta}_n = \frac{\alpha m^2}{\nu^3(m^4 - \nu^4)}\left(\frac{\nu^2}{m}\tan\frac{m}{2} - m\tan\frac{\nu^2}{2m}\right) \tag{13}$$

$$\bar{\Phi}_n = \frac{m^2}{m^4 - \nu^4}\left(\frac{\nu^2}{m}\operatorname{ctan}\frac{\nu^2}{m} - m\operatorname{ctan} m\right)$$

$$\bar{\Psi}_n = \frac{m^2}{m^4 - \nu^4}\left(\frac{m}{\sin m} - \frac{\nu^2}{m\sin\frac{\nu^2}{m}}\right)$$

For $\delta_0 < 4$ denoting

$$\varphi = \frac{\nu}{2}\sqrt{4-\delta_0}\ ; \qquad \psi = \frac{\nu}{2}\sqrt{\delta_0}$$

we bring the general integral of the differential equation (1) to the form:

$$w(\xi,\eta) = \frac{4pb^4}{N\pi^5\alpha^4}\sum_{1,3\ldots}^{\infty}\Big[\frac{1}{n^5} + A_1\cosh\varphi\eta\cos\psi\eta + A_2\cosh\varphi\eta\sin\psi\eta +$$

$$+ A_3\sinh\varphi\eta\cos\psi\eta + A_4\sinh\varphi\eta\sin\psi\eta\Big]\sin\nu\xi$$

in this case the values $\bar{\Theta}_n$, $\bar{\Phi}_n$, $\bar{\Psi}_n$ of the equations (12) are determined by the formulae:

$$\bar{\Theta}_n = \frac{\alpha}{\nu^5}\,\frac{\varphi^2+\psi^2}{2\varphi\psi}\,\frac{\psi\sinh\varphi - \varphi\sin\psi}{\cosh\varphi + \cosh\psi}$$

$$\bar{\Phi}_n = \frac{1}{2\varphi\psi}\,\frac{\psi\sinh\varphi\cosh\varphi - \varphi\cos\psi\sin\psi}{\sinh^2\varphi + \sin^2\psi} \tag{13 a}$$

$$\bar{\Psi}_n = \frac{1}{2\varphi\psi}\,\frac{\varphi\cosh\varphi\sin\psi - \psi\sinh\varphi\cos\psi}{\sinh^2\varphi + \sin^2\psi}$$

Putting in the equations (13, 13a) q = o, we obtain the values $\bar{\Theta}_n, \bar{\Phi}_n, \bar{\Psi}_n$ of the equations (5a)

In the case of simultaneous bending and streching of the plate instead of δ_0 there is to be put $-\delta_0$, i. e. in equations (13a) in the place of φ the value $\frac{\nu}{2}\sqrt{4+\delta_0}$ and in the place of ψ the value $i\psi$ $\quad i = \sqrt{-1}$

In consequence of the continuity equation of the continuous plate

(Fig.3) at the support r-r, i.e. from the condition

$$-\frac{\partial w^r}{\partial y}\Big|_{y=b_r}+\frac{\partial w^{r+1}}{\partial y}\Big|_{y=0}=0$$

we obtain here too an analogous system of equations

$$K_n^{r-1}\,\bar{\Psi}_n^r\mu_r+K_n^r\left(\bar{\Phi}_n^r\mu_r+\bar{\Phi}_n^{r+1}\mu_{r+1}\right)+K_n^{r+1}\,\bar{\Psi}_n^{r+1}\mu_{r+1}+$$

$$+4(p_r b_r^2\mu_r\,\bar{\Theta}_n^r+p_{r+1}\,b_{r+1}^2\mu_{r+1}\,\bar{\Theta}_n^{r+1}=0 \qquad (14)$$

$$(n=1,3,\dots\infty) \qquad (r=1,2,\dots z-1)$$

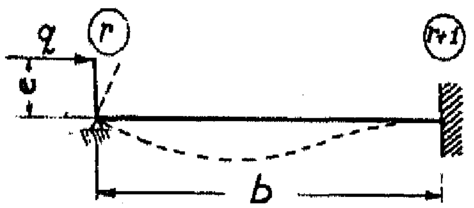
To the equations (14) applies all, that was said on account of equations (7).

Of all particular cases, contained in equation (14) the following two are worth to be mentioned:

a) a single bay plate with built in edges y = o; y = b:

$$K_n=-4pb^2\frac{\bar{\Theta}_n}{\bar{\Psi}_n+\bar{\Phi}_n} \qquad (n=1,3,5,\dots\infty) \qquad (15)$$

b) a single bay plate with built in edge y = b and simply supported edge y = o. The load q uniformly distributed acting on lever arm e. From the equation (14) we obtain:



$$K_n^r\,\bar{\Psi}_n+K_n^{r+1}\,\bar{\Phi}_n=0$$

$$qe=\sum_{1,3,\dots}^{\infty}K_n^r\sin\nu\xi \qquad K_n=\frac{4qe}{\pi n}$$

$$n=1,3,5,\dots$$

Therefore
$$K_n^{r+1} = - \frac{4qe}{n\pi} \frac{\overline{\Psi}_n}{\overline{\Phi}_n} \qquad (n = 1, 3, 5, \dots \infty)$$

We obtain the largest bending moment for $\xi = \frac{1}{2}$; $\eta = 1$

$$M_y = -\frac{4qe}{\pi^k} \sum_{1,3\dots}^{\infty} \frac{1}{n} \frac{\overline{\Psi}_n}{\overline{\Phi}_n} (-1)^{\frac{n-1}{2}}$$

The largest normal stress will be

$$\sigma_{max} = 2 \left(\frac{e}{a}\right)\left(\frac{h}{a}\right) \frac{E\pi\delta_0}{1-\mu^2} \sum_{1,3,\dots}^{\infty} \frac{\overline{\Psi}_n}{n\,\overline{\Phi}_n} (-1)^{\frac{n-1}{2}}$$

For a steel plate:

$$E = 2{,}1 \cdot 10^6 \ kg/cm^2 ; \quad \mu = 0{,}3 ; \quad \frac{h}{a} = \frac{1}{100} ; \quad \frac{e}{h} = \frac{1}{24}$$

is calculated

$\delta_0 =$	4,0	4,2	4,4	4,6	4,8	4,85	
$\sigma_{max} =$	190	420	680	1140	2600	∞	kg/cm^2

For $4{,}7 < \delta_0 < 4{,}85$ σ_{max} will be found in the plastic space; this condition is practically equivalent to the destruction of the plate. As σ_{max} is proportional to e the augmentation of the eccentricity will produce an earlier transgression of the limit of plasticity.

C/ A rectangular plate simply supported along all the edges.

The concentrated force P acts along the axis of symmetry of the plate.

From the many possible cases of the loads acting perpendicularly on the plate, we take an especially simple case, namely the case of the action of a concentrated force Q at the point $(a/_2\ 0)$.

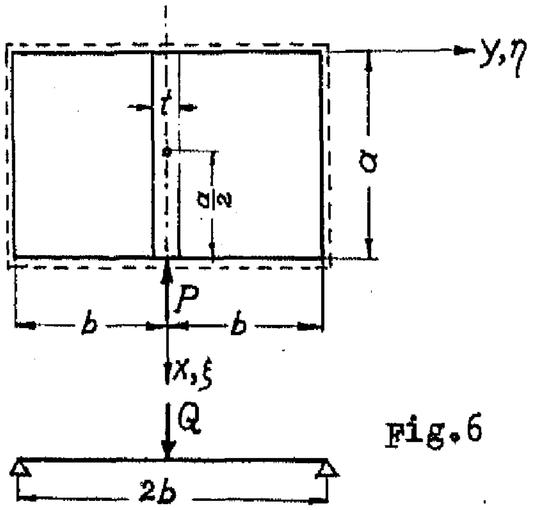


Fig.6

The differential equation of the deformation surface I, II in a dimensionless form

$$\Delta\Delta\,\omega(\xi,\eta) = 0 \qquad (16)$$

$$\eta = \frac{y}{b}\ ; \qquad \xi = \frac{x}{b}\ ;$$

we will transform to a differential equation

$$Y_n^{IV} - 2\nu^2 Y_n'' + \nu^4 Y_n = 0$$

using a single infinite serie:

$$\omega = \sum_{1,2,\dots}^{\infty} Y_n \sin\nu\xi \qquad Y_n = U_1\cosh\nu\eta + U_2\sinh\nu\eta + U_3\,\nu\eta\cosh\nu\eta + U_4\,\nu\eta\sinh\nu\eta \qquad (17)$$

The deflection of a strip of the plate of a breadth t will be represented by the infinite serie: $\omega_0 = \sum_{1,2,\dots}^{\infty} a_n \sin\nu\xi$

From the condition of continuity of the plate I and II by the section $\eta = 0$, we shall obtain two edge conditions:

$$Y_n(0) = a_n \qquad Y_n'(0) = 0$$

At the edge $\eta = 1$ the deflection disappears and the bending moment M_η also, then: $Y_n(1) = 0$; $Y_n''(1) = 0$

After determining the constants $\mathcal{U}_1 \ldots \mathcal{U}_4$ we will find the shearing force P at the section $\eta = 0$

$$T = -N \frac{\partial \Delta \omega}{\partial y} = -\frac{N}{b^3} \sum_{1,2\ldots}^{\infty} Y_n'''(0) \sin \gamma \xi$$

$$T = -\frac{2N}{b^3} \sum_{1,2.}^{\infty} a_n \frac{\gamma^3 \cosh^2 \gamma}{\cosh \gamma \sinh \gamma - \gamma} \sin \gamma \xi \qquad (18)$$

The equilibrium of the strip of breadth t and length a leads to the equation:

$$EJ \frac{d^4 \omega_0}{dx^4} + P \frac{d^2 \omega_0}{dx^2} = -\frac{4N}{b^3} \sum_{1,2\ldots}^{\infty} a_n \frac{\gamma^3 \cosh^2 \gamma}{\cosh \gamma \sinh \gamma - \gamma} \sin \gamma \xi + Q(\xi) \qquad (19)$$

$$J = \frac{th^3}{12}$$

But:

$$Q(\xi) = \frac{2Q}{a} \sum_{1,3\ldots}^{\infty} (-1)^{\frac{n-1}{2}} \sin \gamma \xi$$

From the equation (19) by $t \to 0$ we obtain:

$$a_n = \frac{2Qb^2}{\gamma^2 a} \frac{(-1)^{\frac{n-1}{2}}}{P_k \left(1 - \frac{P}{P_k}\right)} . \qquad n = 1, 3, 5, \ldots \qquad (20)$$

where
$$P_k = \frac{4N}{b} \frac{\gamma \cosh^2\gamma}{\cosh\gamma \sinh\gamma - \gamma}$$

The deflection of the centre of the plate:

$$w\left(\frac{a}{2},0\right) = \frac{2Qb^2}{a} \sum_{1,3\ldots}^{\infty} \frac{1}{\gamma^2 P_k\left(1-\frac{P}{P_k}\right)}$$

By $b \to \infty$ we obtain:

$$w\left(\frac{a}{2},0\right) = \frac{Qa^2}{2N\pi^2} \sum_{1,3,\ldots}^{\infty} \frac{1}{n^3\left(1-\frac{Pa}{4N\pi}\right)} = 0{,}01695\,\frac{Qa^2}{N}\,\frac{1}{1-\frac{Pa}{4N\pi}} \qquad (21)$$

From the formula (21) we observe, that by the simultaneous bending and compression of the plate there is possible only a symmetric form of deflection, whereas by the buckling of the plate there is possibility of asymmetric forms of deflection.

The critical force: $P_k = \frac{4N}{b} \frac{\gamma \cosh^2\gamma}{\cosh\gamma \sinh\gamma - \gamma} \qquad n = 1,2,3,\ldots$

We observe, that only for very small loads Q the force P can approach the value P_k, by large values of loads Q the destruction of the plate will have place by small values of P.
The examples given above lead to the conclusion, that the problem of the bending of a compressed plate is not a problem of stability the equilibrium remains constant up to the destruction of the plate
For each value of q/p or P/Q there is a corresponding form of the equilibrium of the plate.

S u m m a r y

In this paper the author considers a rectangular izotropic plate loaded uniformly with the load p by a simultaneous action of a load q distributed uniformly along the plate edges and directed in the plane of the plate.

There are given the exact solution of the differential equation of the problem for the following types of rectangular plate:

1/ a continuous plate compressed longitudinally

2/ a continuous plate compressed transversally

3/ a rectangular plate simply supported compressed by a concetrated force P.