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SOME DYNAMIC PROBLEMS OF THERMOELASTICITY (II)

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1. Introduction

The present paper, which is a continuation of Ref. [1], contains a detailed study of the action of time-variable forces acting in a thermoelastic space.

The main emphasis will be laid on the determination of Green's displacement functions. By contrast with the Green's function of classical elasto-dynamics, the present solutions are characterized by damping and dispersion of displacements. In addition, there are temperature fields attached to Green's tensor (due to the coupling between the strain and the temperature).

We shall consider concentrated harmonic forces in an infinite thermoelastic body, ranging from the problem of point forces to the two-dimensional problem of linear forces.

The fundamental solutions for concentrated forces enable us to obtain Green's functions in displacements for double forces, concentrated moments and compression centres.

Making use of the reciprocity theorem, as generalized to the coupled thermoelastic problem, we shall obtain a number of equations interrelating the action of concentrated forces and that of heat sources.

As a point of departure, we take the linearized equations of thermoelasticity, [2], [3],

$$\nabla^2\theta - \frac{1}{\varkappa}\frac{\partial\theta}{\partial t} - \eta\frac{\partial}{\partial t}\operatorname{div}\mathbf{u} = -\frac{Q}{\varkappa}, \tag{1.1}$$

$$\mu\nabla^2\mathbf{u} + (\lambda+\mu)\operatorname{grad}\operatorname{div}\mathbf{u} + \mathbf{X} = \gamma\operatorname{grad}\theta + \varrho\frac{\partial^2\mathbf{u}}{\partial t^2}. \tag{1.2}$$

The first of these equations is a generalized heat equation, the second — an equation of the theory of elasticity expressed in displacements (equation of motion). The two equations are coupled

The symbol $\mathbf{u}$ denotes the displacement vector, and $\mathbf{X}$ — that of mass forces; $\theta = T - T_0$ is the difference between the absolute temperature T and the temperature T_0, characterizing the natural state of the body, and Q — a function, characterizing the intensity of a heat source. μ, λ are Lame coefficients in the isothermal state; $\varkappa = \lambda_0/\varrho c_\varepsilon$ is a coefficient in which λ_0 denotes the constant of heat conduction, ϱ — density and c_ε — specific heat with constant strain. Next, $\eta = \gamma T_0/\lambda_0$, where $\gamma = (3\lambda+2\mu)\alpha_t$, and α_t is the coefficient of linear thermal dilatation. Finally $Q =$

$W/\varrho c_\varepsilon$, where W is the quantity of heat produced per unit volume and time. The quantities $\mathbf{u}$, θ, $\mathbf{X}$, Q are functions of location and time.

Resolving the vectors of displacement and force into potential and solenoidal vectors

$$\mathbf{u} = \operatorname{grad} \varphi + \operatorname{rot} \boldsymbol{\psi}, \tag{1.3}$$

$$\mathbf{X} = \varrho(\operatorname{grad} \vartheta + \operatorname{rot} \boldsymbol{\chi}), \tag{1.4}$$

we reduce Eqs. (1.1), (1.2) to the following set of three equations:

$$\left(\nabla^2 - \frac{1}{\varkappa}\partial_t\right)\theta - \eta\,\partial_t\nabla^2\varphi = -\frac{Q}{\varkappa}, \tag{1.5}$$

$$\left(\nabla^2 - \frac{1}{c_1^2}\partial_t^2\right)\varphi = m\theta - \frac{1}{c_1^2}\vartheta, \tag{1.6}$$

$$\left(\nabla^2 - \frac{1}{c_2^2}\partial_t^2\right)\boldsymbol{\psi} = -\frac{1}{c_2^2}\boldsymbol{\chi}, \quad c_1^2 = \frac{\lambda+2\mu}{\varrho}, \quad c_2^2 = \frac{\mu}{\varrho}, \quad \partial_t = \frac{\partial}{\partial t}. \tag{1.7}$$

On eliminating from (1.5) and (1.6) the temperature θ, we obtain two wave equations:

$$\left(\nabla^2 - \frac{1}{\varkappa}\partial_t\right)\left(\nabla^2 - \frac{1}{c_1^2}\partial_t^2\right)\varphi - \frac{\varepsilon}{\varkappa}\partial_t\nabla^2\varphi = -\frac{mQ}{\varkappa} - \frac{1}{c_1^2}\left(\nabla^2 - \frac{1}{\varkappa}\partial_t\right)\vartheta, \tag{1.8}$$

$$\left(\nabla^2 - \frac{1}{c_2^2}\partial_t^2\right)\boldsymbol{\psi} = -\frac{1}{c_2^2}\boldsymbol{\chi}. \tag{1.9}$$

Equation (1.8) describes the propagation of a longitudinal wave, and (1.9) that o a transversal wave.

Let us observe that in an infinite body a heat source and $\mathbf{X}' = \varrho \operatorname{grad} \vartheta$ produce longitudinal (dilatation) waves only, and the forces $\mathbf{X}'' = \varrho \operatorname{rot} \boldsymbol{\chi}$ transversal waves only.

2. Green's Functions for Concentrated Forces

Let us assume that the causes of wave perturbation — that is, heat sources and mass forces — vary harmonically in time;

$$Q(\mathbf{x}, t) = Q^*(\mathbf{x})\varepsilon^{i\omega t}, \quad \vartheta(\mathbf{x}, t) = \vartheta^*(\mathbf{x})e^{i\omega t}, \quad \boldsymbol{\chi}(\mathbf{x}, t) = \boldsymbol{\chi}^*(\mathbf{x})e^{i\omega t}. \tag{2.1}$$

As a consequence, the displacement, temperature and the stress vary harmonically in time. By introducing the notations

$$\theta(\mathbf{x}, t) = \theta^*(\mathbf{x})e^{i\omega t}, \quad \varphi(\mathbf{x}, t) = \varphi^*(\mathbf{x})e^{i\omega t}, \quad \text{etc.} \tag{2.2}$$

Eqs. (1.8) and (1.9) can be reduced to the form

$$(\nabla^2 - k_1^2)(\nabla^2 - k_2^2)\varphi^* = -\frac{mQ^*}{\varkappa} - \frac{1}{c_1^2}(\nabla^2 - q)\vartheta^*, \tag{2.3}$$

$$(\nabla^2 + \tau^2)\boldsymbol{\psi}^* = -\frac{1}{c_2^2}\boldsymbol{\chi}^*, \tag{2.4}$$

where

$$k_1^2+k_2^2 = q(1+\varepsilon)-\sigma^2, \quad k_1^2k_2^2 = -q\sigma^2,$$
$$q = \frac{i\omega}{\varkappa}, \quad \sigma^2 = \frac{\omega^2}{c_1^2}, \quad \tau^2 = \frac{\omega^2}{c_2^2},$$
$$k_j = a_j+ib_j, \quad a_j > 0, \quad j = 1,2,$$

and a_j and b_j are real quantities. The quantities k_1^2 and k_2^2 are the roots of the biquadratic equation

$$k^4+k^2[\sigma^2-q(1+\varepsilon)]-\sigma^2 q = 0.$$

Let us consider a concentrated force in an infinite body, and construct the components of Green's displacement vector. Let us assume first that a concentrated unit force varying in time in a harmonic manner acts at the origin in the x_1-direction.

In general, for any vector field of mass forces, we find the functions ϑ^* and $\boldsymbol{\chi}^*$ from [4]:

$$\vartheta^*(\mathbf{x}) = -\frac{1}{4\pi\varrho}\iiint\limits_{(B)}\Big[X_1^*(\mathbf{x}')\frac{\partial}{\partial x_1}\left(\frac{1}{R(\mathbf{x}',\mathbf{x})}\right)+X_2^*(\mathbf{x}')\frac{\partial}{\partial x_2}\left(\frac{1}{R(\mathbf{x}',\mathbf{x})}\right)$$
$$+X_3^*(\mathbf{x}')\frac{\partial}{\partial x_3}\left(\frac{1}{R(\mathbf{x}',\mathbf{x})}\right)\Big]dV(\mathbf{x}'), \tag{2.5}$$

$$\boldsymbol{\chi}^*(\mathbf{x}) = -\frac{1}{4\pi\varrho}\iiint\limits_{(B)}\Big\{\mathbf{i}\Big[X_2^*(\mathbf{x}')\frac{\partial}{\partial x_3}\left(\frac{1}{R(\mathbf{x}',\mathbf{x})}\right)-X_3^*(\mathbf{x}')\frac{\partial}{\partial x_2}\left(\frac{1}{R(\mathbf{x}',\mathbf{x})}\right)\Big]$$
$$+\mathbf{j}\Big[X_3^*(\mathbf{x}')\frac{\partial}{\partial x_2}\left(\frac{1}{R(\mathbf{x}',\mathbf{x})}\right)-X_1^*(\mathbf{x}')\frac{\partial}{\partial x_3}\left(\frac{1}{R(\mathbf{x}',\mathbf{x})}\right)\Big]$$
$$+\mathbf{k}\Big[[X_1^*(\mathbf{x}')\frac{\partial}{\partial x_2}\left(\frac{1}{R(\mathbf{x}',\mathbf{x})}\right)-X_2^*(\mathbf{x}')\frac{\partial}{\partial x_1}\left(\frac{1}{R(\mathbf{x}',\mathbf{x})}\right)\Big]\Big\}dV(\mathbf{x}'). \tag{2.6}$$

On substituting in (2.5) and (2.6) the quantity

$$X_j^*(\mathbf{x}') = \delta(x_1')\delta(x_2')\delta(x_3')\delta_{1j}, \quad j = 1, 2, 3,$$

which characterizes the action of a concentrated force at the origin and in the x_j-direction, we find successively:

$$\vartheta^*(\mathbf{x}) = -\frac{1}{4\pi\varrho}\frac{\partial}{\partial x_1}\left(\frac{1}{R}\right), \quad \chi_1^* = 0, \tag{2.7}$$
$$\chi_2^* = \frac{1}{4\pi\varrho}\frac{\partial}{\partial x_3}\left(\frac{1}{R}\right), \quad \chi_3^* = -\frac{1}{4\pi\varrho}\frac{\partial}{\partial x_2}\left(\frac{1}{R}\right), \quad R = (x_1^2+x_2^2+x_3^2)^{1/2};$$

therefore, the following equations must be solved

$$(\nabla^2-k_1^2)(\nabla^2-k_2^2)\varphi^* = \frac{1}{4\pi\varrho c_1^2}(\nabla^2-q)\frac{\partial}{\partial x_1}\left(\frac{1}{R}\right),$$
$$(\nabla^2+\tau^2)\psi_1^* = 0,$$
$$(\nabla^2+\tau^2)\psi_2^* = -\frac{1}{4\pi\varrho c_2^2}\frac{\partial}{\partial x_3}\left(\frac{1}{R}\right), \tag{2.8}$$
$$(\nabla^2+\tau^2)\psi_3^* = \frac{1}{4\pi\varrho c_2^2}\frac{\partial}{\partial x_2}\left(\frac{1}{R}\right).$$

We shall concentrate on the first equation of the group (2.8), the solution of the remaining two being known in the form:

$$\psi_1^* = 0, \quad \psi_2^* = \frac{1}{4\pi\varrho\omega^2}\frac{\partial}{\partial x_3}F_0(R,\omega),$$
(2.9)
$$\psi_3^* = -\frac{1}{4\pi\varrho\omega^2}\frac{\partial}{\partial x_2}F_0(R,\omega),$$

where

$$F_0(R,\omega) = \frac{e^{-i\tau R}}{R} - \frac{1}{R}.$$

To solve the first equation of the group (2.8), we make use of two properties of the function φ^*. If the concentrated force acts in the axial direction, the wave and displacement function will be axially symmetric about the x_1-axis. In addition, the function φ^* is antisymmetric in relation to the $x_2 x_3$ plane, in view of the symmetry of displacement in relation to this plane. By treating φ^* as a function of x_1 and the radius $r = (x_2^2+x_3^2)^{1/2}$, we perform on the wave equation first the Hankel transformation and then the Fourier sine transformation. The transformation

$$(2.10) \qquad \tilde{\varphi}(\alpha,\beta) = \sqrt{\frac{2}{\pi}}\int_0^\infty\int_0^\infty \varphi^*(x_1,r)rJ_0(\alpha r)\sin\beta x_1\,dr\,dx_1$$

changes the wave equation into

$$(2.11) \qquad \tilde{\varphi}(\alpha,\beta) = \frac{1}{4\pi\varrho\omega^2}\sqrt{\frac{2}{\pi}}\left[A_2\frac{\beta}{\alpha^2+\beta^2+k_2^2} - A_1\frac{\beta}{\alpha^2+\beta^2+k_1^2} - \frac{\beta}{\alpha^2+\beta^2}\right],$$

where

$$A_2 = \frac{(k_2^2-q)\sigma^2}{k_2^2(k_1^2-k_2^2)}, \qquad A_1 = \frac{(k_1^2-q)\sigma^2}{k_1^2(k_1^2-k_2^2)}.$$

Performing on (2.11) the inverse Hankel-Fourier transformation

$$(2.12) \qquad \varphi^*(x_1,r) = \sqrt{\frac{2}{\pi}}\int_0^\infty\int_0^\infty \tilde{\varphi}(\alpha,\beta)\alpha J_0(\alpha r)\sin\beta x_1\,d\alpha\,d\beta,$$

we obtain for $\varphi^*(x_1,r)$ the following closed-form solution:

$$(2.13) \qquad \varphi^*(x_1,r) = -\frac{1}{4\pi\varrho\omega^2}\frac{\partial}{\partial x_1}F(R,\omega),$$

where

$$F = A_2 I_2 - A_1 I_1 - I_0, \quad I_j(R,\omega) = \frac{1}{R}e^{-k_j R}, \quad j = 1,2; \quad I_0 = \frac{1}{R}.$$

Let us observe that by passing from the coupled to the uncoupled problem — that is, substituting the values: $\eta = 0$, $\varepsilon = 0$, $k_1^2 = q$, $k_2^2 = -\sigma^2$ in the expression for φ^* — we obtain the known solution of the wave equation of classical elastokinetics

$$(2.14) \qquad \varphi^*(x_1,r) = -\frac{1}{4\pi\varrho\omega^2}\frac{\partial}{\partial x_1}\left(\frac{e^{-i\sigma R}}{R} - \frac{1}{R}\right).$$

Let us return to (2.13) and move the concentrated force from the origin to the point (ξ). Let us attach to the symbols φ^* and ψ^* the suffix 1, in order to express the fact that these functions are connected with a force acting in the x_1-direction. We find:

$$\tag{2.15} \varphi^{*(1)}(\mathbf{x}, \xi) = -\frac{1}{4\pi\varrho\omega^2}\frac{\partial}{\partial x_1}F(R, \omega),$$

$$\psi_1^{*(1)}(\mathbf{x}, \xi) = 0, \qquad \psi_2^{*(1)} = \frac{1}{4\pi\varrho\omega^2}\frac{\partial}{\partial x_3}F_0(R, \omega),$$

$$\psi_3^{*}(\mathbf{x}, \xi) = -\frac{1}{4\pi\varrho\omega^2}\frac{\partial}{\partial x_2}F_0(R, \omega),$$

where

$$R = [(x_1-\xi_1)^2+(x_2-\xi_2)^2+(x_3-\xi_3)^2]^{1/2}.$$

On superposing the displacements according to (1.3), we obtain

$$\tag{2.16} u_j^{*(1)} = -\frac{1}{4\pi\varrho\omega^2}\partial_1\partial_j[F(R, \omega)-F_0(R, \omega)]+\frac{1}{4\pi\varrho c_2^2}\delta_{1j}\frac{e^{-i\tau R}}{R}, \quad j = 1, 2, 3.$$

The action of a concentrated force in an infinite body is accompanied by a temperature field which will be determined from (1.6). The amplitude of this field is:

$$\tag{2.17} \theta^{*(1)} = \frac{1}{m}(\nabla^2+\sigma^2)\varphi^{*(1)}+\frac{1}{c_1^2 m}\vartheta^{*(1)}$$

$$= -\frac{1}{4\pi\varrho\omega^2 m}\frac{\partial}{\partial x_1}[(\nabla^2+\sigma^2)F(R, \omega)+\sigma^2 I_0(R)].$$

After some simple computation, we find:

$$\tag{2.18} \theta^{*(1)} = -\frac{q\varepsilon}{4\pi\varrho m c_1^2(k_1^2-k_2^2)}\frac{\partial}{\partial x_1}[I_2(R, \omega)-I_1(R, \omega)].$$

Let us observe that in the non-coupled problem ($\varepsilon = 0$), we have $\theta^{*(1)} = 0$.

In the coupled problem, we find that the function $\theta^{*(1)}$ has a singularity at $(\mathbf{x}) = (\xi)$; therefore, the action of a concentrated force at the point (ξ) produces at that point a concentrated source of heat.

Let the concentrated force at (ξ) act in the successive directions x_1, x_2, and finally x_3. We obtain three groups of components of the displacement vector $\mathbf{u}_j^{*(1)}$, $\mathbf{u}_j^{*(2)}$, $\mathbf{u}_j^{*(3)}$, and the associated temperature fields $\theta^{*(1)}$, $\theta^{*(2)}$, $\theta^{*(3)}$.

The set of all the components of the displacement vector is Green's displacement tensor

$$\tag{2.19} G_j^{(k)}(\mathbf{x}, \xi, t) = \mathrm{Re}\,[u_j^{*(k)}(\mathbf{x}, \xi, \omega)e^{i\omega t}]$$

$$= -\frac{1}{4\pi\varrho\omega^2}\mathrm{Re}\left\{e^{i\omega t}\partial_j\partial_k[F(R, \omega)-F_0(R, \omega)]-\tau^2\delta_{jk}\frac{e^{i(\omega t-\tau R)}}{R}\right\}.$$

The temperature field $\theta^{(k)}$ is expressed by the equation:

$$\tag{2.20} \theta^{(k)}(\mathbf{x}, \xi, t) = \mathrm{Re}\,[\theta^*(\mathbf{x}, \xi, \omega)e^{i\omega t}]$$

$$= -\frac{q\varepsilon}{4\pi\varrho c_1^2 m(k_1^2-k_2^2)}\mathrm{Re}\,\{e^{i\omega t}\partial_k[I_2(R, \omega)-I_1(R, \omega)]\}.$$

The functions $\theta^{*(k)}$ and $u_j^{*(k)}$ being now known, we can determine the amplitudes of stress and strain from the equations:

$$\sigma_{ij}^{*(k)} = 2\mu\varepsilon_{ij}^{*(k)} + (\lambda\varepsilon_{ss}^{*(k)} - \gamma\theta^{*(k)})\delta_{ij},$$

$$\varepsilon_{ij}^{*(k)} = \frac{1}{2}(u_{i,j}^{*(k)} + u_{j,i}^{*(k)}), \quad i, j = 1, 2, 3.$$

Let us consider the double action of a force acting in the x_1— direction. Let a concentrated force P_0 act at the point $((\xi_1, \xi_2, \xi_3)$ in the x_1-direction, another concentrated force P_0 acting at the point $(\xi_1 + \Delta\xi_1, \xi_2, \xi_3)$ in the x_1-direction.

Let $U_j^{*(1)}$ $(j = 1, 2, 3)$ denote the amplitudes of the displacement components. These will be found from the equation:

$$(2.21) \qquad U_j^{*(1)} = P_0\Delta\xi_1 \frac{u_j^{*(1)}(x_1, x_2, x_3, \xi_1 + \Delta\xi_1, \xi_2, \xi_3) - u_j^{*(1)}(x_1, x_2, x_3, \xi_1, \xi_2, \xi_3)}{\Delta\xi_1},$$

where the expression $u_j^{*(1)}$ is given by Eq. (2.16). Assuming that $\lim\limits_{\Delta\xi_1\to 0}(P_0\Delta\xi_1) = 1$, we find for $\Delta\xi_1 \to 0$ from (2.21)

$$(2.22) \qquad U_j^{*(1)}(\mathbf{x}, \boldsymbol{\xi}, \omega) = \frac{\partial}{\partial\xi_1} u_j^{*(1)}(\mathbf{x}, \boldsymbol{\xi}, \omega) = -C\partial_{1'}[\partial_1\partial_j(F - F_0) - \tau^2\delta_{1'j}I_\tau],$$

where

$$C = \frac{1}{4\pi\varrho\omega^2}, \quad I_\tau(R, \omega) = \frac{e^{-i\tau R}}{R}, \quad \partial_{1'} = \frac{\partial}{\partial\xi_1}.$$

The expression (2.22) represents the displacements due to the action of a double force in the x_1-direction.

Let us denote by $\theta^{*(1)}$ the temperature amplitude due to the action of the double force. This will be obtained by the same argument as for $U_j^{*(1)}$ from the equation:

$$(2.23) \qquad \theta^{*(1)}(\mathbf{x}, \boldsymbol{\xi}, \omega) = \frac{\partial}{\partial\xi_1}\theta^{*(1)}(\mathbf{x}, \boldsymbol{\xi}, \omega),$$

$$\theta^{*(1)} = -\frac{q\varepsilon C}{m\, c_1^2(k_1^2 - k_2^2)}\partial_{1'}\partial_1[I_2(R_2, \omega) - I_1(R, \omega)].$$

Our considerations can be generalized to the case of double force action at the point $\boldsymbol{\xi}$ in the x_k-direction. We find:

$$(2.24) \qquad U_j^{*(k)}(\mathbf{x}, \boldsymbol{\xi}, \omega) = \partial_{k'}u_j^{*(k)}(\mathbf{x}, \boldsymbol{\xi}, \omega), \qquad \theta^{*(k)}(\mathbf{x}, \boldsymbol{\xi}, \omega) = \partial_{k'}\theta^{*(k)}(\mathbf{x}, \boldsymbol{\xi}, \omega),$$

where $u_j^{*(k)}$ and $\theta^{*(k)}$ are given by (2.16) and (2.20). Let us consider also the action of a centre of compression at $(\boldsymbol{\xi})$. Let $V_j^*(j = 1, 2, 3)$ denote the amplitude of displacement due to the compression centre. This action being equivalent to that of three double forces acting in the respective directions x_1, x_2 and x_3, the displacement V_j^* takes the form:

$$(2.25) \qquad V_j^* = \partial_{1'}u_j^{*(1)} + \partial_{2'}u_j^{*(2)} + \partial_{3'}u_j^{*(3)} = -C[\partial_k\partial_{k'}\partial_j(F - F_0) + \tau^2\partial_{j'}I_\tau],$$
$$j', j = 1, 2, 3, \quad k, k' = 1, 2, 3.$$

On performing the operations as required, we obtain:

$$(2.26) \qquad V_j^* = C\partial_j(A_2k_2^2I_2 - A_1k_1^2I_1), \quad j = 1, 2, 3.$$

Let θ^* denote the temperature which is related to the action of the compression centre. This function will be obtained from the equation:

$$\theta^* = \partial_{k'}\theta^{*(k)} = \frac{q\varepsilon C\sigma^2}{m(k_1^2-k_2^2)}(k_2^2 I_2 - k_1^2 I_1). \tag{2.27}$$

The compression centre produces longitudinal waves only. Denoting by Φ^* and Ψ^* wave functions in the case of a compression centre, we obtain:

$$\Phi^* = \partial_{j'}\varphi^{*(j)} = -C\partial_k\partial_{k'}F = C(A_2k_2^2I_2 - A_1k_1^2I_1), \quad \Psi_1^* = \Psi_2^* = \Psi_3^* = 0. \tag{2.28}$$

On the other hand, we have

$$V_j^* = \partial_j\Phi^*, \quad \Lambda^* = V_{j,j}^* = \nabla^2\Phi^*. \tag{2.29}$$

From the first equation, we obtain (2.26), the other expresses the dilatation:

$$\Lambda^* = \nabla^2\Phi^* = C(A_2k_2^4I_2 - A_1k_1^4I_1). \tag{2.30}$$

3. Green's Functions of the Two-dimensional Problem

Let us consider mass forces $X_j^* = \delta(x_1)\delta(x_2)\delta_{1j}$ $(j = 1,2)$ acting in an infinite space in the direction of the x_1-axis , and uniformly distributed along the x_3-axis. The displacement and the temperature will be independent of the variable x_3. The equations for the two-dimensional problem will be obtained from those for a concentrated force acting at the point $(0, 0, \xi_3)$ of the infinite space, by integrating the functions φ^*, u_j^* etc. with respect to ξ_3 from $-\infty$ to $+\infty$. Bearing in mind that

$$\int_{-\infty}^{\infty} \frac{\exp\left(-k_1\sqrt{r^2+\xi_3^2}\right)}{\sqrt{r^2+\xi_3^2}}\, d\xi_3 = 2K_0(k_1r), \quad r = (x_1^2+x_2^2)^{1/2}, \tag{3.1}$$

where $K_0(k_1r)$ is modified Bessel function of the second kind, we obtain from (2.16)

$$u_j^{*(1)} = -2C\partial_1\partial_j[A_2K_0(k_2r) - A_1K_0(k_1r) - K_0(i\tau r)] + 2C\tau^2\,\delta_{1j}K_0(i\tau r), \tag{3.2}$$

$$C = \frac{1}{4\pi\varrho\omega^2}, \quad j = 1, 2.$$

In general, for a force acting in the x_k-direction $(k = 1, 2)$, we obtain the equation:

$$u_j^{*(k)} = -2C\{\partial_j\partial_k[A_2K_0(k_2r) - A_1K_0(k_1r) - K_0(i\tau r)] - \tau^2\delta_{kj}K_0(i\tau r)\}, \tag{3.3}$$

$$j,\ k = 1, 2.$$

Similarly, making use of (2.18) and (2.20), we obtain the following expression for the temperature accompanying the strain:

$$\theta^{*(k)} = -\frac{2q\varepsilon C}{mc_1^2(k_1^2-k_2^2)}\,\partial_k[K_0(k_2r) - K_0(k_1, r)], \quad k = 1, 2. \tag{3.4}$$

In the non-coupled problem $(\varepsilon = 0)$, we should have $\theta^{*(k)} = 0$. The functions $u_j^{*(k)}$ and $\theta^{*(k)}$ being determined, we can obtain the stresses from the equations:

$$\sigma_{ij}^{*(k)} = 2\mu\varepsilon_{ij}^{*(k)} + (\lambda\varepsilon_{ss}^{*(k)} - \gamma\theta^{*(k)})\,\delta_{ij}, \quad i, j = 1, 2, \tag{3.5}$$

where

$$\varepsilon_{ij}^{*(k)} = \frac{1}{2}(u_{i,j}^{*(k)} + u_{j,i}^{*(k)}). \tag{3.6}$$

Let us consider, in turn, the displacements produced by the action of double forces. If the double force acts in the x_k-direction, then

$$(3.7) \qquad U_j^{*(k)} = \partial_{k'} u_j^{*(k)}, \quad \Theta^{*(k)} = \partial_{k'} \theta^{*(k)}, \quad j, k = 1, 2, \quad \partial_{k'} = \frac{\partial}{\partial \xi_k},$$

where the functions $u_j^{*(k)}$ and $\theta^{*(k)}$ are given by Eqs. (3.3) and (3.4). The displacements V_j^*, the temperature θ^* and the dilatation Λ^* in the case of a compression centre will be obtained from (2.26) (2.27) by the integration (3.1), performed on expressions obtained for concentrated compression centres. We obtain the equations:

$$(3.8) \qquad V_j^* = 2C\partial_j[A_2 k_2^2 K_0(k_2 r) - A_1 k_1^2 K_0(k_1 r)],$$

$$(3.9) \qquad \theta^* = \frac{2q\varepsilon\sigma^2 C}{m(k_1^2-k_2^2)}[k_2^2 K_0(k_2 r) - k_1^2 K_0(k_1 r)],$$

$$(3.10) \qquad \Lambda^* = 2C[A_2 k_2^4 K_0(k_2 r) - A_1 k_1^4 K_0(k_1 r)].$$

In the case of a compression centre, we are concernerd with a longitudinal cylindrical wave.

4. Green's Functions of the One-dimensional Problem

Let us consider the set of coupled equations of the theory of elasticity in the case of the one-dimensional problem. In this case, the equations (1.1) and (1.2) become simplified to the form:

$$(4.1) \qquad \left(\partial_1^2 - \frac{1}{\varkappa}\partial_t\right)\theta - \eta\partial_t\partial_1 u_1 = -\frac{Q}{\varkappa},$$

$$(4.2) \qquad \left(\partial_1^2 - \frac{1}{c_1^2}\partial_t^2\right)u_1 + \frac{X_1}{\varrho c_1^2} = m\partial_1\theta,$$

where θ and u_1 are functions of x_1 and t only. Let us assume that the mass forces $X_1(x_1, t)$ vary harmonically in time and act in the plane x_2x_3: $X_1(x_1, t) = X_1^* e^{i\omega t}$ With such an assumption, Eqs (4.1) and (4.2) become:

$$(4.3) \qquad (\partial_1^2 - q)\theta^* - \eta\varkappa q\partial_1 u_1^* = 0,$$

$$(4.4) \qquad (\partial_1^2 + \sigma^2)u_1^* = m\partial_1\theta^* - \frac{1}{c_1^2\varrho}X_1^*.$$

On eliminating the temperature, we obtain a plane wave equation determining the displacement:

$$(4.5) \qquad (\partial_1^2 - k_1^2)(\partial_1^2 - k_2^2)u_1^* = -\frac{1}{\varrho c_1^2}(\partial_1^2 - q)X_1^*.$$

In view of the symmetry of the displacement u_1^* and the force X_1^* in relation to the plane $x_1 = 0$, the cosine Fourier transformation should be performed on Eq. (4.5). The solution of (4.5) will take the form:

$$(4.6) \qquad u_1^* = -\frac{1}{c_1^2\varrho\pi}\int_0^\infty \left(\frac{a_2}{\beta^2+k_2^2} - \frac{a_1}{\beta^2+k_1^2}\right)\cos\beta x_1 d\beta$$

$$= -\frac{1}{2c_1^2\varrho}\left(\frac{a_2}{k_2}e^{-k_2x_1} - \frac{a_1}{k_1}e^{-k_1x_1}\right), \quad x_1 > 0,$$

where

$$a_2 = \frac{k_2^2-q}{k_1^2-k_2^2}, \quad a_1 = \frac{k_1^2-q}{k_1^2-k_2^2}.$$

The deformation of the body is accompanied by a temperature field. We find it from (4.2), for $x_1 > 0$, in the form:

$$\theta^* = \frac{1}{2c_1^2 m\varrho}\left[\frac{a_2(k_2^2+\sigma^2)}{k_2^2}e^{-k_2x_1} - \frac{a_1(k_1^2+\sigma^2)}{k_1^2}e^{-k_1x_1}\right], \quad x_1 > 0. \tag{4.7}$$

Bearing in mind that

$$\frac{(k_1^2-q)(k_1^2+\sigma^2)}{k_1^2} = \frac{(k_2^2-q)(k_2^2+\sigma^2)}{k_2^2} = q\varepsilon,$$

the expression for the temperature θ^* becomes:

$$\theta^* = \frac{q\varepsilon}{2c_1^2 m\varrho(k_1^2-k_2^2)}[e^{-k_2x_1} - e^{-k_1x_1}], \quad x_1 > 0, \tag{4.8}$$

Let us observe that for $x_1 = 0$, we obtain, due to the term $X_1^*/\varrho c_1^2$ in (4.2) and $X_1^* = \delta(x_1)$, an infinite local temperature. For $x_1 > 0$, this term is equal to zero.

5. Conclusions Derived from the Reciprocity Theorem

There are some simple relations between the displacements produced by the temperature sources and the temperature due to the action of concentrated forces, which will be explained as conclusions following from the reciprocity condition.

Let us consider first the three-dimensional problem. Let a concentrated unit heat source varying harmonically with time act at the point $(\boldsymbol{\xi})$. This source produces longitudinal waves only, described by the wave function φ_T^*, and propagating in the form of spherical waves. The solution of (1.8) for $\vartheta = 0$ is known [1].It has the form:

$$\varphi_T^* = \frac{m}{4\pi\varkappa(k_1^2-k_2^2)}[I_2(R,\omega) - I_1(R,\omega)]. \tag{5.1}$$

The displacement due to the action of the heat source will be obtained from the equation:

$$u_j^{*T} = \partial_j \varphi_T^*. \tag{5.2}$$

Hence,

$$u_j^{*T} = \partial_j \varphi_T^* = \frac{m}{4\pi\varkappa(k_1^2-k_2^2)}\partial_j[I_2(R,\omega) - I_1(R,\omega)]. \tag{5.3}$$

Let us consider a concentrated force acting at the point $(\mathbf{x}')$ in the x_j-direction.

The temperature produced by this action at the point $(\xi)'$ will be obtained from (2.6):

$$\begin{gathered}\theta^{*(j)}(\boldsymbol{\xi}, \mathbf{x}', \omega) = -\frac{q\varepsilon}{4\pi\varrho m c_1^2(k_1^2-k_2^2)}\partial_j[I_2(R,\omega) - I_2(R,\omega)], \\ R^2 = [(x_1'-\xi_1')^2 + (x_2'-\xi_2') + (x_3'-\xi_3')^2]^{1/2}.\end{gathered} \tag{5.4}$$

Let a unit source of heat act at the point $(\boldsymbol{\xi}')$. The displacement u_j^{*T} at the point $(\mathbf{x}')$, produced by this source and having the direction of x_j is expressed by (5.3). We have

$$u_j^{*T}(\mathbf{x}', \boldsymbol{\xi}', \omega) = \frac{m}{4\pi\varkappa(k_1^2-k_2^2)} \partial_j[I_2(R, \omega)-I_1(R, \omega)]. \tag{5.5}$$

By confronting Eqs. (5.4) and (5.5), we obtain:

$$\theta^{*(j)}(\boldsymbol{\xi}', \mathbf{x}', \omega) = -\frac{i\omega\varkappa\eta}{mc_1^2\varrho} u_j^{*T}(\mathbf{x}', \boldsymbol{\xi}', \omega), \quad j = 1, 2, 3. \tag{5.6}$$

Let us proceed now to solve the two-dimensional problem. Let a concentrated linear heat source act at the point $(\boldsymbol{\xi}')$ varying harmonically in time. This source produces a longitudinal cylindrical wave having the form [1].

$$\varphi_T^* = \frac{m}{2\pi\varkappa(k_1^2-k_2^2)} [K_0(k_2 r)-K_0(k_1 r)]. \tag{5.7}$$

The knowledge of the function φ_T^* will enable the determination of the displacement due to the action of the linear heat source:

$$u_j^{*T}(\mathbf{x}', \boldsymbol{\xi}', \omega) = \partial_j\varphi_T^* = \frac{m}{2\pi\varkappa(k_1^2-k_2^2)} \partial_j[K_0(k_2 r)-K_0(k_1 r)], \quad j = 1, 2; \tag{5.8}$$
$$r = [(x_1'-\xi_1')^2+(x_2^1-\xi_2')^2]^{1/2}.$$

If a force directed in the $x_j (j = 1, 2)$ direction acts at the point $(\mathbf{x}')$, the temperature at the point $(\boldsymbol{\xi})'$ accompanying the deformation due to the action of this concentrated force is determined by (3.4)

$$\theta^{*(j)}(\boldsymbol{\xi}', \mathbf{x}', \omega) = -\frac{q\varepsilon}{2\pi\varrho c_1^2 m(k_1^2-k_2^2)} \partial_j[K_0(k_2 r)-K_0(k_1 r)], \quad j = 1, 2. \tag{5.9}$$

By confronting (5.8) with (5.9), it is seen that

$$\theta^{*(j)}(\boldsymbol{\xi}', \mathbf{x}', \omega) = -\frac{i\omega\varkappa\eta}{mc_1^2\varrho} u_j^{*T}(\mathbf{x}', \boldsymbol{\xi}', \omega), \quad j = 1, 2. \tag{5.10}$$

Let a plane heat source varying harmonically in time act in the $x_1 = \xi_1'$ plane of the infinite space. This action will produce a plane wave and, [1]:

$$\varphi_T^* = -\frac{m}{2\varkappa(k_1^2-k_2^2)} \left\{\frac{1}{k_2} \exp[-k_2(x_1-\xi_1')]-\frac{1}{k_1} \exp[-k_1(x_1-\xi''_1)]\right\}, \tag{5.11}$$
$$\text{for } x_1-\xi_1' > 0.$$

The displacement produced by the plane heat source is

$$u_1^{*T} = \partial_1\varphi_T^* = \frac{m}{2\varkappa(k_1^2-k_2^2)} \{\exp[-k_2(x_1-\xi_1')]-\exp[-k_1(x_1-\xi_1)]\}. \tag{5.12}$$
$$\text{for } x_1-\xi_1' > 0.$$

The mass force $X_1^* = \delta(x_1-x_1')$ acting in the $x_1 = x_1'$ plane gives the temperature:

$$\theta^* = -\frac{q\varepsilon}{2c_1^2 m(k_1^2-k_2^2)} \left\{\exp[-k_2(x_1-x_1')]-\exp[-k_1(x_1-x_1')]\right\}, \quad x_1-x_1' > 0. \tag{5.13}$$

It will be found also in this case that

$$\theta^*(\xi_1', x_1' \,\omega) = -\frac{i\omega\varkappa\eta}{mc_1^2\varrho} u_1^{*T}(x_1', \xi_1', \omega). \tag{5.14}$$

The results obtained may, by confronting a number of solutions, be obtained in a general manner by means of the reciprocity theorem generalized to coupled problems of thermoelasticity, [5, 6]. It is known that Betti's theorem for dynamic problems of thermoelasticity, the temperature being accounted for, has the form:

$$\iiint\limits_{(B)} (X_i^* u_i^{*\prime} - X_i^{*\prime} u_i^*) dV + \gamma \iiint\limits_{(B)} (\theta^* e^{*\prime} - \theta^{*\prime} e^*) dV + \iint\limits_{(S)} (p_i^* u_i^{*\prime} - p_i^{*\prime} u_i^*) dS = 0. \tag{5.15}$$

This equation should be completed with Green's identity

$$\iiint\limits_{(B)} (\theta^* \nabla^2 \theta^{*\prime} - \theta^{*\prime} \nabla^2 \theta^*) dV = \iint\limits_{(S)} \left(\theta^* \frac{\partial \theta^{*\prime}}{\partial n} - \theta^{*\prime} \frac{\partial \theta^*}{\partial n} \right) dS, \tag{5.16}$$

where the relation between the temperature and dilatation must be taken into consideration:

$$(\nabla^2 - q)\theta^* - \eta q \varkappa e^* = -\frac{Q^*}{\varkappa}, \qquad (\nabla^2 - q)\theta^{*\prime} - \eta q \varkappa e^{*\prime} = -\frac{Q^{*\prime}}{\varkappa}. \tag{5.17}$$

Equations (5.16) and (5.17) lead to the relation:

$$\iiint\limits_{(B)} (Q^* \theta^{*\prime} - Q^{*\prime} \theta^*) dV + \eta \varkappa i\omega \iiint\limits_{(B)} (\theta^* e^{*\prime} - \theta^{*\prime} e^*) dV = \iint\limits_{(S)} \left(\theta^* \frac{\partial \theta^{*\prime}}{\partial n} - \theta^{*\prime} \frac{\partial \theta^*}{\partial n} \right) dS. \tag{5.18}$$

Equations (5.15) and (5.18) constitute a generalization of the reciprocity theorem for a thermoelastic body. For the infinite body considered in the present paper, there are no boundary conditions proper and (5.15) and (5.18) reduce to

$$\iiint\limits_{(B)} (X_i^* u_i^{*\prime} - X_i^{*\prime} u_i^*) \, dV + \gamma \iiint\limits_{(B)} (\theta^* e^{*\prime} - \theta^{*\prime} e^*) \, dV = 0, \tag{5.19}$$

$$\iiint\limits_{(B)} (Q^* \theta^{*\prime} - Q^{*\prime} \theta^*) \, dV + \eta \varkappa i\omega \iiint\limits_{(B)} (\theta^* e^{*\prime} - \theta^{*\prime} e^*) \, dV = 0. \tag{5.20}$$

Let us observe that these equations can also be written in the form:

$$\eta \varkappa i\omega \iiint\limits_{(B)} (X_i^* u_i^{*\prime} - X_i^{*\prime} u_i^*) \, dV = \gamma \iiint\limits_{(B)} (Q^* \theta^{*\prime} - Q^{*\prime} \theta^*) \, dV. \tag{5.21}$$

Let us consider the particular case, where

$$X_i^* = \delta(\mathbf{x} - \mathbf{x}') \delta_{ij}, \quad X_i^{*\prime} = 0, \quad Q^{*\prime} = \delta(\mathbf{x} - \xi'), \quad Q^* = 0, \tag{5.22}$$

which describes the action of a concentrated force at the point $(\mathbf{x}')$ in the x_j-direction and a concentrated heat source at $(\boldsymbol{\xi}')$. On substituting (5.22) in (5.21), we obtain:

$$(5.23)\quad \eta\varkappa i\omega \iiint_{(B)} \delta(\mathbf{x}-\mathbf{x}')\delta_{ij}u_1^{*T}(\mathbf{x}, \boldsymbol{\xi}', \omega)\, dV(\mathbf{x}) = -\gamma \iiint_{(B)} \delta(\mathbf{x}-\boldsymbol{\xi}')\theta^{*(j)}(\mathbf{x}, \mathbf{x}', \omega)\, dV(\mathbf{x}).$$

Hence,

$$(5.24)\quad \theta^{*(j)}(\boldsymbol{\xi}', \mathbf{x}', \omega) = -\frac{i\omega\varkappa\eta}{mc_1^2\varrho} u_j^{*T}(\mathbf{x}', \boldsymbol{\xi}', \omega), \quad j = 1, 2, 3.$$

Analogous solutions for the two and one-dimensional problem will be obtained by considering special cases of the reciprocity theorem. Let us consider the case in which

$$(5.25)\quad X_i^* = \delta(\mathbf{x}-\mathbf{x}')\delta_{ij}, \quad X_i^{*\prime} = \delta(\mathbf{x}-\boldsymbol{\xi}')\delta_{ik}, \quad Q^* = 0, \quad Q^{*\prime} = 0.$$

Let us consider two concentrated forces, one of which acts at the point $(\mathbf{x}')$ in the x_j-direction, the other, located at the point $(\boldsymbol{\xi}')$, acting in the x_k-direction.

From (5.21) we find:

$$(5.26)\quad \iiint_{(B)} \delta(\mathbf{x}-\mathbf{x}')\delta_{ij}u_i^{*(k)}(\mathbf{x}, \boldsymbol{\xi}', \omega)\, dV(\mathbf{x}) = \iiint_{(B)} \delta(\mathbf{x}-\boldsymbol{\xi}')\delta_{ik}u_i^{*}(\mathbf{x}, \mathbf{x}', \omega)\, dV(\mathbf{x}),$$

or

$$(5.27)\quad u_j^{*(k)}(\mathbf{x}', \boldsymbol{\xi}', \omega) = u_k^{*(j)}(\boldsymbol{\xi}', \mathbf{x}', \omega), \quad j, k = 1, 2, 3.$$

This relation expresses the reciprocity of displacement. The displacement $u_j^{(k)}$ at $(\mathbf{x}')$, due to the concentrated force at $(\boldsymbol{\xi}')$ acting in the x_j-direction is equal to the displacement $u_k^{*(j)}$ at $(\boldsymbol{\xi}')$ due to the same concentrated force acting at $(\mathbf{x}')$ in the x_j-direction. Let us observe that in addition to (5.27), the following equation resulting from (5.19) should be satisfied:

$$(5.28)\quad \iiint_{(B)} \theta^{*(j)}(\mathbf{x}, \mathbf{x}', \omega)e^{*(k)}(\mathbf{x}, \boldsymbol{\xi}, \omega)\, dV(\mathbf{x}) = \iiint_{(B)} \theta^{*(k)}(,\mathbf{x}\, \boldsymbol{\xi}', \omega)e^{*(j)}(\mathbf{x}, \mathbf{x}', \omega)\, dV(\mathbf{x}), \quad j, k = 1, 2, 3.$$

It can easily be verified, making use of (2.20) and (2.19) and bearing in mind the relation $e^* = \nabla^2\varphi^*$, that (5.28) is satisfied. Let us consider now the case in which

$$(5.29)\quad Q^* = \delta(\mathbf{x}-\mathbf{x}'), \quad Q^{*\prime} = \delta(\mathbf{x}-\boldsymbol{\xi}'), \quad X_i^* = 0, \quad X_i^{*\prime} = 0.$$

Two heat sources act in the infinite space, Q^* at $(\mathbf{x}')$ and $Q^{*\prime}$ at $(\boldsymbol{\xi}')$.

Making use of (5.21), we obtain:

$$(5.30)\quad \iiint_{(B)} \delta(\mathbf{x}-\mathbf{x}')\theta^{*\prime}(\mathbf{x}, \boldsymbol{\xi}', \omega)\, dV(\mathbf{x}) = \iiint_{(B)} \delta(\mathbf{x}-\boldsymbol{\xi}')\theta^{*}(\mathbf{x}, \mathbf{x}', \omega)\, dV(\mathbf{x}).$$

Hence

$$(5.31)\quad \theta^{*\prime}(\mathbf{x}', \boldsymbol{\xi}', \omega) = \theta^{*}(\boldsymbol{\xi}', \mathbf{x}', \omega).$$

We are concerned with the reciprocity of temperatures. At the same time, the following condition resulting from (5.20) should be fulfilled:

$$(5.32)\quad \iiint_{(B)} \theta^{*}(\mathbf{x}, \mathbf{x}', \omega)e^{*}(\mathbf{x}, \boldsymbol{\xi}', \omega)\, dV(\mathbf{x}) = \iiint_{(B)} \theta^{*\prime}(\mathbf{x}, \boldsymbol{\xi}', \omega)e^{*}(\mathbf{x}, \mathbf{x}'\, \omega)\, dV(\mathbf{x}).$$

Let a concentrated heat source act at the point $(\boldsymbol{\xi})$ of the infinite space. The wave function φ_T^* is expressed by Eq. (5.1). It enables us to determine the dilatation produced by the concentrated heat source. We have

$$e_T^*(\mathbf{x}, \boldsymbol{\xi}', \omega) = \nabla^2 \varphi_T^* = \frac{m}{4\pi\varkappa(k_1^2-k_2^2)}[k_2^2 I_2(R, \omega) - k_1^2 I_1(R, \omega)]. \tag{5.33}$$

Let us consider now a compression centre at $(\mathbf{x})$. Its action is accompanied by a temperature field expressed by the equation (2.27):

$$\theta^*(\mathbf{x}, \mathbf{x}', \omega) = \frac{q\varepsilon\sigma^2}{4\pi\varrho m(k_1^2-k_2^2)}[k_2^2 I_2(R, \omega) - k_1^2 I_1(R, \omega)]. \tag{5.34}$$

By confronting (5.33) and (5.34), it is seen that

$$\theta^*(\boldsymbol{\xi}, \mathbf{x}', \omega) = \frac{i\omega\varkappa\eta}{mc_1^2\varrho} e_T^*(\mathbf{x}', \boldsymbol{\xi}', \omega). \tag{5.35}$$

Passing now to the two-dimensional problem, let us consider a linear heat source at $(\boldsymbol{\xi}')$. The dilatation connected with the action of this heat source is:

$$e_T^*(\mathbf{x}, \boldsymbol{\xi}', \omega) = \nabla^2 \varphi_T^* = \frac{m}{2\pi\varkappa(k_1^2-k_2^2)}[k_2^2 K_0(k_2 r) - k_1^2 K_0(k_1 r)], \tag{5.36}$$

where the function φ_T^* has been obtained from Eq. (5.7).

Let us consider a compression centre at $(\mathbf{x}')$. It will produce a temperature, as expressed by the equation:

$$\theta^*(\mathbf{x}, \mathbf{x}', \omega) = \frac{q\varepsilon\sigma^2}{2\pi\varrho m(k_1^2-k_2^2)}[k_2^2 K_0(k_2 r) - k_1^2 K_0(k_1 r)]. \tag{5.37}$$

It is seen that

$$\theta^*(\boldsymbol{\xi}', \mathbf{x}', \omega) = \frac{i\omega\varkappa\eta}{mc_1^2\varrho} e_T^*(\mathbf{x}, \boldsymbol{\xi}', \omega). \tag{5.38}$$

This relation can also be obtained by making use of the reciprocity relation (5.27). This relation has been obtained by considering a concentrated force acting at $(\mathbf{x}')$ in the x_1-direction and a heat source at $(\boldsymbol{\xi}')$. If now the concentrated force is replaced by a compression centre, which is equivalent to summing up the results of action of three double forces, and if it is borne in mind that

$$\theta^*(\mathbf{x}, \mathbf{x}', \omega) = \partial_{1'}\theta^{*(1)}(\mathbf{x}, \mathbf{x}', \omega) + \partial_{2'}\theta^{*(2)}(\mathbf{x}, \mathbf{x}', \omega) + \partial_{3'}\theta^{*(3)}(\mathbf{x}, \mathbf{x}', \omega), \qquad \partial_{j'} = \frac{\partial}{\partial x_j'},$$

and

$$\partial_{1'} u_1^{*T}(\mathbf{x}, \boldsymbol{\xi}', \omega) + \partial_{2'} u_2^{*T}(\mathbf{x}, \boldsymbol{\xi}', \omega) + \partial_{3'} u_3^{*T}(\mathbf{x}, \boldsymbol{\xi}', \omega) = -e_T^*(\mathbf{x}, \boldsymbol{\xi}', \omega), \qquad \partial_{j'} = \frac{\partial}{\partial \xi_j'}$$

we obtain from (5.27)

$$\theta^*(\boldsymbol{\xi}', \mathbf{x}', \omega) = \frac{i\omega\varkappa\eta}{mc_1^2\varrho} e_T^*(\mathbf{x}', \boldsymbol{\xi}', \omega), \tag{5.39}$$

which is a result identical with (5.38).

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Streszczenie

PEWNE PROBLEMY DYNAMICZNE TERMOSPRĘŻYSTOŚCI (II)

W pracy podano w postaci zamkniętej funkcje Greena dla przemieszczeń i temperatury w nieograniczonej przestrzeni termosprężystej tak dla trójwymiarowego jak i dla dwu- i jedno-wymiarowego zagadnienia. Podano szereg związków łączących poszczególne funkcje, a wypływających z rozszerzonego na zagadnienie sprzężone termosprężystości twierdzenia o wzajemności.

Резюме

НЕКОТОРЫЕ ДИНАМИЧЕСКИЕ ЗАДАЧИ ТЕРМОУПРУГОСТИ (II)

Даются в замкнутом виде функции Грина для перемещений и температуры в бесконечном термоупругом пространстве как для трехмерной так и для двухмерной и одномерной задачи. Приводится ряд зависимостей, связывающих отдельные функции, которые вытекают из расширенной на сопряженную задачу термоупругости теоремы о взаимности.

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