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THE PROBLEM OF THE THEORY OF THERMAL STRESSES

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Introduction

In the recent 15 years we have witnessed a tremendous development of the theory of thermal stresses. This development results from the need of engineering and is mainly stimulated by the technical progress in the field of aviation, machines and chemical engineering, above all nuclear.

The subject of the investigations consisted mainly in stationary and quasi-static problems, less attention was devoted to the dynamic problems. Presently the linear theory of thermal stresses is a fully developed field. Methods of solution of the differential equations have been worked out, various general theorems have been proved and numerous particular problems have been solved, resulting in the progress in practical applications.

The most important problems now are the problems of thermoelasticity, concerning the coupling between the fields of deformation and temperature. This topic is of a considerable theoretical interest, since it concerns a synthesis of the theory of elasticity and the theory of thermal conduction.

Nevertheless, certain topics in the theory of thermal stresses still require a certain development; the progress here is both necessary and possible. We mention here the problem of the theory of thermal stresses in anisotropic and homoge - neous bodies.

The problem of the thermal stresses in anisotropic bodies acquires a serious practical aspect in view of a wide application of materials possessing anisotropic structure in machine building and aircraft structures.

The necessity of investigating, the thermal stresses in inhomogeneous bodies follows from the application of high temperatures in the machine building. The inhomogeneity is here produced by the temperature field. The material coefficients become functions of the temperature and therefore in the case of a stationary flow, functions of position.

In this paper we shall briefly outline the fundamental relations and equations of the theory of thermal stresses from various points of view, we shall present the basic achievements and wherever possible indicate the main trends in the investigations.

We confine our considerations to the linear theory of thermal stresses; the geometric or physical nonlinearity will not be dealt with here.

1. The fundamental relations and equations of the theory of thermal stresses

Assume that a homogeneous and isotropic body in its undeformed and free of stress state /no external forces are present/ is at a constant temperature T_0 . This initial state is called the natural state of the body. Owing to the action of external loadings, i.e. the body and surface forces, and heat sources in the interior of the body, and owing to heating of the surface of the body, the latter undergoes a deformation. There arise in the body the displacements $\vec{u}$ and the temperature changes by $\theta = T - T_0$. The deformation of the body is accompanied by the strain ε_{ij} and the stress σ_{ij} . The above mentioned quantities are functions of position and time.

We assume that the change of temperature accompanying the deformation, $\theta = T - T_0$, is small and that the increase of the temperature does not result in any changes of the material coefficients, both elastic and thermal. Furthermore , we assume that the deformations are small, thus confining ourselves to the geometrically linear thermoelasticity. The relation between the deformations and displacements is expressed by the linear formulae

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) \qquad /1.1/$$

It is known that the deformations cannot be arbitrary, but must satisfy the six relations

$$\varepsilon_{ij,kl} + \varepsilon_{kl,ij} - \varepsilon_{jl,ik} - \varepsilon_{ik,jl} = 0 \qquad i,j,k,l = 1,2,3 \qquad /1.2/$$

The constitutive relations, i.e. the relations between the state of stress, state of strain and temperature, are deduced from thermodynamical considerations, taking into account the principle of conservation of energy and the entropy balance [1] + [4]

$$\frac{d}{dt}\int_V \left(U + \frac{1}{2}\rho v_i v_i\right) dV = \int_V X_i v_i \, dV + \int_A p_i v_i \, dA - \int_A q_i n_i \, dA \qquad /1.3/$$

$$\int_V \frac{\partial S}{\partial t}\, dV = -\int_V \frac{q_i n_i}{T}\, dA + \int_V \Theta\, dV \qquad /1.4/$$

Here U is the internal energy, S is the entropy, X_i the components of the body forces, $p_i = \sigma_{ji} n_j$; the components of the stress vector, q_i the components of the vector of heat flux, n_i the components of the normal to the surface A . Further, $v_i = \frac{\partial u_i}{\partial t}$ and the quantity Θ represents the source of entropy, a quantity always positive in a thermodynamically irreversible process.

The terms in the left-hand side of Eq. /1.3/ represent the rate of increase of the internal and kinetic energies. The first term of the right-hand side is the rate of increase of the work of the body forces, and the second the rate of increase of the work of the surface tractions. Finally the last term of the right-hand side of Eq. /1.3/ is the energy acquired by the body by means of the thermal conduction. The left-hand side of Eq. /1.4/ is the rate of increase of the entropy. The first term of the right-hand side of Eq. /1.4/ represents the exchange of entropy with the surroundings and the second term, the rate of production of the entropy due to the heat conduction.

Making use of the equations of motion

$$\sigma_{ji,j} + X_i = \rho \ddot{u}_i \qquad i,j = 1,2,3 \qquad /1.5/$$

and using the divergence theorem to transform the integrals we arrive at the local relations

$$\dot{U} = \sigma_{ji}\,\varepsilon_{ji} - q_{i,i} \qquad \dot{S} = \Theta - \frac{q_{i,i}}{T} + \frac{q_i T_{,i}}{T^2} \tag{1.6}$$

Introducing the Helmholtz free energy $F = U - ST$ and eliminating the quantity $q_{i,i}$ from Eqs /1.6/ we obtain

$$\dot{F} = \sigma_{ji}\,\dot{\varepsilon}_{ji} - \dot{T}S - T\left(\Theta + \frac{q_i T_{,i}}{T^2}\right) \tag{1.7}$$

Since the free energy is a function of the independent variables ε_{ij}, T we have

$$\dot{F} = \frac{\partial F}{\partial \varepsilon_{ji}}\,\dot{\varepsilon}_{ji} + \frac{\partial F}{\partial T}\,\dot{T} \tag{1.8}$$

Assuming that the functions Θ, q_i, σ_{ji} do not explicitly depend on the time derivatives of the functions ε_{ij} and T we obtain comparing /1.7/ and /1.8/

$$\sigma_{ji} = \frac{\partial F}{\partial \varepsilon_{ji}}\,, \qquad S = -\frac{\partial F}{\partial T}\,, \qquad \Theta + \frac{q_i T_{,i}}{T^2} = 0 \tag{1.9}$$

The postulate of the thermodynamics of irreversible processes will be satisfied if $\Theta > 0$, i.e.when $-\frac{1}{T^2}T_{,i}\,q_i > 0$. This condition is satisfied by the Fourier law of heat conduction

$$-q_i = k_{ij}\,T_{,j} \qquad \text{or} \qquad -q_i = k_{ij}\,\theta_{,ij} \qquad \theta = T - T_0 \tag{1.10}$$

For a homogeneous and isotropic body Eq. /1.10/ takes the form

$$-q_i = k\,\theta_{,i} \tag{1.11}$$

Finally, it follows from Eqs $/1.6/_2$ and $/1.9/_3$ that

$$T\dot{S} = -q_{i,i} = k\,\theta_{,jj} \tag{1.12}$$

Here k is the coefficient of heat conduction.

The first two relations /1.9/ imply the constitutive relation. Expanding the free energy $F(\varepsilon_{ij}, T)$ into the Taylor series in the vicinity of the natural state $(\varepsilon_{ij} = 0, T = 0)$ and neglecting terms higher than the second we obtain

$$F = \mu \varepsilon_{ij} \varepsilon_{ij} + \frac{\lambda}{2} \varepsilon_{kk} \varepsilon_{mm} - \gamma \varepsilon_{kk} \theta - \frac{m}{2} \theta^2 \qquad /1.13/$$

The free energy is a scalar and consequently each term of the right-hand side should be a scalar. From the components of the tensor ε_{ij} we can construct two independent invariants of second order, namely $\varepsilon_{ij} \varepsilon_{ij}$ and $\varepsilon_{kk} \varepsilon_{mm}$. The third term of the right-hand side contains the invariant ε_{kk}. This follows from the fact that the tensor ε_{ij} has only one invariant of first order, namely ε_{kk}. Making use of the relations /1.9/ we have

$$\sigma_{ij} = 2\mu \varepsilon_{ij} + (\lambda \varepsilon_{kk} - \gamma \theta) \delta_{ij} \qquad /1.14/$$

$$S = \gamma \varepsilon_{kk} + m \theta \qquad /1.15/$$

Here μ, λ are the Lamé constants measured in the isothermal process. Further, $\gamma = (3\lambda + 2\mu)\alpha_t$ where α_t is the linear coefficient of thermal expansion. The quantity m is determined by means of the condition that dU is a total differential : $m = \frac{c_\varepsilon}{T_0}$ where c_ε is the specific heat at constant deformation.

Relations /1.14/ are called the Duhamel-Neumann relations. Introducing them into the equations of motion /1.5/ and eliminating the deformation ε_{ij} by means of /1.1/ we arrive at the system of the displacement equations

$$\mu \nabla^2 u_i + (\lambda+\mu) u_{k,ki} + X_i = \varrho \ddot{u}_i + \gamma \theta_{,i} \qquad i,j,k=1,2,3 \qquad /1.16/$$

In this system of three equations we have four unknown functions: the three components of the displacement vector $\vec{u}$ and the temperature θ. Eqs /1.16/ should be completed by the heat conduction equation. This is derived by comparing the expressions

$$T\dot{S} = k\,\theta_{,jj} \qquad \dot{S} = \gamma\,\dot{\varepsilon}_{kk} + \frac{c_\varepsilon}{T_0}\,\theta \qquad /1.17/$$

The second of the above equations follows from /1.15/. Assuming that $\left|\frac{\theta}{T_0}\right| \ll 1$ and comparing $/1.17/_1$ and $/1.17/_2$ we obtain after a linearization

$$\nabla^2\theta - \frac{1}{\varkappa}\dot{\theta} - \eta\,\dot{\varepsilon}_{kk} = 0, \quad \varkappa = \frac{k}{c_\varepsilon}, \quad \eta = \frac{\gamma T_0}{k} \qquad /1.18/$$

Taking into account the action of heat sources of intensity W per unit time and volume we arrive at the non-homogeneous equation

$$\nabla^2\theta - \frac{1}{\varkappa}\dot{\theta} - \eta\,\dot{\varepsilon}_{kk} = -\frac{Q}{\varkappa}, \quad Q = \frac{W}{\lambda} \qquad /1.19/$$

Eq. /1.19/ is the generalized equation of heat conduction. Eqs /1.16/ and /1.19/ are coupled and constitute the system of equations of the coupled thermoelasticity. This system of equations is most complicated and its solution encounters difficiulties of mathematical nature. However, it turns out that the term $\eta\,\dot{\varepsilon}_{kk}$ very little influences the distribution of stresses and temperature. The disregarding of this term constitutes the basis of a simplified theory, <u>the theory of</u>

thermal stresses. Neglecting the term $\eta \varepsilon_{kk}$ we obtain the independent systems of equations

$$\mu \nabla^2 u_i + (\lambda+\mu) u_{k,ki} = \varrho \ddot{u}_i + \gamma \theta_{,i} \qquad /1.20/$$

$$\left(\nabla^2 - \frac{1}{\varkappa} \partial_t \right) \theta = - \frac{W}{\varkappa} \qquad /1.21/$$

Eq. /1.21/ is the classical heat conduction equation.

Taking into account the prescribed boundary and initial conditions we determine by means of Eq. /1.21/ the temperature $\theta(x,t)$ and this known function is then introduced into the system of the displacement equations /1.20/. In the case of a stationary temperature field the displacement and the temperature become independent of time. The corresponding system of equations has the form

$$\mu \nabla^2 u_i + (\lambda+\mu) u_{k,ki} = \gamma \theta_{,i} \,, \quad \nabla^2 \theta = - \frac{Q}{\varkappa} \qquad /1.22/$$

In the following considerations we shall deal only with the simplified theory, i.e. the linear theory of thermal stresses described by the system of equations /1.21/, /1.20/ for the non-stationary state, and the system /1.22/ for the stationary case.

2. General theorems of the theory of thermal stresses

The important role of the variational theorems in the theory of elasticity is well known. They make it possible to derive the differential equations for complex systems /e.g.bending of plates and shells/ together with the appropriate "natural" boundary conditions, and, moreover, to construct approximate solutions.

Thus, the principle of virtual work for the variation of the displacement

$$\int_V (X_i - \rho \ddot{u}_i)\delta u_i \, dV + \int_A p_i \delta u_i \, dA = \int_V \sigma_{ij} \delta \varepsilon_{ij} \, dV \qquad /2.1/$$

for the linear Duhamel-Neumann relations, i.e. for the problems of the theory of thermal stresses, takes the form

$$\int_V (X_i - \rho \ddot{u}_i)\delta u_i \, dV + \int_A p_i \delta u_i \, dA = \delta W_\varepsilon - \gamma \int_V \theta \delta \varepsilon_{kk} \, dV \qquad /2.2/$$

Here $W_\varepsilon = \int \left(\mu \varepsilon_{ij} \varepsilon_{ij} + \frac{\lambda}{2} \varepsilon_{kk} \varepsilon_{nn}\right) dV$.

Eq. /2.2/ may be written as follows

$$\int_V (X_i - \rho \ddot{u}_i - \gamma \theta_{,i})\delta u_i \, dV + \int_A (p_i + n_i \gamma \theta)\delta u_i \, dA = \delta W_\varepsilon \qquad /2.3/$$

Comparing this expression with the virtual work performed by the forces X_i^*, p_i^* on the displacements δu_i in the body of the same shape and volume and assuming that the process is isothermal we have $(\theta^* = 0)$

$$\int_V (X_i^* - \rho \ddot{u}_i)\delta u_i \, dV + \int_A p_i^* \delta u_i \, dA = \delta W_\varepsilon \qquad /2.4/$$

We have assumed here that the external forces X_i^*, p_i^* are chosen in such a way that the displacement field u_i is identical with that produced by the action of the forces X_i, p_i and the temperature field θ. Comparing Eqs /2.3/ and /2.4/ we obtain

$$X_i^* = X_i - \gamma \theta_{,i} \quad x \in V \; ; \quad p_i^* = p_i + n_i \gamma \theta \quad x \in A \qquad /2.5/$$

This is the so-called analogy of body forces [5]. Relations /2.5/ make it possible to reduce the problem of the theory of thermal stresses to the problems of the theory of elasticity.

Making use of the analogy of the body forces we are in a position to state some variational theorems for the theory of thermal stresses. Thus, the theorem of minimum of potential energy generalized to the theory of thermal stresses has the form

$$\delta \pi = 0 \, , \qquad \pi = \mathrm{Min} . \qquad /2.6/$$

where

$$\pi = W_\varepsilon - \int_V X_i u_i \, dV - \int_{A_\sigma} p_i u_i \, dA - \gamma \int_V \theta \varepsilon_{kk} \, dV \, .$$

Here A_σ is the part of the surface A bounding the body, on which the tractions p_i are known.

The theorem of minimum of the complementary work in the theory of thermal stresses has the form

$$\delta \Gamma = 0 \qquad \Gamma = \mathrm{Min} . \qquad /2.7/$$

where

$$\Gamma = W_\sigma - \int X_i u_i \, dV - \int_{A_u} p_i u_i \, dA + \alpha_t \int \theta \sigma_{kk} \, dV$$

Here

$$W_\sigma = \int_V \left(\mu' \sigma_{ij} \sigma_{ij} + \frac{\lambda'}{2} \sigma_{kk} \sigma_{nn} \right) dV \, , \quad \mu' = \frac{1}{4\mu} \, ,$$

$$\lambda' = - \frac{\lambda}{2\mu (3\lambda + 2\mu)}$$

A_u denotes the part of the surface A on which the displacements are prescribed.

Let us now return to the variational principle /2.3/. If we assume that the virtual increments $\delta u_i, \delta\varepsilon_{ij}$ are identical with the real increments occuring in passing from the instant t to the instant $t+dt$ and bearing in mind that $\delta u_i = \dot{u}_i dt$, $\delta\varepsilon_{ij} = \dot{\varepsilon}_{ij} dt$, $\delta W_\varepsilon = \dot{W}_\varepsilon dt$ we have

$$\frac{d}{dt}(W_\varepsilon + K) = \int_V (X_i - \gamma\theta_{,i}) v_i \, dV + \int_A (p_i + n_i \gamma\theta) v_i \, dA , \quad v_i = \dot{u}_i \qquad /2.8/$$

Here $K = \frac{\varrho}{2}\int_V v_i v_i \, dV$ is the kinetic energy. Eq. /2.8/ constitutes the fundamental energy theorem of the theory of thermal stresses. From this theorem we can deduce the uniqueness of the solution of the differential equations of the theory of thermal stresses for a simply connected body, and moreover, the generalization of the Kirchhoff theorem of the elasticity theory. [6]

The principle of virtual work /2.2/ makes it possible to derive the Hamilton principle for the theory of thermal stresses

$$\delta \int_0^t (U - K)\, dt = \int_0^t \delta L \, dt \qquad /2.9/$$

where

$$U = W_\varepsilon - \gamma \int_V \theta \varepsilon_{kk} \, dV , \quad \delta L = \int_V X_i \delta u_i \, dV + \int_A p_i \delta u_i \, dA$$

In the case of conservative forces $\delta A = -\delta V = \frac{\partial V}{\partial u_i}\delta u_i$

we obtain from /2.9/

$$\delta \int_0^t (U - K - V)\, dt = 0 \qquad /2.10/$$

One of the most interesting theorems of the theory of elasticity is the reciprocity theorem. This theorem implies not only the symmetry of the fundamental solutions /the Green functions/ but it makes it possible to deduce methods of solution of the differential equations of the theory of thermal stresses. The Duhamel-Neumann relations yield the identity

$$\sigma_{ij}\varepsilon'_{ij} - \sigma'_{ij}\varepsilon_{ij} = \gamma(\theta'\varepsilon_{kk} - \theta\varepsilon'_{kk}) \qquad /2.11/$$

Integrating over the volume of the body the expressions /2.11/ containing the stress and strain belonging to two different systems of causes and results, after simple transformations, using the equations of motion, we obtain

$$\int_V [X_i(x)\, u'_i(x) - X'_i(x)\, u_i(x)]\, dV(x) + \int_A [p_i u'_i - p'_i u_i]\, dA = \qquad /2.12/$$
$$= \gamma \int_V (\theta\varepsilon'_{kk} - \theta'\varepsilon_{kk})\, dV$$

Eq. /2.12/ was first given by W.M.Maysel [7].

Let us consider a particular case of this theorem. Consider a bounded body fixed on the surface A_u and free of tractions on the surface A_σ ; $A = A_u + A_\sigma$. The displacement $u_k(\xi)$ due to the heating of the body is obtained from the formula following from the theorem /2.12/

$$u_k(\xi) = \gamma \int_V \theta(x)\, U^{(k)}_{j,j}(x,\xi)\, dV(x) \qquad /2.13/$$

Here $u_j' = U_j^{(k)}(x,\xi)$ is the field of displacement occuring in a body of the same shape and the same boundary conditions, in the isothermal process $\theta' = 0$. The displacements $U_j^{(k)}$ are due to the action of a concentrated force located at the point ξ and directed parallelly to the x_k -axis . Formula /2.13/ given by W.M.Maysel constitutes a method of solution of the equations of the theory of thermal stresses by means of the Green function. This method was applied by Maysel to the solution of a number of examples concerning thermal stresses in plates and shells. In these cases the determination of the Green function for various shapes and boundary conditions does not encounter any serious difficulties.

The reciprocity theorem can be generalized to the dynamic theory of thermal stresses. It has here the form

$$\int_0^t d\tau \int_V \left[X_i(x,t-\tau)\, u_i'(x,\tau) - X_i'(x,\tau)\, u_i(x,t-\tau)\right] dV(x) +$$
$$\int_0^t d\tau \int_A \left[p_i(x,t-\tau)\, u_i'(x,\tau) - p_i'(x,\tau)\, u_i(x,t-\tau)\right] dA(x) = \qquad /2.14/$$
$$= \gamma \int_0^t d\tau \int_V \left[\theta(x,t-\tau)\, \varepsilon_{kk}'(x,\tau) - \theta'(x,\tau)\, \varepsilon_{kk}(x,t-\tau)\right] dV(x)$$

A procedure analogous to that in the stationary case leads to the formula for the displacements

$$u_k(\xi,t) = \gamma \int_0^t d\tau \int_V \theta(x,t-\tau)\, U^{(k)}_{j,j}(x,\xi,\tau)\, dV(x) \qquad /2.15/$$

Formula /2.15/ concerns a heated body, fixed on A_u and free of tractions on A_σ. The displacement $U_i^{(k)}(x,\xi,t)$ refers to the action of a concentrated instantaneous force applied at the point ξ and acting along the x_k-axis. This displacement occurs in a body fixed on A_u and free of tractions on A_σ, in the isothermal process state, $\theta' = 0$.

A variant of the reciprocity theorem /2.14/ is the following theorem given by W.M.Maysel [8]:

$$\int_V (X_i u_i' - X_i' u_i)\,dV + \int_A (p_i u_i' - p_i' u_i)\,dA - \varrho \int_V \ddot{u}_i u_i'\,dV = \int_V (\theta' \varepsilon_{kk} - \theta \varepsilon_{kk}')\,dV \qquad /2.16/$$

Here the quantities without primes refer to the dynamic problem while the primed concern the static case.

The formula analogous to /2.15/ takes the form

$$u_k(\xi,t) + \varrho \int_V \ddot{u}_i(x,t)\, U_i^{(k)}(x,\xi)\,dV(x) = \gamma \int_V \theta(x,t)\, U_{j,j}^{(k)}(x,\xi)\,dV(x) \qquad i,k = 1,2,3 \qquad /2.17/$$

It concerns a heated body, fixed on A_u, free of stresses on A_σ. The displacement $U_i^{(k)}(x,\xi)$ is that due to the action of a static concentrated force at the point ξ, acting in the direction x_k in the body in the isothermal process /the Green function/.

The reciprocity theorem /2,14/ constitutes the basis of the method of W.Nowacki [9] of solving problems with mixed boundary conditions by means of the Green function satis -

fying homogeneous boundary conditions.

The reciprocity theorem /2.12/ implies an interesting result concerning the change of volume of a body. The increment of the volume of a simply connected body, heated and free of tractions on its surface A is given by the formula [10]

$$\Delta V = 3\alpha_t \int_V \theta(x)\, dV(x) \qquad /2.18/$$

The formula /2.18/ yields the statement that the mean values of the stress invariant vanish [11] :

$$\int_V \sigma_{kk}\, dV = 0 \;.$$

3. Methods of solving differential equations of the theory of thermal stresses. Stationary and quasi-static problems

Consider the system of displacement equations of the theory of thermal stresses

$$\mu \nabla^2 u_i + (\lambda+\mu) u_{k,ki} = \gamma \theta_{,i} \qquad i,k = 1,2,3 \qquad /3.1/$$

The temperature θ appearing in these equations is a known function, obtained by solving the heat conduction equation in the case of a stationary flow of heat

$$\nabla^2 \theta = -\frac{Q}{\varkappa} \qquad /3.2/$$

completed by the appropriate boundary condition.

Assume that the boundary conditions for Eqs /3.1/ are homogeneous.

Set

$$u_i(x) = 0 \quad x \in A_u \ ; \quad p_i = \sigma_{ji}(x) n_j(x) = 0 \quad x \in A_\sigma \qquad /3.3/$$

The solution of the system /3.1/ can be represented in the form

$$u_i = u_i' + u_i'' \qquad /3.4/$$

where u_i' is the particular solution of the non-homogeneous system of equations /3.1/ and u_i'' is the general solution of the homogeneous system /3.1/. The particular solution can easily be found by introducing the potential of thermo-elastic deformation Φ related to the displacement u_i' by means of [12] [13]

$$u_i' = \Phi_{,i} \qquad /3.5/$$

substituting /3.4/ and /3.5/ into the system of equations /3.1/ we are led to the system of equations

$$\nabla^2 \Phi = m\Theta, \quad m = \frac{\gamma}{\lambda + 2\mu} \qquad /3.6/$$

$$\mu \nabla^2 u_i'' + (\lambda + \mu) u_{k,ki}'' = 0 \qquad /3.7/$$

The problem has been reduced to the solution of the Poisson equation and the system of the displacement equations of the theory of elasticity.

In an infinite elastic space the solution of the Poisson equation /3.6/ is the final solution

$$\Phi(\xi) = -\frac{m}{4\pi} \int_V \frac{\Theta(x)\, dV(x)}{R(x,\xi)} \qquad /3.8/$$

Here $R(x,\xi)$ is the distance between the points (x) and (ξ). The displacements, deformations and stresses are expressed in terms of the function Φ by the relations

$$u_i = \Phi_{,i} \quad , \quad \varepsilon_{ij} = \Phi_{,ij} \quad \sigma_{ij} = 2\mu(\Phi_{,ij} - \delta_{ij}\Phi_{,kk}) \qquad /3.9/$$

In a bounded body the function Φ can satisfy at most a part of the conditions and therefore always an additional solution is necessary. The solution of Eqs /3.1/ for a bounded body can be represented in various forms. If we apply to the solution of Eqs /3.7/ the Papkovitch-Neuber functions, the displacement u_i is assumed in the form

$$\vec{u} = \operatorname{grad} \phi + (\varphi + \vec{R}\cdot\vec{\psi}) - 4(1-\nu)\vec{\psi} \qquad /3.10/$$

where the functions φ, ψ_i satisfy the harmonic equation and the function Φ Eq. /3.6/

If we introduce the Galerkin functions, then

$$u_i = \Phi_{,i} + \frac{\lambda+2\mu}{\mu}\nabla^2\chi_i\,\delta_{ij} - \frac{\lambda+\mu}{\mu}\partial_i\partial_j\chi_j \qquad /3.11/$$

where the function Φ satisfies Eq./3.6/ and the func - tions χ the biharmonic equations

$$\nabla^2\nabla^2\chi_i = 0 \qquad i = 1,2,3 \qquad /3.12/$$

In problems possessing axial symmetry with respect to the z-axis it is convenient to use the Love functions. In the system of cylindrical coordinates (r, ϑ, z) we express the displacement as follows:

$$u_r = \frac{\partial \phi}{\partial r} - \frac{\partial^2 \chi}{\partial r \partial z} \quad , \quad u_z = \frac{\partial \phi}{\partial z} + 2(1-\nu)\nabla^2 \chi - \frac{\partial^2 \chi}{\partial z^2} \qquad /3.13/$$

here the function ϕ satisfies Eq. /3.6/ and the Love function the biharmonic equation

$$\nabla^2 \nabla^2 \chi(r, z) = 0 \qquad /3.14/$$

To determine the thermal stresses in bodies of simple shape /elastic space, semi-space, elastic layer, etc./ the method of integral transforms has been successfully applied to Eqs /3.1/ [14] [15].

One more method of solving problems of the theory of thermal stresses is worth mentioning here. It consists in the determination of the Green function as the solution of the equation

$$\nabla^2 \phi^*(x, \xi) = m\delta(x-\xi) \qquad /3.15/$$

Here $\delta(x-\xi)$ is the Dirac delta function. The solution ϕ^* of Eq. /3.15/ does not satisfy all boundary conditions and therefore, in the case of a bounded body an additional solution $u_i'' = U(x, \xi)$ is needed, which satisfies the system of homogeneous equations

$$\mu \nabla^2 U_i + (\lambda + \mu) U_{k,ki} = 0 \qquad /3.16/$$

Knowing the function $U_i(x, \xi)$ we are in a position to determine the displacements $u_i(x)$ produced by the action of the temperature field $\theta(x)$. They are determined from the formula

$$u_i(\xi) = \int_V \theta(x)\, U_i(x,\xi)\, dV(x) \qquad /3.17/$$

This procedure is of a considerable importance in the case of a discontinous temperature field, when the function θ does not satisfy the heat conduction equation. The discontinuous temperature fields are encountered in certain specific cases, for instance when a part of the body is heated to a constant temperature $\hat{\theta}^{(e)}$ and a part to the temperature $\theta^{(i)}$. The discontinuous temperature field is also obtained in the case of a body with different thermal properties and uniform elastic properties, heated to a constant temperature θ_0.

The functions Φ^*, U_i have been determined for bodies of simple shapes, such as elastic semi-space, sphere, infinite cylinder, layer, etc. [17], [19].

Another method of the determination of the thermal stresses consists in making use of the Beltrami-Michell equations, in which in accordance with the body forces analogy the body forces are replaced by the quantity $-\gamma\theta_{,i}$. Thus, we obtain the system of equations

$$\nabla^2 \sigma_{ij} + \frac{2(\lambda+\mu)}{3\lambda+2\mu}\sigma_{kk,ij} + 2\mu\alpha_t\left(\theta_{,ij} + \frac{3\lambda+2\mu}{\lambda+2\mu}\theta_{,kk}\delta_{ij}\right) = 0 \qquad /3.18/$$

$$i,j,k = 1,2,3$$

Except for certain few cases [20], [21] these equations for spatial problems have not been widely applied.

On the other hand the above method has successfully been used in two-dimensional problems, in the problems of the plane state of stress and strain. Introducing the Airy stress function related to the stresses by means of the expressions

$$\sigma'_{\alpha\beta} = -\partial_\alpha \partial_\beta F + \delta_{\alpha\beta} \nabla_1^2 F \qquad \nabla_1^2 = \partial_1^2 + \partial_2^2 \qquad \alpha,\beta = 1,2 \tag{3.19}$$

we obtain after the elimination of the temperature, the equation

$$\nabla_1^2 \nabla_1^2 F = \beta_1 Q \tag{3.20}$$

where for the plane state of stress $\beta = E\alpha_t$, while for the plane state of strain $\beta = \frac{E\alpha_t}{1-\nu}$. Eq. /3.20/ should be completed by the boundary conditions for the boundary free of tractions

$$F = 0 \qquad \frac{\partial F}{\partial n} = 0 \tag{3.21}$$

It follows from Eq. /3.20/ and the boundary conditions /3.21/ that for a simply connected region and the plane state of stress we have $F \equiv 0$, i.e. the body deforms without stresses. In the plane state of strain, besides the stresses $\sigma_{\alpha\beta}$ given by the formulae /3.19/ there appear the stresses

$$\sigma_{33} = \nu \nabla_1^2 F - 2\mu m \theta \tag{3.22}$$

Consequently, if there are no heat sources in a simply connected infinite cylinder /plane state of strain/, then

$F = 0$ and the only non-vanishing stress is $\sigma_{33} = -2\mu m\theta$ /Muskelishvili [22]/. In the case of presence of heat sources and a multiply connected body the solution of Eq. /3.20/ can be deduced by using the complex variable functions /Muskhelishvili [23] , Gatewood [24] /. There is an interesting observation by Dubas [25] and Tremmel [26] , concerning the analogy between Eq. /3.20/ and the equation of deflection of a thin plate fixed on the boundary. It is known that this deflection is described by the equation

$$\nabla_1^2 \nabla_1^2 w = \frac{p}{N} \qquad /3.23/$$

and the boundary conditions

$$w = 0 \qquad \frac{\partial w}{\partial n} = 0 \qquad /3.24/$$

This "plate analogy" makes it possible to make use of the numerous solutions in the theory of plates to determine the thermal stresses in discs.

Every non-stationary problem of the theory of thermal stresses is a dynamic problem, but in the case of a slow variation of the temperature in time the influence of the inertia forces is negligible and the problem may be regarded as quasi-static, disregarding the inertia terms in the equations of motion. Thus, in the quasi-static case we are faced with the system of equations

$$\mu \nabla^2 u_i + (\lambda + \mu) u_{k,ki} = \gamma \theta_{,i} \qquad /3.25/$$

$$\left(\nabla^2 - \frac{1}{\varkappa}\partial_t\right)\theta = -\frac{Q(x,t)}{\varkappa} \qquad /3.26/$$

From Eq. /3.26/ completed by the appropriate boundary and initial conditions we find the temperature θ as a function of the variables x, t and as a known function it is introduced into Eq. /3.25/. The solution of /3.25/ yields the displacements $u_i(x,t)$, the time however which appears in the displacements is regarded as a parameter. The method of solving Eqs /3.25/ is the same as for the stationary case. The function $\Phi(x,t)$, however, can be determined in a different way, found by J.N.Goodier [13]

$$\Phi = m\varkappa\int_0^t \theta\, dt + \Phi_0 + t\,\Phi_1$$

here Φ_1 is a harmonic function and $\Phi_0 = \Phi(x,0)$ is the displacement potential corresponding to the initial temperature $\theta_0 = \theta(x,0)$. The function Φ_0 should satisfy the equation

$$\nabla^2\Phi_0^{(x)} = m\,\theta_0(x)$$

The stationary and quasi-static problems of the theory of thermal stresses have already been treated in numerous scientific papers. Important methods of solutions and important papers have been dealt with in some monographs, such as E.Melan and H.Parkus "Wärmespannungen infolge stationärer Temperaturfelder" /1953/, H.Parkus: "Instationäre Wärme - spannungen" /1959/, B.A.Boley and J.H.Weiner "Theory of thermal stresses" /1960/, W.Nowacki "Thermoelasticity"/1962/; therefore we shall not deal here with particular problems.

4. The dynamic problem of the theory of thermal stresses.

In numerous important engineering problems we encounter a sudden heating of the surface of the body or a sudden action of the heat sources. In these cases it is inadmissible to neglect the influence of the inertia terms. Here we are faced with the system of hyperbolic equations

$$\mu \nabla^2 u_i + (\lambda+\mu) u_{k,ki} = \varrho \ddot{u}_i + \gamma \theta_{,i} \qquad /4.1/$$

The temperature θ is a known function, obtained by solving the heat conduction equation

$$\left(\nabla^2 - \frac{1}{\varkappa}\partial_t\right)\theta = -\frac{Q}{\varkappa} \qquad /4.2/$$

Decomposing the displacement vector into its potential and solenoidal parts

$$u_i = \Phi_{,i} + \epsilon_{ijk}\psi_{j,k} \qquad /4.3/$$

we arrive at the system of wave equations

$$\square_1^2 \phi = m\theta \qquad \square_2^2 \psi_i = 0 \qquad i = 1,2,3 \qquad /4.4/$$

We have introduced here the notations

$$\square_\alpha^2 = \nabla^2 - \frac{1}{c_\alpha^2}\partial_t^2\,, \quad \alpha = 1,2\,; \qquad c_1 = \left(\frac{\lambda+2\mu}{\varrho}\right)^{1/2}, \quad c_2 = \left(\frac{\mu}{\varrho}\right)^{1/2}$$

Eliminating the temperature from Eqs /4.2/ and $/4.4/_1$ we obtain

$$\square_1^2 D \phi = -\frac{m}{\varkappa} Q\,, \quad \square_2^2 \psi_i = 0 \qquad /4.5/$$

where $D = \nabla^2 - \frac{1}{\varkappa}\partial_t$

The first equation refers to the longitudinal wave while the second to the transverse wave. It is evident that in an infinite space under the action of temperature /heat sources/, only longitudinal waves arise. In a finite region both types of waves occur, for the functions Φ and ψ_i are connected by means of the boundary conditions.

There exist also other methods of solving the system of equations /4.1/. Representing the solution of Eqs /4.1/ in the form

$$u_i = u_i' + u_i'' \qquad /4.6/$$

where

$$u_i' = \Phi_{,i} \qquad u_i'' = \frac{\lambda+2\mu}{\mu}\Box_1^2 \varphi_i \delta_{ij} - \frac{\lambda+\mu}{\mu}\partial_i\partial_j\varphi_j \qquad /4.7/$$

we have to solve the system of equations

$$\Box_1^2\Phi = m\Theta\,, \quad \Box_1^2\Box_2^2\varphi_i = 0 \qquad i=1,2,3 \qquad /4.8/$$

Here $u_i' = \Phi_{,i}$ is the particular solution of the non-homogeneous system of equations /4.1/ and $\vec{\varphi}$ is the solution of the homogeneous system

$$\mu\nabla^2 u_i'' + (\lambda+\mu)\, u_{k,ki}'' = \varrho\ddot{u}_i'' \qquad /4.9/$$

The vector $\vec{\varphi}$ is the familiar function of M.Iacovache [27].

Consider now the longitudinal thermoelastic wave produced in an infinite body. To this end einvisage the homogeneous equation /4.5/$_1$ which in the new variables $\tau = \frac{c_1^2}{\varkappa} t$, $\xi_i = \frac{c_1}{\varkappa} x_i$ takes the form

$$(\nabla^2 - \partial_\tau^2)(\nabla^2 - \partial_\tau)\phi(\xi,\tau) = 0 \qquad /4.10/$$

Consider first the motion harmonic in time

$$\phi(x,t) = e^{-i\omega t}\phi^*(x), \quad \phi(\xi,\tau) = e^{-i\chi\tau}\phi^*(\xi) \qquad /4.11/$$

$$\chi = \frac{\omega\varkappa}{c_1^2}$$

Taking into account /4.11/, Eq. /4.10/ takes the form

$$(\nabla^2 + \chi^2)(\nabla^2 + i\chi)\phi^*(\xi) = 0 \qquad /4.12/$$

The solution of this equation depending only on the radius ϱ connecting the points ξ and η, and satisfying the radation conditions /Sommerfeld conditions/, has the form

$$\phi = \frac{A_1}{\varrho} e^{-i\chi(\tau-\varrho)} + \frac{A_2}{\varrho} e^{-\varrho\sqrt{\frac{\chi}{2}}} e^{-i\chi\left(\tau - \frac{\varrho}{\sqrt{2\chi}}\right)} \qquad /4.13/$$

$$\varrho^2 = (\xi_j - \eta_j)(\xi_j - \eta_j)$$

This solution consists of two terms, the first of the nature of an elastic wave /undamped and propagated with a constant velocity/ and the second of the diffusion nature. The second type of wave motion undergoes a dispersion /it is propagated with a velocity $\sqrt{2\varkappa\omega}$ and undergoes a damping expressed by the term $e^{-\varrho\sqrt{\frac{\chi}{2}}}$

In the case of action of a concentrated heat source

$$Q(x,t) = Q_0\,\delta(R)\,e^{-i\omega t}, \quad R = (x_1^2 + x_2^2 + x_3^2)^{1/2}$$

we obtain for the function Φ the expression

$$\phi = - \frac{Q_0 m}{4\pi c_1 \chi(1+\chi^2)\varrho} \Big\{ \chi \cos \chi (\tau-\varrho) - \sin \chi (\tau-\varrho) - \qquad /4.14/$$

$$- e^{-\varrho\sqrt{\frac{\chi}{2}}} \Big[\chi \cos \chi \Big(\tau - \frac{\varrho}{\sqrt{2\chi}}\Big) + \sin \chi \Big(\tau - \frac{\varrho}{\sqrt{2\chi}}\Big) \Big] \Big\}$$

Here, too, the above mentioned types of waves are evident, i.e. the elastic wave and the thermal diffusional one.

In the case of an aperiodic motion the solution of Eq. /4.10/ depending only on the radius ϱ takes the form

$$\Phi(\xi,\tau) = \frac{f(\tau-\varrho)}{\varrho} + \int_0^{\tau} \frac{g(\tau-\tau')}{\pi(\tau'^3)^{1/2}} \exp\Big(-\frac{\varrho^2}{4\tau'}\Big) d\tau \qquad /4.15/$$

In the particular case of a concentrated and instantaneous heat source

$$Q(x,t) = Q_0 \delta(R)\, \delta(t)$$

we obtain for the function Φ the expression

$$\Phi(\xi,\tau) = \frac{m}{4\pi c_1 \varrho} \Big\{ (e^{\tau-\varrho} - 1)\, H(\tau-\varrho) - \Big[\mathcal{U}(\varrho,\tau) - \operatorname{erf}\Big(\frac{\varrho}{2\sqrt{\tau}}\Big) \Big] \Big\} \qquad /4.16/$$

where

$$\mathcal{U}(\varrho,\tau) = \frac{e^{\tau}}{2} \Big[e^{\varrho} \operatorname{erfc}\Big(\frac{\varrho}{2\sqrt{\tau}} + \sqrt{\tau}\Big) + e^{-\varrho} \operatorname{erfc}\Big(\frac{\varrho}{2\sqrt{\tau}} - \sqrt{\tau}\Big) \Big]$$

and

$$H(\tau-\varrho) = \begin{cases} 0 & \text{for} \quad \tau < \varrho \\ 1 & \text{for} \quad \tau > \varrho \end{cases}$$

Here as well we encounter the two above mentioned types of waves.

It is also possible to determine the thermal stresses on the basis of the differential equations in stresses. They can be deduced by means of the compatibility equations or by means of an appropriate transformation of the displacement equations [20].

Here we have the system of equations

$$\Box_2^2 \sigma_{ij} + \frac{2(\lambda+\mu)}{3\lambda+2\mu}\sigma_{kk,ij} + \left(\frac{1}{c_2^2}-\frac{1}{c_1^2}\right)\frac{\lambda\delta_{ij}}{3\lambda+2\mu}\ddot{\sigma}_{kk} +$$

$$+2\mu\alpha_t\left(\theta_{,ij} + \frac{3\lambda+2\mu}{\lambda+2\mu}\theta_{,kk}\delta_{ij}\right) - \frac{5\lambda+4\mu}{\lambda+2\mu}\alpha_t\varrho\,\delta_{ij}\ddot{\theta} = 0 \qquad /4.17/$$

$$i,j,k = 1,2,3$$

which should be completed by the boundary and initial conditions.

As in the case of stationary problems we can use the concept of the nucleus of thermoelastic strain in order to determine the dynamic thermal stresses. By this concept we understand the displacement field $u_i(x,\xi,t)$ produced by the temperature θ^* in the form of the Dirac delta function $\theta^* = \delta(x-\xi)\delta(t)$

Denoting by $[S^*(x,\xi,t]$ the solution of this problem, the solution $[S(x,t)]$ referring to the temperature field $\theta(x,t)$ is given by the formula

$$[S(x,t)] = \int_0^t d\tau \int_V [S^*(\xi,x,t-\tau)]\,\theta(\xi,\tau)\,dV(\xi) \qquad /4.18/$$

This method of solution is particularly convenient when the temperature field is discontinuous both as a function of position and time, so that is does not satisfy the heat conduction equation /4.2/. J.Ignaczak and W.Piechocki [28] [29] obtained by means of this method many interesting results.

There is an extensive literature concerning thy dynamic problems of the theory of thermal stresses. So far numerous one-dimensional and two-dimensional problems have been solved. The first solutions here given by V.I.Danilovskaya dates 1950 [30], [31]. They concern the propagation of thermoelastic waves in an elastic semi-space, due to a sudden heating of the bounding surface. Here, too, we find the two types of waves, elastic and diffusional. On the front of the elastic wave there occurs a jump of stress /change of sign/. The problem of propagation of a spherical wave in an infinite space was discussed by W.Nowacki [32], while the cylindrical wave was dealt with by H.Parkus /33/. A practically important case of a sudden heating of the boundary of a spherical cavity in an infinite elastic space was investigated by E.Sternberg and J.C.Chakravarty [34]. J.Ignaczak examined the action of a concentrated instantaneous heat source in an infinite elastic space with a spherical cavity [35]. A concentration of stresses around a spherical and cylindrical cavities was dealt with by J.Ignaczak and W.Nowacki [36]. The problem of heat sources moving with a constant velocity in an infinite elastic

space was the subject of the paper by M.Żórawski [37].
Finally we mention the paper of B.A.Boley and A.D.Barbar concerning the vibrations of a thin plate, produced by a sudden heating or cooling [38].

5. Thermal Stresses in Anisotropic Bodies

The theory of thermal stresses in homogeneous isotropic bodies has been treated in detail in the scientific literature, problems of thermal stresses in anisotropic bodies, however, have received very little attention. This fact is due not only to the mathematical difficulties of the problem, but also to the lack of wide practical applications. More and more frequently, however, engineering structures contain materials of macroscopically anisotropic structure /plates, discs, shells, thick-walled pipes, etc./ the properties of which /both elastic and thermal/ are different in different directions.

Here we shall give only a review of problems solved so far, and certain general relations; the reader interested in details will find them in the cited papers.

The equation of heat conduction in a body of general rectilinear anisotropy has the form /we disregard the coupling of the deformation and temperature fields/

$$\lambda_{ij} T_{,ij} - \varrho c \dot{T} = -W , \qquad ij = 1,2,3 \qquad /5.1/$$

where $\lambda_{ij} = \lambda_{ji}$ are the coefficients of heat conduction.
The stress-strain-temperature relations which consitute a generalization of the Hooke-Duhamel law are [40]

$$\varepsilon_{ij} = a_{ijkl}\sigma_{kl} + \alpha_{ij}T, \qquad i,j,k,l = 1,2,3 \qquad /5.2/$$

The number of eighty one elastic compliance constants is reduced by certain symmetry properties

$$a_{ijkl} = a_{jikl} = a_{ijlk} = a_{klij} \qquad /5.3/$$

so that the final number is 36. These coefficients, however, do not determine directly the material constants, since their values vary with the direction of the coordinate axes. Application of the theory of invariants to the transformation of the above linear forms, and assumption of the existence of a homogeneous quadratic function of potential energy, permits a further reduction of the number of the coefficients a_{ijkl} ; thus, for a body of the most general anisotropy /triclinic structure/ there exist 21 independent coefficients. The quantities α_{ij} entering into /5.2/, called the coefficients of linear thermal expansion, consitute a symmetric tensor, i.e. $\alpha_{ij} = \alpha_{ji}$.

Symmetry implies simpler structures. Thus, for a monoclinic structure, we have 13 coefficients, for orthorhombic 9, for hexagonal 5, for cubic 3 ; finally, for an isotropic material, there exist two independent elastic compliance constants.

Solving /5.2/ with respect to σ_{ij} , we obtain

$$\sigma_{ij} = A_{ijkl}\varepsilon_{kl} + \beta_{ij}T , \quad i,j,k,l = 1,2,3 \qquad /5.4/$$

where

$$A_{ijkl} = A_{ijlk} = A_{jikl} , \qquad \beta_{ij} = \beta_{ji} .$$

The quantities A_{ijkl} are called the stiffness constants . Introducing /5.4/ into the equations of motion

$$\sigma_{ij,j} = \varrho \ddot{u}_i , \qquad /5.5/$$

and expressing the deformations in terms of the displace - ments by means of the relations

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) \qquad /5.6/$$

we obtain after a rearrangement the following displacement equations

$$\frac{1}{2} A_{ijkl}\left(u_{k,l} + u_{l,k}\right) + \beta_{ij}T_{,j} = \varrho \ddot{u}_i \qquad /5.7/$$

$$i,j,k,l = 1,2,3$$

or

$$L_{ij}(u_j) + \beta_{ij}T_{,j} = 0 \qquad i,j = 1,2,3 . \qquad /5.7.1/$$

where L_{ij} are certain linear second-order differential operators of the spatial variables and time.

The solutions of /5.7/ can be represented in the form of a sum where the first component $\bar{u}_i$ satisfies the non-homogeneous system /5.7.1/ and the second component $\bar{\bar{u}}_i$ the homogeneous system

$$L_{ij}(\bar{\bar{u}}_j) = 0 . \qquad /5.8/$$

The $\bar{\bar{u}}_i$ can be represented in terms of three functions χ_i $(i=1,2,3)$ which satisfy the homogeneous equation

$$|L_{ij}|\,\chi_i = 0 \qquad i,j = 1,2,3 \tag{5.9}$$

They can be regarded as Galerkin functions generalized to the case of anisotropic bodies.

Although it is possible to obtain a formal solution of the system of equations /5.1/ ÷ /5.7/ by applying the quadruple Fourier integral transformation, it has not been possible so far to obtain solutions in a form which is suitable for the evaluation of a three-dimensional problem with either general or orthogonal anisotropy /orthotropy/ [41].

The displacement equations for a medium with an arbitrary curvilinear orthotropy have been derived and examined in the paper by J.Nowinski, W.Olszak and W.Urbanowski [42]; these authors solved three examples, the first of which relates to a nonuniformly heated thick-walled cylinder of cylindrical orthotropy, the second to an analogous problem for a disc, and the third to the non-uniform heating of a spherical shell with spherical orthotropy.

Interesting investigations on stress free states in anisotropic bodies have been carried out by W.Olszak [43]. He proved that in bodies which deform freely and possess rectilinear anisotropy only linear temperature distributions will cause no stresses. However, in the case of bodies of curvilinear anisotropy, the compatibility equations represent much stronger limitations than for bodies of rectilinear anisotropy. For instance, for bodies of spherical orthotropy, only a constant distribution of temperature produces no stress, while for bodies of spherical orthotropy any non-vanishing temperature field induces a state of stress.

Among three-dimensional problems, the problems of stationary and quasi-static thermal stresses in bodies of transverse orthotropy have been treated in greatest detail. Thus, B.Sharma [44] investigated thermal stresses due to the heating of a plane bounding an elastic half-space; he arrived at a solution by introduction of two stress functions which satisfy second order differential equation. A different method was devised by Z. Mossakowska and W. Nowacki [45] who introduced three functions to thermoelastic problems in anisotropic bodies. The problem was solved and closed expressions derived for the thermal stresses produced by heat sources in an infinite elastic space and an elastic half-space, for various static and thermal boundary conditions. Similar solutions were given in the case of the effect of nucleus of thermoelastic strain. The cases of stationary heating of an elastic half-space and a layer were also examined. Further, it was proved that the stresses with a vector perpendicular to the plane bounding a semi-space do not vanish, which was the case in the problem of E.Sternberg and E.L.MacDowell [46], but tend to zero as the transverse isotropy passes into isotropy. Finally, solutions were given for a few quasi-static problems relating to the effect of instantaneous heat sources in an elastic space and half-space. A number of problems on axisymmetric thermal stresses in a semi-space of transverse isotropy were solved by A.Singh [47] who employed two displacement functions.

Two-dimensional problems have been dealt with fairly extensively. Thus, W.H.Pell [48] examined the problem of simultaneous bending and compression of an anisotropic plate,produced by a staionary temperature field varying linearly with the thickness of the plate; in particular, he investigated in detail the circular plate.

J.Mossakowski [49] applied the complex variable method to derive a number of solutions for the effect of a heat source in a semi-infinite disc of isogonal anisotropy. It is convenient here to introduce a stress function analogous to the Airy function in isotropic discs [50]. A method of solution for orthotropic discs which uses a function of the Airy type [50] and displacement functions has been presented for static and dynamic problems [51]. Some static and quasi-static problems were solved by C.I.Bors [54] + [55]. The only dynamic problem solved so far is the problem of the determination of the stresses in an anisotropic half-space, when in a plane parallel to the boundary $x_3=0$ there acts a plane non-stationary heat source. The temperature field, the stresses and the displacements depend only on the variables x_3 and t [52], [53].

6. Thermal stresses in isotropic non-homogeneous bodies

Another field of elasticity undergoing a rapid development is by the theory of elasticity of isotropic non-homogeneous bodies. The non-homogeneity is understood in the macroscopic sense. The mechanical quantities - the elasticity moduli E, G Poisson's ratio ν and the density ϱ , are functions of position. In general, they will vary continuously, in the particular case of piecewise variability we are faced with a layered medium.

The first papers dealing with non-homogeneous media were concerned with the propagation of elastic waves in problems of seismology [56] ÷ [61]. /The earth's crust has density and mechanical properties which vary with depth/. The static problems of the elasticity of non-homogeneous media became the subject of investigation by numerous authors [62] ÷ [64] and of a Symposium held by IUTAM in Warsaw in 1958.

Let us consider a non-homogeneous body in which both the mechanical and thermal properties are functions of position but are independent of time and temperature; the variability of these quantities is due to technological processes during manufacture /concrete, iron, casting, etc./. The stress-strain relations have the form

$$\sigma_{ij} = 2\mu\,\varepsilon_{ij} + (\lambda\,\varepsilon_{kk} - \gamma T)\,\delta_{ij} . \qquad /6.1/$$

where the quantities $\mu, \lambda, \gamma = (3\lambda+2\mu)\alpha_t, \alpha_t$ are functions of position, i.e. they depend on the coordinates / x_p / .
Introducing /6.1/ into the equations of motion

$$\sigma_{ij,j} + F_i = \varrho \ddot{u}_i \qquad /6.2/$$

and expressing the deformations by the displacements, we arrive at the displacement equations

$$\begin{gathered}\mu u_{i,kk} + \left[(\lambda+\mu) u_{k,ki}\right]_{,i} - \mu u_{k,ki} - 2 u_{k,k} \mu_{,i} + \\ + (u_{i,k} + u_{k,i}) \mu_{,k} + F_i - (\gamma T)_{,i} = \varrho \ddot{u}_i, \qquad i = 1,2,3,\end{gathered} \qquad /6.3/$$

which can be written in vector form

$$\begin{gathered}\mu \nabla^2 \vec{u} - \mu \operatorname{grad} \operatorname{div} \vec{u} - 2(\operatorname{grad} u)(\operatorname{div} \vec{u}) + \\ + 2(\operatorname{grad} \mu \cdot \varphi) + \operatorname{grad}\left[(\lambda+2\mu)\operatorname{div} \vec{u}\right] + \vec{F} - \operatorname{grad}(\gamma T) = \varrho \ddot{\vec{u}},\end{gathered} \qquad /6.4/$$

where φ is the deformation tensor. The heat conduction equation takes the form

$$(\lambda' T_{,k})_{,k} - c\varrho \dot{T} = - W \qquad /6.5/$$

where λ' denotes the coefficient of heat conduction.

The displacement equations /6.3/, and the heat conduction e uation /6.5/ are linear equations with variable coeffi - cients; the latter is mainly responsible for the mathematical difficulties, whence only in very few cases it is possible to obtain solutions in terms of known functions . Apparently investigations concerning non-homogeneous bo - dies will aim at seeking approximate solutions, on the basis of variational and orthogonalization methods.

It is evident that the general principles of d'Alembert , Hamilton and Castigliano remain valid for non-homogeneous bodies, where the quantities $\mu, \lambda, \alpha_t, \lambda'$ are to be regarded as variable.

In recent years, in view of the wide use of structural elements subject to elevated temperatures, a new type of investigations has arisen, in which the influence of the temperature on the mechanical and thermal properties of a body is taken into account. The body becomes non-homogeneous owing to the presence of the temperature field; hence, its properties depend on the position. In this case the relations /6.1/ are replaced by

$$\sigma_{ij} = 2\mu\varepsilon_{ij} + \left[\lambda\varepsilon_{kk} - (3\lambda+2\mu)\int_0^T \alpha_t(\eta)\,d\eta\right]\delta_{ij}\,, \quad i,j=1,2,3 \qquad /6.6/$$

where the quantities λ, μ, α_t depend on the temperature, i.e.

$$\mu = \mu\left[T(x_i)\right], \qquad \lambda = \lambda\left[T(x_i)\right].$$

Introducing the stresses into the equilibrium equations in the case of the stationary problem we arrive at the system of equations /6.3/ in which the inertia terms have been neglected. Moreover, we have

$$\mu_{,i} = \frac{\partial\mu}{\partial T}\,\frac{\partial T}{\partial x_i}\,, \qquad \lambda_{,i} = \frac{\partial\lambda}{\partial T}\,\frac{\partial T}{\partial x_i}\,,$$

and

$$\gamma T = (3\lambda+2\mu)\int_0^T \alpha_t(\eta)\,d\eta\,.$$

Finally the equation of heat conduction is non-linear

$$\left[\lambda'(T)T_{,k}\right]_{,k} = -W \;. \qquad /6.7/$$

Note that, if we introduce the auxiliary function

$$G(T) = \frac{1}{\lambda'_0}\int_0^T \lambda'(\tau)\,d\tau = G\left[T(x_r)\right] \;, \qquad /6.8/$$

we can transform /4.8/ to the form

$$\nabla^2 G(x_r) = -\frac{1}{\lambda'_0} W(x_r), \qquad \lambda'_0 = \text{const.} \qquad /6.9/$$

Solving this equation we obtain the temperature field in the implicite form /6.8/.

In the case under consideration the principal difficulty constists in the solution of the heat conduction equation /6.7/. Knowing the temperature field and the dependence of the coefficients μ, λ, α_t , on the temperature and hence also on the position, we have to solve the displacement equations which are linear partial differential equations with variable coefficients. In view of the existing difficulties in solving these equations it is necessary to introduce a number of simplifications. For instance, it may be assumed that Poisson's ratio is constant, or one may set $\nu = 1/2$ i.e. regard the body as incompressible. A further simplification replaces the thermal extension $\varepsilon^0 = \int_0^T \alpha_t(\eta)\,d\eta$ by its mean value $\varepsilon^0 = \alpha_t^* T$.

So far, only a few one-dimensional cases have been solved [65] ÷ [70] , which concern mainly the states of stress in a circular disc, in a hollow cylinder and hollow sphere.

Under the assumption of the independence of ν on the temperature and the mean value of the thermal expansion we have the following displacement equations:

a/ for a plane state of stress and axisymmetric temperature field

$$\frac{\partial}{\partial r}\left\{E\left[\frac{\partial u_r}{\partial r}+\frac{1}{r}u_r-(1+\nu)\alpha_t^* T\right]\right\}=(1-\nu)\frac{u_r}{r}\frac{\partial E}{\partial r}, \quad /6.10/$$

b/ for a plane state of strain

$$\frac{\partial}{\partial r}\left\{E\left[\frac{\partial u_r}{\partial r}+\frac{1}{r}u_r-\frac{1-\nu}{1+\nu}\alpha_t^* T\right]\right\}=\frac{1-2\nu}{1-\nu}\frac{u_r}{r}\frac{\partial E}{\partial r}, \quad /6.11/$$

c/ for spherically symmetrical problems

$$\frac{\partial}{\partial R}\left\{E\left[\frac{\partial u_R}{\partial R}+\frac{2u_R}{R}-\frac{1+\nu}{1-\nu}\alpha_t^* T\right]\right\}=\frac{2(1-2\nu)}{1-\nu}\frac{u_R}{R}\frac{\partial E}{\partial R}. \quad /6.12/$$

It follows from the above equations that the assumption $\nu=1/2$ leads to a considerable simplification, because in this case the right-hand sides of /6.11/ and /6.12/ vanish. This fact, obviously, greatly simplifies the solution of the equations.

J.Nowiński [66] examined the thermal stresses in a thick-walled cylinder under the assumption that $E(T)=E_0 e^{-\beta T}$ or $E(r)=E_0 \varkappa r$. S.A.Shesterikov making the same assumption investigated thermal stresses in a disc. In another paper J.Nowiński [71] considered the state of stress in a full sphere and in a hollow sphere in which $\nu = 1/2$ and $\varepsilon^{t} = \int_{0}^{T} \alpha(\eta)\, d\eta$; he obtained the solution in a closed form.

It follows from the numerical example in which α_t, E, λ vary linearly with T that the thermal stresses $\sigma_{RR}, \sigma_{\varphi\varphi}$ are nearly equal to the stresses $\sigma^{*}_{RR}, \sigma^{*}_{\varphi\varphi}$ determined on the basis of the mean values of α varying along the thickness of the sphere.

R.Trostel in the above quoted papers assumes that $\nu = 1/2$; hence it is possible to solve /6.11/ exactly. In his next paper, he presents a perturbation method for the solution of the displacement equations, making the assumption that the modulus E varies slowly with T,

$$\frac{1}{E}\,\frac{dE}{dT} = \frac{d}{dT}\left(\log\frac{E}{E_0}\right) = \varepsilon\,\Phi(T)$$

where ε is a small parameter. Using the perturbation method, R.Trostel solved the problem of thermal stresses in a thick-walled pipe in which $\nu = \text{const} \neq 1/2$ and $\lambda(T)$, $\alpha_t(T)$ are linear functions, while $E(T)$ is a quadratic function of the temperature. M.Sokołowski [70] investigated the thermal stresses in an infinite cylinder and in a phere due to heating of the outer surface and heat sources, assuming that ν = const. and that the derivatives of E with respect to the radius are so small that it is admissible to disregard the right-hand sides of equations /6.10/ + /6.12/. The attention is focused on considering the variability of the conduction coefficient $\lambda'(T)$ and on establishing rules of behaviour of the thermal stresses in terms of the nature of variability of $\lambda'(T)$ and the direction of the heat flow.

There is no difficulty in extending the d'Alembert and Hamilton principles to bodies in which the non-homogeneity is due to elevated temperatures; it should only be borne in mind that the quantities μ, λ depend on the temperature.

The Betti-Rayleigh reciprocal theorem has been generalized to the case of elastic bodies with temperature dependent properties by J.Nowiński [72]. In order to apply this theorem to thermoelastic problems involving temperature-dependent properties, a related elastic problem for an "inherently" non-homogeneous body must first be solved.

Betti's reciprocal theorem has the form

$$\int_V F'_i u_i \, dV + \int_\Gamma p'_i u_i \, d\Gamma = \qquad /6.13/$$

$$= \int_V \left(\int_0 \alpha_t(\tau) \, d\tau \right) \Lambda'^{(i)} \, dV + \int_V F_i u'_i \, dV + \int_\Gamma p_i u'_i \, d\Gamma ,$$

where the displacements u'_i are produced by the forces p'_i, F'_i in a non-homogeneous body in which $T=0$, while the displacements u_i are due to the action of the forces p_i, F_i and the temperature field in the same non-homogeneous body, $\Lambda'^{(i)}$ denotes the sum of normal stresses produced by the forces p'_i, F'_i. For instance the thermoelastic displacement in a body V is given by the general formula

$$u_i(x_r) = \int_V dV \int_0 \alpha_t(\tau)\, d\tau\, \Lambda'$$

in which the sum of the stresses $\Lambda'(x_r, \xi_r)$ has to be found from the solution of the classical problem for a non-homogeneous body subject to a concentrated uniform force at the point ξ_r of V in the direction of the x_i axis.

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