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PROBLEMS OF THERMOELASTICITY

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DYNAMICAL PROBLEMS OF THERMOELASTICITY

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We known from experiment that deformation of a body is associated with a change of heat content in it. The time varying loading of a body causes in it not only displacements but also temperature distribution changing in time. Conversely, the heating of a body produces in it deformation and temperature change. The motion of a body is characterized by mutual interaction between deformation and temperature fields. The domain of science dealing with the mutual interaction of these fields is called the thermoelasticity.

Owing to the coupling between these fields, the temperature terms appear in the displacement equations of motion, whereas the deformation terms - in the equation of thermal conductivity.

The coupling between deformation and temperature fields was first postulated by J.M.C.Duhamel, the originator of the theory of thermal stresses who has introduced the dilatation term in the equation of thermal conductivity. However, this equation was not well grounded in the thermodynamical way. Then, the attempt of the thermodynamical justification of this equation was undertaken by W.Voigt [2] and H. Jeffreys [3]. However, just lately as in 1956, M.A.Biot [4] gave the full justification of the thermal conductivity equation on the foundation of thermodynamics of irreversible processes [5] . M.A.Biot also presented the fundamental methods for solving the thermoelasticity equation as well as the variational theorem.

The thermoelasticity describe a broad range of phenomena, it is the generalization of the classical theory of elasticity and of the theory of thermal conductivity. Now, the thermoelasticity is a domain of science fully formed. The fundamental relations and differential equations have been formulated. A number of methods for solving the thermoelasticity equations and basic energy and variational theorems have been developed. Scientific workers have solved some problems concerning the propagation of thermoelastic waves.

As it is known, the research work in the field of thermoelasticity was preceded by wide investigations in the framework of the so-callod theory of thermal stresses. Under this name we mean the investigation of strains and stresses produced by heating a body with the simplifying assumption that thermal conductivity does not affect the deformation of an elastic body.

In this theory originating from the beginnings of the elasticity theory and recently being intensively developed owing to its growing practical significance, the classical equation of thermal conductivity, not containing the term associated with the body deformation, has been used.

The elastokinetics has been used simultaneusly with the theory of thermal stresses. In this case, the simplifying assumption has been introduced which postulates that the heat exchange among body parts, caused by heat conductivity, is so slow that the motion can be thought of as an adiabatic process.

The domains mentioned here constitute now the particular case of the more general theory, namely, of elasticity. The particular theorems and methods of the theory of thermal conductivity and of the classical theory of elasticity are comprised in general theorems and methods of thermoelasticity.

It should be noted that solutions obtained within framework of the thermoelasticity differ slightly from solutions of the classical theory of elasticity or the theory of thermal conductivity. The coupling between the deformation and temperature field is weak. But the qualitative differences are fundamental. This is seen, even if, on the examples of elastic waves which within the framework of thermoelasticity are damped and dispersed, whereas in the framework of elastokinetics, only undamped waves appear. The thermoelasticity is of fundamental significance in those cases in which the investigation of elastic dissipation is a main aim. The meaning of thermoelasticity consists chifly in cognizing and generalizing value of this theory.

In the present paper of survey character, the attention is focused at foundations of thermodynamical theories, at the differential equations of thermoelasticity and more important methods for solving them and at general energy and variational theorems.

Smaller attention is devoted to solving concrete problems and the reader is referred to literature enlisted at the end of the work. When writing function relations and equations, we shall apply the index tensor notation in the Cartesian system of coordinates.

2. Fundamental assumptions and relations of the linear thermoelasticity

In the present section, we shall consider homogeneous anisotropic elastic bodies. For these bodies, we shall derive general relations and extended equations of thermal conductivity, and after that, we shall deal with a homogeneous isotropic body which will the subject of the further sections of the present work.

Let a body be in the temperature T_0 in an undeformed and unstressed state /with absence of external forces /. This starting state will be called the natural state, assuming that entropy equals to zero for this body. Owing to the action of external forces, i.e. body and surface forces, and under the influence of heat sources and heating /or cooling/ the body surface, the medium will be subjected to deformation and temperature change. The displacements u will appear in the body and the temperature change can be written as $\theta = T - T_0$, where T is the absolute temperature of a point x of the body. The temperature change is accompanied by arising stresses ε_{ij} and strains σ_{ij} . The quantities u , θ , ε_{ij} , σ_{ij} are the functions of position x and time t .

We assume that the temperature change $\theta = T - T_0$ accompanying deformation is small and increase in the temperature does not result in essential variations of material coefficients both elastic and thermal. These coefficients will regarded as independent of T .

To the introduced assumption $|\theta / T_0| \ll 1$ let us add others concerning small strains. Namely, we assume that second powers and products of the components of strains may be neglected as small quantities compared with the strains ε_{ij} . Thus, we restrict further considerations to the thermoelasticity geometrically linear. The dependency among strains and displacements is confined to the linear relation.

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right), \qquad i,j = 1,2,3. \qquad /2.1/$$

As it is known, strains can not be arbitrary functions, they must satisfy six relations, the so-called relations of geometrical inseparability

$$\varepsilon_{ij,kl} + \varepsilon_{kl,ij} - \varepsilon_{jl,ik} - \varepsilon_{ik,jl} = 0, \qquad i,j,k,l = 1,2,3. \qquad /2.2/$$

The main task becomes the obtaining of state equations relating the components of stress tensor σ_{ij} with the components of strain tensor ε_{ij} and of temperature θ

Let us note that the mechanical and thermal state of the medium is, at a given time instant, completely described by the distribution of stresses ε_{ij} and temperature θ. We hence conclude that for the isothermal change of state $(T=T_0)$, we encounter the process elastically and thermodynamically reversible. However, in processes in which the temperature changes take place, we observe two interrelated phenomena, namely, the reversible elastic process and the irreversible thermodynamical process. The latter is caused by spontaneous and thereby irreversible process of carrying the heat by means of thermal conductivity.

Thermoelastic disturbances can not be described with the help of classical themodynamics and we have to use the relations of the thermodynamics of irreversible processes [5], [6].

To obtain the state equations, we should consider the energy of the system. We start from the differential relation originating from the first principle of thermodynamics

$$du=\sigma_{ij}\,d\varepsilon_{ij}+dQ \qquad /2.3/$$

The relation shows that a small change du in the internal energy is equal to a sum of strain work and an increment of heat amount introduced to the considered infinitely small volume of the body. A change of heat quantity equals to Tds, where s is an entropy, and thus Eq. [2.3] takes the form

$$du = \sigma_{ij}\, d\varepsilon_{ij} - T ds. \qquad /2.3'/$$

It should be added that the increment of the internal energy u is a total differential. The independent variables are in /2.3'/ strains ε_{ij} and entropy s, so that $u = u(\varepsilon_{ij}, s)$.

It is more convenient to replace the function u by the free energy $f = u - sT$, as a function of variables ε_{ij} and T

$$df = \sigma_{ij}\, d\varepsilon_{ij} - s dT. \qquad /2.4/$$

The df is a total differential, too.

The relations /2.3/ and /2.4/ permit to determine stresses as the function of independent variables ε_{ij} and s or ε_{ij} and T. Taking into account that

$$du = \left(\frac{\partial u}{\partial \varepsilon_{ij}}\right)_s d\varepsilon_{ij} + \left(\frac{\partial u}{\partial s}\right)_\varepsilon ds, \qquad /2.5/$$

$$df = \left(\frac{\partial f}{\partial \varepsilon_{ij}}\right)_T d\varepsilon_{ij} + \left(\frac{\partial f}{\partial T}\right)_\varepsilon dT, \qquad /2.6/$$

we obtain, from equating /2.3'/ with /2.5/ and /2.4/ with /2.6/, the following expressions.

$$\sigma_{ij} = \left(\frac{\partial u}{\partial \varepsilon_{ij}}\right)_s, \quad T = \left(\frac{\partial u}{\partial s}\right)_\varepsilon, \quad \sigma_{ij} = \left(\frac{\partial f}{\partial \varepsilon_{ij}}\right)_T, \quad s = -\left(\frac{\partial f}{\partial T}\right)_\varepsilon. \qquad /2.7/$$

In further considerations, we shall make use of the third equation of /2.7/, aiming at presenting the stresses σ_{ij}

as the function of strains ε_{ij} and of T .

Let us expand the function $f(\varepsilon_{ij},T)$ into an infinite series in the neighbourhood of the natural state $f(0,T_0)$:

$$f(\varepsilon_{ij},T)=f(0,T_0)+\frac{\partial f(0,T_0)}{\partial\varepsilon_{ij}}\varepsilon_{ij}+\frac{\partial f(0,T_0)}{\partial T}(T-T_0)+ \qquad /2.8/$$

$$+\frac{1}{2}\left[\frac{\partial^2 f(0,T_0)}{\partial\varepsilon_{ij}\partial\varepsilon_{kl}}\varepsilon_{ij}\varepsilon_{kl}+2\frac{\partial^2 f(0,T_0)}{\partial\varepsilon_{ij}\partial T}\varepsilon_{ij}(T-T_0)+\frac{\partial^2 f(0,T_0)}{\partial T^2}(T-T_0)^2\right]+\dots$$

From the expansion of $f(\varepsilon_{ij},T)$ we retain only the linear and quadratic terms, confining ourselves to only linear relations among stresses σ_{ij} , strains and temperature change θ.

Taking into account that for $\varepsilon_{ij}=0$, $T=T_0$, we consider the natural state, it can be assumed that $f(0,T_0)=0$ The term $\partial f(0,T_0)/\partial T$ will be also equated to zero. Since it results from equating Eqs. /2.4/ and /2.6/ that $(\partial f/\partial T)_\varepsilon=-s$ therefore, for the natural state, there is

$$\frac{\partial f(0,T_0)}{\partial T}=-s(0,T_0)=0.$$

Let us now take advantage of the third one of expressions /2.7/

$$\sigma_{ij}(\varepsilon_{ij},T)=\left(\frac{\partial f}{\partial\varepsilon_{ij}}\right)_T=\frac{\partial f(0,T_0)}{\partial\varepsilon_{ij}}+\frac{\partial^2 f(0,T_0)}{\partial\varepsilon_{ij}\partial\varepsilon_{kl}}\varepsilon_{kl}+\frac{\partial^2 f(0,T_0)}{\partial\varepsilon_{ij}\partial T}(T-T_0). \qquad /2.9/$$

Thus we have obtained the linear relation for small strains which agrees with the introduced assumption $|\theta/T_0| \ll 1$. It should be put $\partial f(0,T_0)/\partial\varepsilon_{ij}=0$ in Eq. /2.9/ since, for the natural state $\varepsilon_{ij}=0$, $T=T_0$, it should be $\sigma_{ij}=0$. Introducing denotation

$$\frac{\partial^2 f(0,T_0)}{\partial\varepsilon_{ij}\partial\varepsilon_{kl}}=c_{ijkl}, \qquad \frac{\partial^2 f(0,T_0)}{\partial\varepsilon_{ij}\partial T}=-\beta_{ij}, \qquad \frac{\partial^2 f(0,T_0)}{\partial T^2}=n$$

we present the relations /2.8/ and /2.9/ in the form

$$f(\varepsilon_{ij},T)=\frac{1}{2}c_{ijkl}\varepsilon_{ij}\varepsilon_{kl}-\beta_{ij}\varepsilon_{ij}\theta+\frac{n}{2}\theta, \qquad /2.10/$$

$$\sigma_{ij}=\frac{1}{2}(c_{ijkl}+c_{klij})\varepsilon_{kl}-\beta_{ij}\theta. \qquad /2.11/$$

Let us note additionally that

$$\left(\frac{\partial\sigma_{ij}}{\partial\varepsilon_{kl}}\right)_T=c_{ijkl}, \qquad \left(\frac{\partial\sigma_{ij}}{\partial T}\right)_\varepsilon=-\beta_{ij}. \qquad /2.12/$$

In the relations /2.11/, we recognize the Hooke's law generalized for thermoelastic problems. The /2.11/ are called the Duhamel-Neumann relations for an anisotropic body. The constants c_{ijkl}, β_{ij}, concerning the isothermal state play the role of material constants [7]. The quantities c_{ijkl} are the components of the tensor of elastic stiffness.

In the elasticity theory of an anisotropic body, the following symmetry properties of tensor are proved

$$c_{ijkl} = c_{jikl}, \qquad c_{ijkl} = c_{ijlk}, \qquad c_{ijkl} = c_{klij}. \qquad /2.13/$$

These relations lead to reduction of quantity of constants from 81 to 21 of mutually independent constants for a body with general anisotropy.

Let us solve the systems of equations /2.11/ for deformations

$$\varepsilon_{ij} = s_{ijkl}\,\sigma_{kl} + a_{ij}\theta. \qquad /2.14/$$

The quantities s_{ijkl} are called the coefficients of elastic susceptibility. Also for these quantities, the symmetry relations hold

$$s_{ijkl} = s_{jikl}, \qquad s_{ijkl} = s_{ijlk}, \qquad s_{ijkl} = s_{klij}.$$

Let us now consider a volume element of the anisotropic body free of stresses on its surface. Then, according to /2.14/, we obtain for this element

$$\mathring{\varepsilon}_{ij} = a_{ij}\theta. \qquad /2.15/$$

The relation /2.15/ describes the familiar physical phenomenon, namely, the proportionality of the element deformations to the increment of temperature θ . The quantities a_{ij} are the coefficients of linear thermal expansion. The a_{ij} is a symmetric tensor what follows from the symmetry of the tensor ε_{ij} . It should added that the coefficient of volume thermal expansion a_{jj} is an invariant.

From the relations /2.11/ and /2.14/ we get the following expressions

$$\left(\frac{\partial \sigma_{ij}}{\partial \varepsilon_{kl}}\right)_T = c_{ijkl}, \quad \left(\frac{\partial \sigma}{\partial T}\right)_s = -\beta_{ij} = -a_{kl} c_{ijkl}, \left(\frac{\partial \varepsilon_{ij}}{\partial T}\right)_\sigma = a_{ij}. \qquad /2.16/$$

In the further considerations concerning derivation of the extended equation of thermal conductivity it will be necessary to present the internal energy and entropy as a function of deformation and temperature. The starting point are the total differentials

$$du = \sigma_{ij} d\varepsilon_{ij} + T ds, \qquad /2.17/$$

$$ds = \left(\frac{\partial s}{\partial \varepsilon_{ij}}\right)_T d\varepsilon_{ij} + \left(\frac{\partial s}{\partial T}\right)_\varepsilon dT. \qquad /2.18/$$

Inserting /2.18/ into /2.17/, we obtain

$$du = \left[T\left(\frac{\partial s}{\partial \varepsilon_{ij}}\right)_T + \sigma_{ij}\right] d\varepsilon_{ij} + T\left(\frac{\partial s}{\partial T}\right)_\varepsilon dT. \qquad /2.19/$$

The necessary and sufficient condition in order the quantity du to be a total differential is

$$\frac{\partial}{\partial T}\left[T\left(\frac{\partial s}{\partial \varepsilon_{ij}}\right)_T + \sigma_{ij}\right] = \frac{\partial}{\partial \varepsilon_{ij}}\left[T\left(\frac{\partial s}{\partial T}\right)_s\right].$$

From this condition, the relation results

$$\left(\frac{\partial s}{\partial \varepsilon_{ij}}\right)_T + \left(\frac{\partial \sigma_{ij}}{\partial T}\right)_\varepsilon = 0$$

or taking into account the second term in the group /2.16/

$$\left(\frac{\partial s}{\partial \varepsilon_{ij}}\right)_T = \beta_{ij}. \qquad /2.20/$$

On the other hand, we utilize the thermodynamical relation

$$T\left(\frac{\partial s}{\partial T}\right)_\varepsilon = \left(\frac{\partial u}{\partial T}\right)_\varepsilon = c_\varepsilon , \qquad /2.21/$$

where c_ε is a specific heat related to unit volume at constant deformation. Substituting /2.20/ and /2.21/ into /2.18/ and /2.19/, we obtain

$$ds = \beta_{ij} d\varepsilon_{ij} + \frac{c_\varepsilon}{T} dT, \qquad /2.22/$$

$$du = \sigma_{ij} d\varepsilon_{ij} + T\beta_{ij} d\varepsilon_{ij} + c_\varepsilon dT. \qquad /2.23/$$

Inserting the relations /2.11/ into /2.23/ and integrating the expressions /2.20/ and /2.23/ with the assumption that for the natural state $(T=T_0, \varepsilon_{ij}=0, \sigma_{ij}=0)$ there is $s=0$, $u=0$, we have

$$s = \beta_{ij}\varepsilon_{ij} + c_\varepsilon \ln\left(1+\frac{\theta}{T_0}\right), \qquad /2.24/$$

$$u = \frac{1}{2} c_{ijkl}\varepsilon_{ij}\varepsilon_{kl} + T_0\beta_{ij}\varepsilon_{ij} + c_\varepsilon\theta. \qquad /2.25/$$

In the expression for entropy, the first term on the right-hand side is due the coupling of deformation field with temperature field, the second term expresses the entropy caused by the heat flow. The purely elastic term does not appear in this expression. It hence results that deformation process is, in the isothermal conditions, reversible and does not cause increament in the entropy. In the expression /2.25/ for the internal energy, three terms appear. The first of them is of purely elastic character, it represents the strain work, the second - heat content in a unit volume, the middle term is a result of mutual interaction between deformation and temperature fields. For the particular case of isothermal process, there is $u=\varepsilon_{ij}\sigma_{ij}/2+T_0\beta_{ij}\varepsilon_{ij}$.

Let us return to the expression /2.24/. In virtue of the introduced assumption $|\theta/T_0|\ll 1$, the function $\ln(1+\theta/T_0)$ can be expanded into an infinite series and only one term of expansion can be taken into account. Thus, we obtain

$$s=\beta_{ij}\varepsilon_{ij}+\frac{c_\varepsilon}{T_0}\theta. \qquad /2.26/$$

For the internal energy $f=u-sT$, we have

$$f\approx\frac{1}{2}c_{ijkl}\varepsilon_{ij}\varepsilon_{kl}-\beta_{ij}\varepsilon_{ij}\theta-\frac{c_\varepsilon}{2T_0}\theta^2 \qquad /2.27/$$

In this way, we have determined the $n=-c_\varepsilon/T_0$, involved in /2.10/.

It remains to interrelate the entropy with the thermal conductivity. In a solid body, the heat transfer is realized through the thermal conductivity meant as a transfer of heat from spots with higher temperature to those with lower one. The equation of thermal conductivity is derived from the principle of energy conservation expressed in the form of entropy flow. This law constituting the local formulation of the second principle of thermodynamics can be written in the form

$$T\frac{ds}{dt} = -\operatorname{div} q, \qquad \frac{ds}{dt} = -\frac{1}{T} q_{i,i}. \qquad /2.28/$$

By q let us denote the vector of the energy flow which in our case, is equal to the heat flow.

Let us consider a body containing the region V enveloped by the surface A . Then the integral

$$\frac{dS}{dt} = \int_V \frac{ds}{dt} dV = -\int_V \frac{q_{i,i}}{T} dV \qquad /2.29/$$

denotes the entropy increment in the volume V in a time unit which is caused by the heat flow.

The relation /2.29/ can also be presented in the form

$$\frac{dS}{dt} = -\int_A \frac{q_i n_i}{T} dA - \int_V \frac{q_i T_{,i}}{T^2} dV. \qquad /2.29'/$$

The entropy increment in time consists here of two main parts viz., of the surface integral expressing increment /positive or negative/ of entropy due to the heat exchange with enviroment, and of the integral associated with the generating of entropy in the region V.

Let us return to the relation /2.28/ which can be presented in the form

$$\frac{ds}{dt} = -\left(\frac{q_i}{T}\right)_{,i} - \frac{q_i T_{,i}}{T^2} . \qquad /2.30/$$

This expression describes, in a local manner, the entropy increment in time.

It is seen from comparing /2.29'/ and /2.30/ that the first term in /2.30/ concerns the exchange of entropy with environment, the second term is related to generating entropy in an elementary volume of the body. The relation /2.30/ can be presented in the form

$$\frac{ds}{dt} = -\operatorname{div}\left(\frac{q}{T}\right) + \sigma, \qquad /2.30'/$$

where $\sigma = -q_i T_{,i}/T^2$ is a source of entropy.
By ds_e/dt let us denote the exchange of entropy with environment, by ds_i/dt the rate of generating entropy.
Then [5]

$$\frac{ds_e}{dt} = -\operatorname{div}\left(\frac{q}{T}\right) = -\left(\frac{q_i}{T}\right)_{,i}, \quad \frac{ds_i}{dt} = -\frac{q_i T_{,i}}{T^2} = \sigma. \qquad /2.31/$$

The local formulation of the second principle of thermodynamics of irreversible processes requires in order to be in each element of the body

$$\frac{ds}{dt} = \frac{ds_e}{dt} + \frac{ds_i}{dt} > 0 , \qquad \frac{ds_i}{dt} = \sigma > 0.$$

The entropy source σ is in an irreversible process always and everywhere greater than zero, in an reversible it is equal to zero. We shall utilize this theorem in further considerations.

The source of entropy is associated with origins of irreversible processes, with the so-called intensive quantities or thermodynamical stimuli F_i . This interconnection can be written as

$$\sigma = F_i q_i . \qquad /2.32/$$

The entropy source is equal to a sum of products of thermodynamical stimuli and components of heat flow coupled with them. From equating /2.31/ and /2.32/ it is seen that

$$F_i = -\frac{T_{,i}}{T^2} . \qquad /2.33/$$

Then the temperature gradient is a thermodynamical stimuli for the thermal conductivity.

On the other hand, among the components of heat flow vector q and thermodynamical stimuli there exists the function relation

$$q_i = q_i(F_1, F_2, F_3). \qquad /2.34/$$

For laminar flows which will be considered here, we can assume that the relation /2.34/ is linear i.e. that

$$q_i = L_{ij} F_j . \qquad /2.35/$$

These are the phenomenological equations of the energy flow. The quantities L_{ij} appearing in them are the constants satisfying the Onsager's relations

$$L_{ij} = L_{ji} . \qquad /2.36/$$

Substituting /2.33/ into /2.35/, we obtain

$$q_i = -\frac{L_{ij} T_{,i}}{T^2} . \qquad /2.37/$$

This equation agrees with the Fourier's law for the thermal conductivity in an anisotropic body. For the entropy source, we get

$$\sigma = L_{ij} \frac{T_{,i} T_{,j}}{T^4} > 0. \qquad /2.38/$$

Since it always must be $\sigma > 0$, then the quantities L_{ij} must be positive. Introducing the quantities $\lambda_{ij} = L_{ij}/T^2 > 0$ /coefficients of thermal conductivity/, we obtain the following law of the heat flow in an anisotropic medium

$$q_i = -\lambda_{ij} T_{,j} . \qquad /2.39/$$

Combining the relations /2.39/ and /2.28/ and differentiating the /2.26/ with respect to time, we arrive at the equations

$$T \frac{ds}{dt} = \lambda_{ij} T_{,ij} , \qquad /2.40/$$

$$T \frac{ds}{dt} = T \beta_{ij} \frac{d\varepsilon_{ij}}{dt} + \frac{c_\varepsilon}{T_0} T \frac{d\theta}{dt} . \qquad /2.41/$$

Equating these equations yields the equation of thermal conductivity

$$\lambda_{ij} T_{,ij} = T\beta_{ij} \frac{d\varepsilon_{ij}}{dt} + \frac{c_\varepsilon}{T_0} T \frac{d\theta}{dt}, \quad \theta = T - T_0 . \qquad /2.42/$$

Let us note that this is a nonlinear equation on account of its right-hand side. Putting $T = T_0$ in the right-hand side of /2.42/, we linearize this equation. Finally, we obtain

$$\lambda_{ij} \theta_{,ij} - c_\varepsilon \dot{\theta} - T_0 \beta_{ij} \dot{\varepsilon}_{ij} = 0. \qquad /2.42'/$$

In this extended equation of thermal conductivity, the term $T_0 \beta_{ij} \dot{\varepsilon}_{ij}$, appears which characterizes the coupling of deformation field with temperature field. The dot above the function denotes the derivative of this function with respect to time. If, sources of heat act in the body, we should add to /2.40/ the quantity, which determines the amount of heat produced in a unit of volume and time

$$T \frac{ds}{dt} = \lambda_{ij} T_{,ij} + W.$$

Eq. /2.42'/, in the case of the appearance of heat source in the body, is extended to the form

$$\lambda_{ij} \theta_{,ij} - c_\varepsilon \dot{\theta} - T_0 \beta_{ij} \dot{\varepsilon}_{ij} = -W. \qquad /2.42''/$$

On the foundation of Duhamel-Neumann relations derived for an anisotropic body, we shall easily go on to anisotropic body applying the following relation

$$c_{ijkl} = \mu\,[\delta_{ij}\,\delta_{jl} + \delta_{il}\,\delta_{jk}] + \lambda\,\delta_{ij}\,\delta_{kl},$$
$$s_{ijkl} = \mu'\,[\delta_{ik}\,\delta_{jl} + \delta_{il}\,\delta_{jk}] + \lambda'\,\delta_{ij}\,\delta_{kl},$$
$$\beta_{ij} = \gamma\,\delta_{ij}, \qquad \alpha_{ij} = \alpha_t\,\delta_{ij} \tag{2.43}$$

Here, μ, λ are Lamé's constants for an isothermal state, and

$$\gamma = (3\lambda + 2\mu)\alpha_t, \quad \mu' = \frac{1}{4\mu},$$
$$\lambda' = -\frac{\lambda}{2\mu(3\lambda + 2\mu)}$$

The quantity α_t is the coefficient of linear thermal expansion. In this way, the relations /2.11/ and /2.14/ transform in the Duhamel-Neumann relations for an isotropic body

$$\sigma_{ij} = 2\mu\,\varepsilon_{ij} + (\lambda\,\varepsilon_{kk} - \gamma\,\theta)\delta_{ij}, \tag{2.44}$$

$$\varepsilon_{ij} = \alpha_t\,\theta\,\delta_{ij} + 2\mu'\,\sigma_{ij} + \lambda'\,\delta_{ij}\,\sigma_{kk}, \tag{2.45}$$

For an isotropic $\lambda_{ij} = \lambda_0\,\delta_{ij}$. Thus, the equation of thermal conductivity /2.42''/ assumes the form / 4 / :

$$\lambda_0\,\theta_{,jj} - c_\varepsilon\,\dot{\theta} - T_0\,\gamma\,\dot{\varepsilon}_{kk} = -W$$

or

$$\theta_{,jj} - \frac{1}{\varkappa}\,\dot{\theta} - \eta\,\dot{\varepsilon}_{kk} = -\frac{Q}{\varkappa}, \tag{2.46}$$

where

$$\varkappa = \frac{\lambda}{c_t}, \quad \eta = \frac{\beta T_0}{\lambda}, \quad \delta = \frac{l}{}$$

Let us give, moreover, the expressions of u, f, s for an anisotropic body. We obtain

$$u = \frac{1}{2}\sigma_{ij}\varepsilon_{ij} + \frac{1}{2}\beta\varepsilon_{kk}(\theta + 2T_0) + c_t\theta,$$

$$f = \mu\varepsilon_{ij}\varepsilon_{ij} + \frac{\lambda}{2}e^2 - \beta e\theta - \frac{c_t}{2T_0}\theta^2, \qquad /2.47/$$

$$s = \beta e + c_t\frac{\theta}{T_0}, \quad e = \varepsilon_{kk}.$$

The state equation and the equations of thermal conductivity derived in this section should be joined with the equations of motion of a solid deformable body. In this way, we shall obtain a full set of thermoelasticity equations.

Let us attract attention to the fact the coupling of temperature and deformation fields vanishes when external forces or heating the body is stationary. In this case, the time derivatives disappear in the equation of thermal conductivity, Eq. /2.46/ transforms into Poisson's equation.

3. Differential equations of thermoelasticity and methods for solving them

The full set of the differential equations of thermoelasticity is composed of the equations of motion and the equations of thermal conductivity. The equations of motion

$$\sigma_{ij,j} + X_i = \rho \ddot{u}_i(x,t), \quad x \in V,\ t > 0, \qquad /3.1/$$

can be transformed, making use of the state equations

$$\sigma_{ij} = 2\mu \varepsilon_{ij} + (\lambda \varepsilon_{kk} - \gamma \theta)\delta_{ij}, \quad x \in V + \Sigma,\ t > 0, \qquad /3.2/$$

and of the relations among displacements and deformations

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right), \quad x \in V + \Sigma,\ t > 0, \qquad /3.3/$$

into the three equations containing displacements u_i and temperature θ as unknown functions

$$\mu u_{i,jj} + (\lambda + \mu) u_{j,ji} + X_i = \rho \ddot{u}_i + \gamma \theta_{,i}, \quad x \in V,\ t > 0 \qquad /3.4/$$

The above equations and those of thermal conductivity

$$\theta_{,ii} - \frac{1}{\varkappa}\dot{\theta} - \eta \dot{u}_{k,k} = -\frac{Q}{\varkappa}, \quad x \in V,\ t > 0 \qquad /3.5/$$

are coupled each other. Body forces, heat sources, heating and heat flow through the surface enveloping the region and initial conditions are the causes of arising both displacements and temperature accompanying them in a body.

Boundary conditions of a mechanical type are given in the form of either given displacements u_i or loadings $p_i = \sigma_{ji} n_j$ on the surface Σ. Thermal conditions can be, in a general way, written in the form

$$\alpha \frac{\partial\theta}{\partial n} + \beta\theta = f(x,t), \quad x \in \Sigma, t > 0, \quad \alpha, \beta = \text{const.}, \qquad /3.6/$$

determining the heat flow through the surface Σ. If $\beta = \infty$, then the temperature θ on boundary is equal to zero; if $\alpha = \infty$, then we have the case of the surface Σ thermally isolated. The initial conditions hints that at an initial time instant, e.g. for $t = 0$, displacement u_i, velocity of these displacements and temperature are the known functions

$$u_i(x,t)_{t=0} = f_i(x), \quad \dot{u}_i(x,t)_{t=0} = g_i(x), \quad \theta(x,t)_{t=0} = h(x) \qquad /3.7/$$

The system of Eqs /3.4/ and /3.5/ is greatly complicated and the tendency is obvious to lead this system to a system of simpler equations viz., wave equations. The essential simplification is obtained by decomposition of the displacement vector and the vector of body forces into potential part and solenoidal part. Substituting then, into Eq. /3.4/ and /3.5/, the formulae

$$u_i = \Phi_{,i} + \epsilon_{ijk} \Psi_{k,j}, \quad X_i = \rho(\vartheta_{,i} + \epsilon_{ijk} \chi_{k,j}), \qquad /3.8/$$

where φ i ϑ are the scalar functions, whereas ψ_i and χ_i vector functions, we lead the thermoelasticity equations to the following system of equations / 8 /

$$\Box_1^2 \varphi - m\theta = -\frac{1}{c_1^2}\vartheta, \qquad /3.9/$$

$$\Box_2^2 \psi_i = -\frac{1}{c_2^2}\chi_i, \qquad /3.10/$$

$$D\theta - \eta\nabla^2\dot{\varphi} = -\frac{Q}{\varkappa}, \quad c_1 = \left(\frac{\lambda+2\mu}{\rho}\right)^{1/2}, \; c_2 = \left(\frac{\mu}{\rho}\right)^{1/2}, \; m = \gamma/\rho c_1^2. \qquad /3.11/$$

The following denotations are introduced here

$$\Box_\alpha^2 = \nabla^2 - \frac{1}{c_\alpha^2}\partial_t^2, \quad D = \nabla^2 - \frac{1}{\varkappa}\partial_t,$$

$$\partial_t = \frac{\partial}{\partial t}$$

Eqs. /3.9/ and /3.11/ are coupled each other in a direct manner. Elimination of the function θ leads to the equation of a longitudinal wave

$$\left(\Box_1^2 D - \eta m \partial_t \nabla^2\right)\phi = -\frac{mQ}{\varkappa} - \frac{1}{c_1^2}D\vartheta. \qquad /3.12/$$

Eqs. /3.10/ describes a transverse wave. Let us note that the functions ϕ and ψ_i are joined mutually through boundary conditions which will be expressed, in every case, by displacements u_i and derivatives of these functions and by temperature θ.

Eliminating the function ϕ out of Eqs. /3.9/ and /3.11/, we obtain the equation

$$\left(\Box_1^2 \Box_2 - \eta m \partial_t \nabla^2\right)\theta = -\frac{m}{\kappa}\Box_1^2 Q - \frac{1}{c_1^2}\eta\,\partial_t \nabla^2 \vartheta \qquad /3.13/$$

We see that Eqs. /3.12/ and /3.13/ have the same form. The structure of these formulae, what will be discussed later, indicates that we consider a wave damped and subjected to disperssion. In an unbounded thermoelastic space, the longitudinal and transverse waves propagate independently of each other. Let us assume that heat sources Q and body forces $X_i = \rho \vartheta_{,i}$ are the source of motion. Under the assumption that $\chi_i = 0$ and that initial conditions connected with Eq. /3.10/ are equal to zero, we have $\psi_i \equiv 0$ in the whole space.

In the unbounded space, there will arise only longitudinal waves of dilatation character.

Taking into account /3.2/ and /3.8/, we have

$$u_i = \Phi_{,i}\,, \quad \epsilon_{ij} = \Phi_{,ij}\,, \quad \epsilon_{kk} = \nabla^2\Phi,$$

and

$$\sigma_{ij} = 2\mu\left(\Phi_{,ij} - \delta_{ij}\Phi_{,kk}\right) + \rho\,\delta_{ij}\left(\ddot{\Phi} - \vartheta\right).$$

If, in the unbounded space the body forces $X_i = \rho\, \epsilon_{ijk}\, \chi_{k,j}$ act, whereas $Q = 0$, $\vartheta = 0$ and $\Phi(x,0) = 0$ $\dot{\Phi}(x,0) = 0$ then only the functions ψ_i are different from zero, but $\Phi \equiv 0$, $\theta \equiv 0$ in the whole region. Only the transverse waves are propagated and their velocity is $c_2 = \left(\frac{\mu}{\rho}\right)^{1/2}$ These waves are not accompanied by heat production.

Let us observe that for transverse waves there is

$$u_i = \epsilon_{ijk}\psi_{k,j}\,, \quad u_{k,k} = 0, \quad \vartheta = 0, \quad \sigma_{ij} = 2\mu\varepsilon_{ij} = \mu(u_{i,j} + u_{j,i}).$$

In a bounded body, there appear simultaneously, in principle, two kinds of waves. The solution for Eqs. /3.10/ and /3.12/ will be constructed of two parts viz., of the particular integrals of these equations φ^0 , ψ_i^0 and of the general integrals of homogeneous equations

$$(\Box_1^2 D - \eta m \dot{c}_t \nabla^2)\varphi' = 0, \quad \Box_2^2 \psi_i' = 0;$$

where, the functions φ' and ψ_i' should be chosen so as to satisfy all possible boundary conditions.

The next method used for solving the differential equations of thermoelasticity is the method of disjoining the equations which consists in leading Eq. /3.4/ and /3.5/ to a system of four equations decoupled. Only one unknown function appears in each equation. Presumably, this method was first used by Hilbert [9] for the differential equations of optics. A certain its variant in the operator form developed by G.Moisil [10] was applied for the quasi-static equations of thermoelasticity by V.Ionescu-Cazimir [12] . S.Kaliski [11] has disjoined the dynamical equations of thermoelasticity on another way. This result was repeated, using other else manner, by J.S.Postrigacz [13] and D.Rüdiger [14] .

Omitting details of this method, we shall present only the final result. We introduce one scalar function ψ and one vector function φ_i and with the help of them we express displacement and temperature as follows

$$u_i = (\Omega\delta_{ij} - \Gamma\partial_i\partial_j)\varphi_j + \mu_0\partial_i\psi, \qquad /3.14/$$

$$\theta = \eta\partial_t\partial_j\square_2^2\varphi_j + (1+a)\square_1^2\psi, \qquad /3.15/$$

where

$$\Omega = (1+a)\square_1^2 D - \mu_0\eta\partial_t\nabla^2; \quad \Gamma = aD - \mu_0\eta\partial_t,$$

$$a = \frac{\lambda+\mu}{\mu}, \quad \mu_0 = \frac{\gamma}{\mu}.$$

Substituting u_i and θ into Eqs. /3.4/ and /3.5/ we obtain four already decoupled equations for the functions φ_i and ψ

$$\square_2^2\left(\square_1^2 D - \eta m\partial_t\nabla^2\right)\varphi_i + \frac{X_i}{\rho c_1^2} = 0, \qquad /3.16/$$

$$\left(\square_1^2 D - m\eta\partial_t\nabla^2\right)\psi + \frac{Q\mu}{\varkappa c_1^2\rho} = 0. \qquad /3.17/$$

To these equations, we should add boundary and initial conditions. In the boundary conditions, there appear, of course, the functions φ_i and ψ. The simplicity of the differential equations /3.16/ and /3.17/ is, however, ransomed with the complicated form of boundary conditions. Therefore, Eqs. /3.16/ and /3.17/ will be applied, first of all, in a unbounded space, where boundary conditions in strict sense do not exist and they are replaced by the requirement of zero values of displacements and temperature in infinity. This postulate is fulfilled if distribution of body forces and of heat sources is restricted to a finite region.

The interesting way for solving the differential equations of thermoelasticity was given by H.Zorski [15]. This way leads to transforming the system of differential equation /3.4/ and /3.5/ into a system of three differential equations for displacements u_i. We shall present it shortly in reference to an unbounded space with the assumption of homogeneous initial conditions. We write the conductivity equation in such a form that the term containing dilatation velocity is on the right-hand side of equation

$$\theta_{,jj} - \frac{1}{\varkappa}\dot{\theta} = \eta \dot{u}_{j,j} \qquad /3.18/$$

Regarding the function $\eta \dot{u}_{j,j}$ as a heat source, we can give the solution of Eq. /3.18/ using the Green's function valid for the classical equation of thermal conductivity

$$\begin{aligned} &G_{,jj} - \frac{1}{\varkappa}\dot{G} = -\frac{1}{\varkappa}\delta(x-\xi)\delta(t), \\ &G(x,\xi,t) = \frac{\exp\left(-\frac{\rho^2}{4\varkappa t}\right)}{8(\pi\varkappa t)^{3/2}} \end{aligned} \qquad /3.19/$$

Inserting the solution of Eq. /3.18/

$$\theta(x,t) = -\eta\varkappa\int_0^t d\tau\int_V G(\xi,x,t-\tau)\frac{\partial}{\partial\tau}\operatorname{div} u(\xi,\tau)\,dV(\xi), \quad \rho^2 = (\xi_i - x_i)(\xi_i - x_i)$$

into the displacement equations /3.4/, we get the following differential-integral equation

$$\mu\nabla^2 u + (\lambda+\mu)\operatorname{grad}\operatorname{div} u - \rho\ddot{u} = -\frac{\eta\varkappa\gamma}{\ell}\operatorname{grad}\int_0^t d\tau\int_V G(\xi,x,t-\tau)\frac{\partial}{\partial\tau}\operatorname{div} u(\xi,\tau)\,dV(\xi) \qquad /3.20/$$

If the displacement vector is decomposed according to formula /3.8/, then Eq. /3.20/ disintegrates into the system of equations

$$\square_1^2\phi + \frac{\mu\eta}{c_1^2\gamma}\int_0^t d\tau\int_V G(\xi,x,t-\tau)\frac{\partial}{\partial\tau}\nabla^2\phi(\xi,\tau)\,dV(\xi) = 0, \qquad /3.21/$$

$$\square_2^2\psi = 0 \qquad /3.22/$$

The differential-integral equation /3.21/ is equivalent to Eqs. /3.9/ and /3.11/.

In certain cases, especially when boundary conditions are given in terms of stresses, it is useful to utilize the equations analogous to Beltrami-Michell equations. These equations for uncoupled problems have been derived by J. Ignaczak [16], for coupled problems by E. Soós [17]. Another method of solutions in terms of stresses in reference to a plane state of deformation was given by W. Nowacki [18].

If the variability of body forces, heat sources, surface loadings and heatings is slow, then the inertial terms in the equations of motion can be deleted and the problem can be regarded as quasi-static. The quasi-static equations of thermoelasticity

$$\mu u_{i,jj} + (\lambda+\mu)u_{j,ji} + X_i = \gamma \vartheta_{,i}, \qquad /3.23/$$

$$\theta_{,jj} - \frac{1}{\varkappa}\dot{\theta} - \eta \dot{u}_{j,j} = -\frac{Q}{\varkappa}, \qquad /3.24/$$

continue to be coupled. A solution for this system of equations is particularly simple for an unbounded thermoelastic medium in which heat sources Q and body forces of the potential type $X_i = \rho \vartheta_{,i}$ act. By introducing the thermoelastic potential of displacement Φ , we obtain, from /3.23/ and /3.24/, the disjoined system of equations / 15 /

$$\nabla^2\theta - \frac{1}{\varkappa_1}\dot{\theta} = -\frac{Q}{\varkappa} - \frac{\eta}{c_1^2}\dot{\vartheta}, \quad \nabla^2\Phi = m\theta - \frac{\vartheta}{c_1^2}, \qquad /3.25/$$

$$\varkappa_1 = \frac{\varkappa}{1+\varepsilon}, \quad \varepsilon = \eta m \varkappa.$$

The temperature θ is determined here from a parabolic differential equation whose structure is similar to the classical equation of thermal conductivity.

For disjoining the system of Eqs. /3.23/ and /3.24/ we can also apply the manner presented previously /equations /3.14/ - /3.17/ which consists in neglecting the inertial terms appearing there.

The manner given by M.A.Biot [4] is also interesting.

By introducing the expression for entropy

$$s = \gamma e_{kk} + \frac{c_\varepsilon}{T_0}\vartheta. \qquad /3.26/$$

into the Eqs. /3.23/ and /3.24/ with the assumption $Q=0$, $X_i=0$, we obtain the system of equations

$$\mu u_{i,jj} + (\lambda+\mu+\delta)u_{j,ji} = \gamma\beta s_{,i}\,, \qquad /3.27/$$

$$s_{,jj} - \frac{1}{\varkappa_2}\dot{s} = 0, \quad \delta = \gamma^2\beta, \quad \beta = \frac{T_0}{c_\varepsilon}, \quad \varkappa_2 = \varkappa\frac{\lambda+2\mu}{\lambda+2\mu+\beta^2} \qquad /3.28/$$

These equations are disjoined and the entropy fulfills parabolic equations. The solution of Eqs. /3.27/ can be written in the form of Papkowich-Boussinesq potentials

$$u_i = -(\psi_{0,i} + x_j\psi_{j,ij}) + B\psi_i, \quad B = 2\,\frac{\lambda+2\mu+\delta}{\lambda+\mu+\delta}; \qquad /3.29/$$

with the assumption that the vector function ψ_i is harmonic. To determine the functions ψ_0, ψ_i we have at our disposal the following equations

$$\nabla^2\psi_i = 0, \quad \nabla^2\psi_0' = 0, \quad \left(\nabla^2 - \frac{1}{\varkappa_2}\partial_t\right)\psi_0'' = 0, \qquad /3.30/$$

where $\psi_0 = \psi_0' + \psi_0''$.

After determining the functions ψ_0, ψ_i and taking into account boundary and initial conditions, we shall obtain displacements from the formula /3.29/.

As we have mentioned at the very beginning, the thermoelasticity comprises full divisions of the directions developed so far separately: classical elastokinetics, thermal conductivity theory and thermal stresses theory. We shall arrive at the differential equations of classical elastokinetics assuming that the motion executes in adiabatic conditions i.e. without heat exchange among particular parts of body. Since, for an adiabatic process, there is $\dot{s} = 0$, then we obtain from the formula /3.26/ $\dot{\theta} = -\eta \varkappa \dot{\varepsilon}_{kk}$ or after integrating and assuming homogeneous initial conditions

$$\theta = -\eta_T \varkappa \varepsilon_{kk} . \qquad /3.31/$$

This equation replaces the equation of heat conduction. Inserting /3.31/ into /3.4/, we obtain the displacement equation of classical elastokinetics

$$\mu_s u_{i,jj} + (\lambda_s + \mu_s) u_{j,ji} + X_i = \rho \ddot{u}_i , \qquad /3.32/$$

where

$$\lambda_s = \lambda_T + \gamma_T \eta_T \varkappa , \quad \mu_T = \mu_s$$

The quantities λ_s, μ_s are the Lamé's constants measured in adiabatic conditions. The state equations after substituting /3.31/ into /3.2/ take the form

$$\sigma_{ij} = 2\mu_s \varepsilon_{ij} + \lambda_s \varepsilon_{kk} \delta_{ij} . \qquad /3.33/$$

In the theory of thermal stresses in which the influence of body surface heating and heat sources action on deformation and stress state of a body is considered, the influence of the term $\eta \dot{e}$ appearing in the thermal conductivity equation on the body deformation is assumed to be very small and negligible pratcically. This simplification leads to the system of two equations independent of each other

$$\mu u_{i,jj} + (\lambda+\mu) u_{j,ji} = \rho \ddot{u}_i + \gamma \theta_{,i} \qquad /3.34/$$

$$\theta_{,ii} - \frac{1}{\kappa}\dot{\theta} = -\frac{Q}{\kappa} \qquad /3.35/$$

The temperature θ is determined from /3.35/, i.e. from the classical equation of thermal conductivity. When we know the temperature distribution, we are able to determine displacements from Eqs. /3.34/.

Wide literature is devoted to theory of thermal stresses. Many practical problems both quasi-static and dynamic have been solved so far. The methods for solving the system of equations /3.34/ and /3.35/ have been elaborated in details. The reader can find them in monographs [19] - [22].

In the case of steady flow of heat, the production of entropy is compensated by the exchange of entropy with environment. This exchange is negative and its absolute value is equal to entropy production in a body. In the equations of thermoelasticity /3.4/ and /3.5/ the derivatives with respect to time disappear. Eq /3.4/ becomes the equation of elastostatics

$$\mu u_{i,jj} + (\lambda+\mu) u_{j,ji} + X_i = \gamma \theta_{,i} \qquad /3.36/$$

and the thermal conductivity equation transforms in the equation of parabolic type, the Poisson's equation

$$\theta_{,jj} = -\frac{Q}{\varkappa} \qquad /3.37/$$

In virtue of the familiar analogy of body forces [23], the determination of thermal stresses is reduced here to the solutions of the classical theory of elasticity.

4. Variational theorems of thermoelasticity

It is known how great part is played by the variational theorems in the elasticity theory with variation of deformation state or stress state. They permit not only to derive the differential equations describing the bending of plates shells, discs, membranes, etc., but also to construct approximative solutions. In what follows, we shall present the variational theorem with the variation of deformation state for thermoelasticity. This method was devised by M.A.Biot [4] . This theorem will be consisted of two parts; first of them utilizes the d'Alembert's principle familiar in the elasticity theory

$$\int_V \sigma_{ij}\,\delta\varepsilon_{ij}\,dV = \int_V (X_i - \rho\ddot{u}_i)\,\delta u_i\,dV + \int_A p_i\,\delta u_i\,dA \qquad /4.1/$$

In this equation δu_i are the virtual increments of displacements, $\delta\varepsilon_{ij}$ the virtual increments of deformations.

We assume that δu_i and $\delta \varepsilon_{ij}$ are arbitrary continuous functions independent of time and complying with the conditions constraining the body motion.

The d'Alembert's principle is valid irrespectively of body material, i.e. for all forms of dependency of the stress state on deformation state. Supplementing /4.1/ with the state equation and introducing the quantity

$$W = \int_V \left(\mu \varepsilon_{ij} \varepsilon_{ij} + \frac{\lambda}{2} (\varepsilon_{kk})^2 \right) dV, \qquad /4.2/$$

where the integrand is a quadratic form positive definite, we obtain from /4.1/ the following equation

$$\delta W = \int_V (X_i - \rho \ddot{u}_i) \delta u_i \, dV + \int_\Sigma p_i \delta u_i \, dV + \gamma \int_V \theta \delta e \, dV, \quad e = \varepsilon_{kk} \qquad /4.3/$$

The second part of the variational theorem should take advantage of the laws governing the heat flow. Therefore , we shall utilize the expressions interrelating the heat flow, temperature and entropy

$$q_i = -\lambda_0 \theta_{,i}, \quad -q_{i,i} = \dot{s} T_0 = \gamma \dot{\varepsilon}_{kk} T_0 + c_\varepsilon \dot{\theta}. \qquad /4.4/$$

These relations can be written in the form more convenient further studies by introducing the vector function interrelated with entropy and flow in the following way

$$s = -S_{i,i}, \quad q_i = T_0 \dot{S}_i \ . \qquad /4.5/$$

Combining /4.4/ and /4.5/ each other we obtain

$$T_0 \dot{S}_i = -\lambda_0 \theta_{,i}\,, \quad -T_0 \dot{S}_{i,i} = c_\varepsilon \dot{\theta} + T_0 \gamma \dot{\varepsilon}_{kk} \qquad /4.6/$$

Let us multiply the first one of Eqs. /4.6/ by the virtual increment δS_i and integrate over the body region

$$\int_V \left(\theta_{,i} + \frac{T_0}{\lambda_0} \dot{S}_i\right) \delta S_i \, dV = 0. \qquad /4.7/$$

Through transforming this integral and taking into account the second one of the relations /4.6/, we obtain equation

$$\frac{c_\varepsilon}{T_0} \int_V \theta \delta\theta \, dV + \frac{T_0}{\lambda_0} \int_V \dot{S}_i \delta S_i \, dV + \int_\Sigma \theta n_i \delta S_i \, d\Sigma + \gamma \int_V \theta \delta e \, dV = 0, \qquad /4.8/$$

in which there is involved the term $\int_V \theta \delta e \, dV$ identical to the appearing in /4.3/. Eliminating this term from Eqs. /4.4/ and /4.8/, we get the final form of the variational theorem

$$\delta(W + P + D) = \int_V (X_i - \rho \ddot{u}_i) \delta u_i \, dV + \int_\Sigma p_i \delta u_i \, d\Sigma - \int_\Sigma \theta n_i \delta S_i \, d\Sigma. \qquad /4.9/$$

We have introduced here the following denotations

$$P = \frac{c_\varepsilon}{2T_0} \int_V \theta^2 dV, \quad D = \frac{T_0}{2\lambda_0} \int_V (\dot{S}_i)^2 dV \qquad /4.10/$$

The function P is called the thermal potential, D the dissipation function. Let us consider, moreover, the particular cases. If we assume $\theta = -\eta_r \varkappa \varepsilon_{kk}$ in Eq. /4.3/, what corresponds to assumption of adiabatic process, then /4.3/ transforms into

$$\delta W_s = \int_V (X_i - \rho \ddot{u}_i)\delta u_i \, dV + \int_\Sigma p_i \delta u_i \, d\Sigma, \qquad /4.11/$$

where

$$W_s = \int_V \left(\mu_s e_{ij} e_{ij} + \frac{\lambda_s}{2} \varepsilon_{kk} \varepsilon_{rr}\right) dV,$$

and μ_s, λ_s are the Lamé's adiabatic constants. The Eq./4.1/ constitutes the d'Alembert's principle for classical elastokinetics.

In the theory of thermal stresses we neglect the mutual interaction of the deformation and temperature fields what is expressed by deleting the term $\gamma \dot{\varepsilon}_{kk} T_0$ in the second one of the Eqs, /4.4/. Neglecting this term leads to modified Eq. /4.8/. We obtain here

$$\delta(P+D) + \int_\Sigma \theta m_i \delta S_i \, d\Sigma = 0. \qquad /4.12/$$

Eq. /4.12/ expresses the variational theorem for the classical uncoupled problem of thermal conductivity. In the theory of thermal stress, we have at our disposal two equations viz., Eqs./4.12/ and /4.3/ in which the function θ is thought of as the known function.

Let us now return to the general variational theorem of thermoelasticity /4.9/ and assume that the virtual increments δu_i , $\delta \varepsilon_{ij}$, δS_i , etc. coincide with the increments really occuring when the process pass from a time instant t to $t+dt$. Then

$$\delta u_i = \frac{\partial u_i}{\partial t} dt = v_i dt, \quad \delta S_i = \frac{\partial S_i}{\partial t} = \dot{S}_i dt, \quad \delta W = \dot{W} dt, \qquad /4.13/$$

and so forth.

Putting /4.13/ into /4.9/, we obtain

$$\frac{d}{dt}\left(K + W + P\right) + \chi_T = \int_V X_i v_i \, dV + \int_\Sigma p_i v_i \, d\Sigma + \frac{\lambda_0}{T_0}\int_\Sigma \theta \theta_{,n} \, d\Sigma, \qquad /4.14/$$

where $K = \frac{\rho}{2}\int_V v_i v_i \, dV$ is the kinetic energy, χ_T dissipation function, where

$$\chi_T = \lambda_0 T_0 \int_V \left(\frac{\theta_{,i}}{T_0}\right)^2 dV = \lambda_0 T_0 \int_V \left(\frac{q_i}{\lambda_0 T_0}\right)^2 dV.$$

The equation /4.14/ is called the basic energy theorem of thermoelasticity. This theorem can be utilized to determine the uniqueness of solutions for the thermoelasticity equations [21], [24]. Proceeding in like manner as in the elasticity theory, we assume that the thermoelasticity equations are satisfied by two groups of functions u_i' , θ' and u_i'' , θ'' . Constructing the difference among these functions $\overset{\circ}{u}_i = u_i' - u_i''$, $\overset{\circ}{\theta} = \theta' - \theta''$ and inserting it into Eqs. /3.4/ and /3.5/, we see that these equations are homogeneous and satisfy homogeneous boundary and initial conditions. To the functions $\hat{u}_i$, $\hat{\vartheta}$ there then corresponds a thermodynamical body, in interior of which is free of heat sources and body forces and which is not loaded on its surface and it finds itself in the conditions of zero temperature $\hat{\vartheta}$. The formula /4.14/ will answer the question whether or not displacements $\hat{u}_i$ and temperature $\hat{\vartheta}$ will appear in the body interior. Eq. /4.14/ takes the form

$$\frac{d}{dt}\int_V\left(\frac{\varrho}{2}\hat{v}_i\hat{v}_i+\mu\hat{\varepsilon}_{ij}\hat{\varepsilon}_{ij}+\frac{\lambda}{2}\hat{\varepsilon}_{kk}\hat{\varepsilon}_{nn}+\frac{\mu}{2\eta\varkappa}\hat{\vartheta}^2\right)dV=-\frac{\lambda_0}{T_0}\int_V(\hat{\vartheta}_{,i})^2\,dV\leq 0. \qquad /4.15/$$

The integral appearing on left-hand side of the equation is equal to zero at initial time instant since the functions $\hat{u}_i, \hat{v}_i, \hat{\varepsilon}_{ij}, \hat{\vartheta}$ satisfy the homogeneous initial conditions . On the other side, the derived inequality indicates that the left-hand side of equation either decreases assuming negative values, or it is equal to zero.

Since the expression under the integral sign is a sum of the second powers and the integrand is equal to zero for $t=0$, so only the second one of the mentioned alternatives is possible. As a result we obtain $\hat{v}_i=0$, $\hat{\varepsilon}_{ij}=0$, $\hat{\vartheta}=0$ for $t\geq 0$. Since the stresses $\hat{\sigma}_{ij}$ are linearly related to the quantities $\hat{\varepsilon}_{ij}$, $\hat{\vartheta}$ then also $\hat{\sigma}_{ij}=0$ for $t\geq 0$.

In consequence, we obtain

$$u_i' = u_i'', \quad \theta' = \theta'', \quad \sigma_{ij}' = \sigma_{ij}'' \quad \text{for } t \geq 0. \qquad /4.16/$$

Then, there exists only one solution for the thermoelasticity equations.

5. The reciprocity theorem

One of the most interesting theorems of the thermoelasticity theory is the E.Betti's theorem on reciprocity. Since not only the symmetry of fundamental solutions /of Green's function/ follows from this theorem, but also it provides foundation for developing further methods for integrating the differential equations of the thermoelasticity theory.

The extended theorem on reciprocity concerning the thermoelasticity problems has been fully formulated by V.Ionescu-Cazimir [25] . The elements of this theorem, altough expressed in a less general form can be found in works by M.A.Biot [26].

We shall present the reciprocity theorem in its main outlines emphasizing numerous its applications.

Let two systems of forces act in an isotropic body. We assume that inside the body V , the heat sources and body forces operate, and on body surface, the loadings p_i and temperature $\vartheta = \vartheta$ are given. We denote these causes in abbreviation $I = \{X_i, p_i, Q, \vartheta\}$ and the consequences following from them - by the symbol $C = \{u_i, \theta\}$. The second system of causes and consequences is denoted by $I'=\{X_i', p_i', Q', \vartheta'\}$ and $C'=\{u_i', \theta'\}$. The initial conditions are assumed to be homogeneous. Starting from motion equations, thermal conductivity equations and Duhamel-Neumann relations written for both systems, suitably adding those systems and integrating over the region V, we obtain two equations of reciprocity for the transforms of functions involved in both systems

$$\int_V (\bar{X}_i \bar{u}_i' - \bar{X}_i' \bar{u}_i)\,dV + \int_\Sigma (\bar{p}_i \bar{u}_i' - \bar{p}_i' \bar{u}_i)\,d\Sigma + \mu \int_V (\bar{\theta}\bar{e}' - \bar{\theta}'\bar{e})\,dV = 0, \qquad /5.1/$$

$$\int_V (\bar{Q}'\bar{\theta} - \bar{Q}\bar{\theta}')\,dV + \eta \varkappa p \int_V (\bar{\theta}'\bar{e} - \bar{\theta}\bar{e}')\,dV + \varkappa \int_\Sigma (\bar{\vartheta}\bar{\theta}_{,n}' - \bar{\vartheta}'\bar{\theta}_{,n})\,d\Sigma = 0, \qquad /5.2/$$

where

$$\bar{u}_i(x,p) = \int_0^\infty u_i(x,t)\, e^{-pt}\,dt, \qquad \text{on so forth.}$$

The first of these equations arises from employing the motion equations and state equations with applying the Green's transformation. Eliminating from these equations the common terms, we get the following equation

$$\eta \varkappa p \left[\int_V (\bar{X}_i \bar{u}_i' - \bar{X}_i' \bar{u}_i)\,dV + \int_\Sigma (\bar{p}_i \bar{u}_i' - \bar{p}_i' \bar{u}_i)\,d\Sigma \right] = \qquad /5.3/$$

$$= \varkappa \mu \int_\Sigma (\bar{\vartheta}'\bar{\theta}_{,n} - \bar{\vartheta}\bar{\theta}_{,n}')\,d\Sigma + \mu \int_V (\bar{Q}\bar{\theta}' - \bar{Q}'\bar{\theta})\,dV.$$

The Eq. /5.3/ should be subjected to the Laplace's inverse transformation. After utilizing the theorem on convolution, we have

$$\eta\varkappa\Big\{\int_V dV(x)\int_0^t\Big[X_i(x,t-\tau)\frac{\partial u_i'(x,\tau)}{\partial\tau}-X_i'(x,t-\tau)\frac{\partial u_i(x,\tau)}{\partial\tau}\Big]d\tau + \int_\Sigma d\Sigma(x)\int_0^t\Big[p_i(x,t-\tau)\frac{\partial u_i'(x,\tau)}{\partial\tau}-p_i'(x,t-\tau)\frac{\partial u_i(x,\tau)}{\partial\tau}\Big]d\tau\Big\} = \qquad /5.4/$$

$$= \mu\int_V dV(x)\int_0^t\Big[Q(x,t-\tau)\theta'(x,\tau)-Q'(x,t-\tau)\theta(x,\tau)\Big]d\tau + \mu\varkappa\int_\Sigma d\Sigma(x)\int_0^t\Big[\vartheta'(x,t-\tau)\theta_{,n}(x,\tau)-\vartheta(x,t-\tau)\theta'_{,n}(x,\tau)\Big]d\tau.$$

The Eq. /5.4/ is valid for both dynamical and quasi-static problems. But in both cases, the functions u_i , θ and u_i', θ' have different meaning. We have assumed in our considerations that on the surface Σ , the loading p_i and temperature $\theta = \vartheta$ are given. It is seen from the structure of Eq. /5.4/ that we can assume that on Σ displacements and heat flow are proportional to the gradient of temperature $\theta_{,n} = \vartheta_{,n}$.

The Eqs. /5.4/ are also satisfied for mixed boundary conditions.

The Eq. /5.4/ assumes particularly simple form for an unbounded body, because, in this case, the surface integrals vanish.

If we encounter vibration harmonically varying in time $X_i(x,t) = X_i^*(x)e^{i\omega t}$, $p_i(x,t) = p_i^*(x)e^{i\omega t}$ and so forth then the equation of reciprocity takes the form

$$\eta \varkappa i\omega\Big[\int_V (X_i' v_i'^{*} - X_i'^{*} v_i^{*})\,dV + \int_\Sigma (p_i' v_i'^{*} - p_i'^{*} v_i^{*})\,d\Sigma\Big] = \qquad /5.4'/$$
$$= \varkappa\mu\int_\Sigma (\vartheta'^{*}\theta_{,n}^{*} - \vartheta^{*}\theta_{,n}'^{*})\,d\Sigma + \mu\int_V (Q^{*}\theta'^{*} - Q'^{*}\theta^{*})\,dV.$$

We shall obtain from Eq. /5.4/ a number of interesting conclusions. Let us assume that at the point ξ of the region V, the instantaneous force $X_i = \delta(x-\xi)\delta(t)\delta_{ij}$ acts which is directed along x_j-axis, whereas at the point ξ' the concentrated force $X_i' = \delta(x-\xi')\delta(t)\delta_{ik}$ which is directed along x_k-axis. If we assume that the boundary conditions are homogeneous, the relation /5.4/ gives

$$\frac{\partial u_j'(\xi,\xi',t)}{\partial t} = \frac{\partial u_k(\xi',\xi,t)}{\partial t}.$$

For the heat source $Q = \delta(x-\xi)\delta(t)$ and the source $Q' = \delta(x-\xi')\delta(t)$, we have

$$\theta'(\xi,\xi',t) = \theta(\xi',\xi,t).$$

If the concentrated and instantaneous force $X_i = \delta(x-\xi)\delta(t)\delta_{ij}$ is applied at the point ξ, and the heat source $Q' = \delta(x-\xi')\delta(t)$ at the point ξ', then the following relation is obtained from Eq. /5.4/

$$\theta(\xi',\xi,t) = -\frac{\eta\varkappa}{\mu}\,\frac{\partial u_j'(\xi,\xi',t)}{\partial t}.$$

Let the heat source $Q = \delta(x_1)\delta(x_2)\delta(x_3 - vt)$ moves with a constant velocity v in the direction of x_3-axis. Assuming that in the system of causes with "primes" $Q' = \delta(x - \xi')\delta(t)$, we get from /5.4/

$$\vartheta(\xi_1, \xi_2, \xi_3, t) = \int_0^t \vartheta'(0, 0, v\tau; \xi_1, \xi_2, \xi_3, t - \tau)\, d\tau.$$

The above formula permits to determine the temperature caused by moving heat source making use of the expression for temperature caused by the action of instantaneous but not moving heat source.

From Eqs. /5.1/, /5.2/ or /5.3/, we can obtain particular forms of the reciprocity theorem which concern the classical elastokinetics and the thermal stresses theory.

If we assume that deformation takes place in adiabatic conditions, then it should be put $\vartheta = -\eta_T \varkappa \varepsilon_{kk}$, $\vartheta' = -\eta_T \varkappa \varepsilon_{kk}'$ in /5.1/. Then, the following equation remains

$$\int_V (\bar{X}_i \bar{u}_i' - \bar{X}_i' \bar{u}_i)\, dV + \int_\Sigma (\bar{p}_i \bar{u}_i' - \bar{p}_i' \bar{u}_i)\, d\Sigma = 0 \qquad /5.5/$$

The Eq. /5.2/ disappears since in the elastokinetics we assume that heat sources do not exist in a body and the body surface is thermally isolated.

In the theory of thermal stresses, we neglect the dilatation term in the thermal conductivity equation. This omitting is formally equivalent to putting $\eta = 0$ in Eq. /5.2/.

Thus, we obtain the equations

$$\int_V(\bar X_i\bar v_i'-\bar X_i'\bar v_i)dV+\int_\Sigma(\bar p_i\bar u_i'-\bar p_i'\bar u_i)d\Sigma+\mu\int_V(\bar\theta\bar e'-\bar\theta'\bar e)dV=0, \qquad /5.6/$$

$$\int_V(\bar Q\bar\theta'-\bar Q'\bar\theta)dV+\varkappa\int_\Sigma(\bar\vartheta\bar\theta'_{,n}-\bar\vartheta'\bar\theta_{,n})d\Sigma=0. \qquad /5.7/$$

The Eq. /5.6/ has been derived by W.M.Majziel / 27 / . The Eq. /5.7/ is the equation of reciprocity for the classical equation of thermal conductivity.

We shall, moreover, consider the case in which the causes $I=\{X_i,p_i,Q,\vartheta\}$ and consequences $C=\{u_i,\theta\}$ refer to a coupled problem of thermoelasticity, and the causes $I'=$ $=\{X_i',p_i',Q',\vartheta'\}$ and consequences $C'=\{u_i',\theta'\}$ to a uncoupled problem. Taking into account the difference in the thermal conductivity equations for coupled and uncoupled problems

$$\begin{aligned}&\bar\theta_{,kk}-\frac{p}{\varkappa}\bar\theta-\eta p\bar e=-\frac{\bar Q}{\varkappa},\\ &\bar\theta'_{,kk}-\frac{p}{\varkappa}\bar\theta'=-\frac{\bar Q'}{\varkappa},\end{aligned} \qquad /5.8/$$

we obtain instead of Eq. /5.8/ the following equation

$$\int_V(\bar Q'\bar\theta-\bar Q\bar\theta')dV+\varkappa\eta p\int_V\bar\theta'\bar e\,dV+\varkappa\int_\Sigma(\bar\vartheta\bar\theta'_{,n}-\bar\vartheta'\bar\theta_{,n})d\Sigma=0 \qquad /5.9/$$

Eliminating the term $\int_V\bar\theta'\bar e\,dV$ out of Eqs. /5.1/ and /5.9/ we get the reciprocity theorem in the form

$$\begin{aligned}&\varkappa\eta p\left[\int_V(\bar X_i\bar v_i'-\bar X_i'\bar v_i)dV+\int_\Sigma(\bar p_i\bar u_i'-\bar p_i'\bar u_i)d\Sigma+\mu\int_V\bar\theta\bar e'dV\right]=\\ &=\varkappa\mu\int_\Sigma(\bar\vartheta\bar\theta'_{,n}-\bar\vartheta'\bar\theta_{,n})d\Sigma-\varkappa\int_V(\bar Q\bar\theta'-\bar Q'\bar\theta)dV\end{aligned} \qquad /5.10/$$

Let us now assume that only concentrated and instantaneous heat source acts in the system with "primes" and boundary conditions are homogeneous. Inserting then into Eq. /5.10/

$$Q' = \delta(x-\xi)\delta(t), \quad X_i' = 0, \quad p_i' = 0, \quad \vartheta' = 0$$

we have

$$\bar{\theta}(\xi,p) + \eta\varkappa p \int_V \bar{\theta}(\xi,p)\,\bar{e}'(x,\xi,p)\,dV(x) = \bar{M}(\xi,p) \qquad /5.11/$$

where

$$\bar{M}(\xi,p) = \int_V \bar{Q}(x,p)\,\bar{\theta}'(x,\xi,p)\,dV(x) - \varkappa\int_\Sigma \bar{\vartheta}(x,p)\,\bar{\theta}'_{,n}(x,\xi,p)\,d\Sigma(x) -$$

$$-\frac{\eta\varkappa}{\gamma}\Big[\int_\Sigma (\bar{p}_i(x,p)\,\bar{u}_i(x,\xi,p)\,d\Sigma(x) + \int_V \bar{X}_i(x,p)\,\bar{u}_i'(x,\xi,p)\,dV(x)$$

Since the functions u_i', θ' are known as solutions for the differential equations of the thermal stresses theory, and the functions $\bar{Q}, \bar{\vartheta}, \bar{p}_i, \bar{X}_i$ are given, then the function $\bar{M}(\xi,p)$ is known. The Eq. /5.11/ is the Fredholm's integral nonhomogeneous equation of the second kind in which the temperature θ appears as an unknown function. Also displacements can be obtained in the like manner.

The procedure here presented was proposed by V.Ionescu-Casimir [25] and applied for determining the Green's function in an unbounded elastic region [28], [29].

6. The methods for integrating thermoelasticity equations following from the reciprocity theorem

In the elastostatics, the expression is derived which interrelates displacement $u_i(x,t)$, $x \in V$, $t > 0$ inside a body with displacements u_i and loadings p_i on the body surface. Those relations are familiar as the Somiglian's and Green's theorem [30] . We shall present below the theorems of such a kind extended for the thermoelasticity problems.

Let us assume that causes producing deformations and temperature in the body are expressed solely by initial conditions. The initial conditions are assumed to be homogeneous. The equations describing the body motion are of the form

$$\sigma_{ij,j} = \rho \ddot{u}_i , \quad \theta_{,ii} - \frac{1}{\varkappa}\dot{\theta} - \eta \dot{e} = 0, \; x \in V, t > 0 \qquad /6.1/$$

We add the state equations to these equations

$$\sigma_{ij} = 2\mu \varepsilon_{ij} + (\lambda \varepsilon_{kk} - \gamma \theta)\delta_{ij} . \qquad /6.2/$$

We consider the second system of equations with "primes" concerning an unbounded thermoelastic body

$$\sigma'_{ij,j} = \rho \ddot{u}'_i , \quad \theta'_{,ii} - \frac{1}{\varkappa}\dot{\theta}' - \eta \dot{e}' = -\frac{1}{\varkappa}\delta(x-\xi)\delta(t) \qquad /6.3/$$
$$x \in V, \; t > 0$$

and the Duhamel-Neumann equations

$$\sigma'_{ij} = 2\mu \varepsilon'_{ij} + (\lambda \varepsilon'_{kk} - \gamma \theta')\delta_{ij} . \qquad /6.4/$$

In Eqs. /6.1/ ÷ /6.4/ we perform the Laplace's transformation with taking into account homogeneous initial conditions, and next we add suitably these equations and accomplish integration over the region V .

After a number transformations which are omitted here, we obtain finally / 31 /

$$\bar{\vartheta}(x,p) = -\frac{\eta \varkappa p}{\gamma} \int_{\Sigma} \left[\bar{p}_i(\xi,p)\,\bar{u}_i'(\xi,x,p) - \bar{p}_i'(\xi,x,p)\,u_i(\xi,p) \right] d\Sigma(\xi) - \tag{6.5}$$

$$- \varkappa \int_{\Sigma} \left[\bar{\vartheta}'(\xi,x,p)\,\bar{\vartheta}_{,m}(\xi,p) - \bar{\vartheta}(\xi,p)\,\bar{\vartheta}_{,m}'(\xi,x,p) \right] d\Sigma(\xi).$$

This formula can also be derived from the reciprocity theorem /5.3/ assuming $Q' = \delta(x-\xi)\delta(t),\ X_i = 0,\ X_i' = 0,\ Q = 0.$

Let us consider, in turn, the second system of equations

$$\sigma_{ji,j}^{s} = \rho \ddot{u}_i^{s} - \delta(x-\xi)\delta_{is}\delta(t), \tag{6.6}$$

$$\vartheta_{,jj}^{s} - \frac{1}{\varkappa}\dot{\vartheta}^{s} - \eta \dot{e}^{s} = 0, \tag{6.7}$$

$$\sigma_{ij}^{s} = 2\mu\,\varepsilon_{ij}^{s} + (\lambda \varepsilon_{kk}^{s} - \gamma \vartheta^{s})\delta_{ij}. \tag{6.8}$$

The functions $u_i^s,\ \vartheta^s$ are assigned to a unbounded thermoelastic region. They are induced by action of instantaneous concentrated force $X_i' = \delta(x-\xi)\delta(t)\delta_{is}$ directed towards the x_s -axis. Putting $X_i' = \delta(x-\xi)\delta(t)\delta_{is},\ X_i = 0,\ Q = 0,$ $Q' = 0$ in the reciprocity theorem, we obtain the following expression for displacements u_s [31]

$$u_s(x,p) = \int_{\Sigma} \left[\bar{p}_i(\xi,p)\,\bar{u}_i^s(\xi,x,p) - \bar{p}_i^s(\xi,x,p)\,u_i(\xi,p) \right] d\Sigma(\xi) - \qquad /6.9/$$
$$- \frac{\mu}{\eta p} \int_{\Sigma} \left[\bar{\vartheta}_{,n}(\xi,p)\,\bar{\theta}^s(\xi,x,p) - \bar{\vartheta}(\xi,p)\,\bar{\theta}^s_{,n}(\xi,x,p) \right] d\Sigma(\xi).$$

The Eqs. /6.5/ and /6.9/ should be subjected additionally to the Laplace's inverse transformation. It leads to convolution expressions which are omitted here.

The Eqs. /6.5/ and /6.9/ consitute the generalization of Somiglian's equations for the thermoelasticity problems. Using them, we are able to express the functions $u_i(x,t)$, $\theta(x,t)$, $x \in V$, $t > 0$ in terms of surface integrals in which the functions u_i, θ and their derivatives appear.

If the Green's functions $\bar{u}_i'$, $\bar{\theta}'$ and $\bar{u}_i^s$, $\bar{\theta}^s$ are chosen so as they refer to a body occupying the region V bounded by the surface Σ and if it is assumed that the following boundary conditions should be satisfied on Σ

$$\bar{u}_i' = 0, \quad \bar{\vartheta}' = 0, \quad \bar{u}_i^s = 0, \quad \bar{\theta}^s = 0 \qquad \text{on } \Sigma$$

then Eqs. /6.5/ and /6.9/ are simplified to the form

$$\theta(x,p) = \varkappa \int_{\Sigma} \bar{\vartheta}(\xi,p)\,\bar{\theta}'_{,n}(\xi,x,p)\,d\Sigma(\xi) + \frac{\eta \varkappa p}{\mu} \int_{\Sigma} \bar{p}_i'(\xi,x,p)\,\bar{u}_i(\xi,p)\,d\Sigma(\xi) \qquad /6.10/$$

$$u_s(x,p) = -\int_{\Sigma} \bar{p}_i^s(\xi,x,p)\,\bar{u}_i(\xi,p)\,d\Sigma(\xi) + \qquad /6.11/$$
$$+ \frac{\mu}{\eta p} \int_{\Sigma} \bar{\vartheta}(\xi,p)\,\bar{\theta}^s_{,n}(\xi,x,p)\,d\Sigma(\xi).$$

These formulae constitute the solution of the first boundary problem in which displacements u_i and temperature θ

are given on Σ . If the functions $\bar{u}_i', \bar{\theta}'$ and $\bar{u}_i^1, \bar{\theta}^1$ were related to a body occupying the region V free from loadings and temperature on the surface Σ, it should be added to Eqs. /6.5/ and /6.9/

$$\bar{p}_i' = 0, \quad \bar{\vartheta}' = 0, \quad \bar{p}_i^1 = 0, \quad \bar{\theta}^1 = 0 \text{ on } \Sigma.$$

In this case, the formulae /6.5/ and /6.9/ assume the form

$$\theta(x,p) = -\frac{\eta\varkappa p}{\mu}\int_\Sigma \bar{p}_i(\xi,p)\,\bar{u}_i'(\xi,x,p)\,d\Sigma(\xi) + \varkappa\int_\Sigma \bar{\vartheta}(\xi,p)\,\bar{\theta}'_{,m}(\xi,x,p)\,d\Sigma(\xi), \qquad /6.12/$$

$$\bar{u}_1(x,p) = \int_\Sigma \bar{p}_i(\xi,p)\,\bar{u}_i^1(\xi,x,p)\,d\Sigma(\xi) + \frac{\mu}{\eta p}\int_\Sigma \bar{\vartheta}(\xi,x,p)\,\bar{\theta}'_{,m}(\xi,x,p)\,d\Sigma(\xi). \qquad /6.13/$$

and they consitute the solution of the second boundary problem in which loadings p_i and temperature ϑ are given on the surface Σ . However, the application of formulae /6.10/ - /6.13/ is restricted owing to the difficulties associated with obtaining the Green's functions $u_i', \theta', u_i^1, \theta^1$ satisfying the boundary conditions given in advance. In the analogous way as for the extended Somiglian's and Green's formulae, we can construct the solution of thermoelasticity equations for mixed boundary conditions. One of manners, being the extension of the W.M. Majziel methods from the thermal problems theory to thermoelasticity problems, can be found in the previously cited work by V.Ionescu-Cazimir [25]. It consists in using the Green's functions satisfying at once mixed boundary conditions. The second manner devised by W.Nowacki [32] consists in making use of the Green's auxiliary functions fulfilling continuous boundary conditions and leading the problem to solving the system of Fredholm's integral equations of the first order.

7. Harmonic waves

In discussion of a wave of the simplest type i.e. the plane harmonic wave, the essential properties of the propagation of elastic waves, their character, velocity of wave propagation, wave dispersion and damping are revealed at once. Also the fundamental differences among thermoelastic waves and elastic and thermal waves will be disclosed [33] and [34].

Let us consider a harmonic plane wave, moving in the direction of x_1 -axis, induced by a cause of mechanical or thermal nature. Since displacements u_j and temperature θ depend solely on the variables x_1 and t, the displacement equations and the thermal conductivity equations, after taking into account that

$$u_j = \mathrm{Re}\left[u_j^*(x_1,\omega)e^{-i\omega t}\right], \quad \theta = \mathrm{Re}\left[\theta^*(x_1,\omega)e^{-i\omega t}\right], \qquad /7.1/$$

assume the form

$$
\begin{aligned}
&(\partial_1^2+\sigma^2)u_1^* = m\partial_1\theta, \\
&(\partial_1^2+q)\theta^* + \eta\varkappa p\,\partial_1 u_1^* = 0 \\
&(\partial_1^2+\tau^2)u_2^* = 0, \qquad (\partial_1^2+\tau^2)u_3^* = 0,
\end{aligned} \qquad /7.2/
$$

where

$$\sigma^2 = \frac{\omega^2}{c_1^2}, \quad \tau^2 = \frac{\omega^2}{c_2^2}, \quad q = \frac{i\omega}{\varkappa}.$$

Eliminating the temperature θ^* from the two first equations, we have

$$[(\partial_1^2+\sigma^2)(\partial_1^2+q)+q\varepsilon\partial_1^2]u_1^* = 0, \quad (\partial_1^2+\tau^2)u_2^* = 0, \quad (\partial_1^2+\tau^2)u_3^* = 0 \qquad /7.3/$$

The first equation refers to a longitudinal wave, two remai--ning ones to transverse waves.

If we insert

$$u_1^* = u^0 e^{ikx_1}, \qquad \theta^* = \theta^0 e^{ikx_1}$$

into the two first of Eqs. /7.2/, we obtain

$$\frac{u^0}{\theta^0} = \frac{mik}{\sigma^2-k^2}, \quad \frac{\theta^0}{u^0} = -\frac{\eta q\varkappa ik}{q-k^2}$$

After eliminating the quantity u^0/θ^0 from these relation, we obtain the following algebraic equation

$$k^4 - k^2[\sigma^2+q(1+\varepsilon)] + q\sigma^2 = 0, \quad \varepsilon = \eta m\varkappa, \qquad /7.4/$$

from which, we determine the roots

$$\left.\begin{matrix} k_1^2 \\ k_2^2 \end{matrix}\right\} = \frac{1}{2}\left\{\sigma^2+q(1+\varepsilon) \pm [(\sigma^2+q(1+\varepsilon))^2 - 4q\sigma^2]^{1/2}\right\}$$

These roots are the functions of the parameter ε: $k_1 = k_1(\varepsilon)$, $k_2 = k_2(\varepsilon)$. For $\varepsilon = 0$, we have

$$k_1(0) = \lambda_1 = \sigma, \qquad k_2(0) = \lambda_2 = \sqrt{q}$$

The following functions are the solutions of the two first Eqs. /7.2/

$$u_1 = u_+^0 \exp[-i\omega t + ik_1x_1] + u_-^0 \exp[-i\omega t - ik_1x_1] + \\ + \frac{m i k_2}{\sigma^2 - k_2^2}\{\theta_+^0 \exp[-i\omega t + ik_2x_1] - \theta_-^0 \exp[-i\omega t - ik_2x_1]\}, \qquad /7.5/$$

$$\theta = \theta_+^0 \exp[-i\omega t + ik_2x_1] + \theta_-^0 \exp[-i\omega t - k_2 i x_1] + \\ + \frac{\eta \varkappa q i k_1}{k_1^2 - q}\{u_+^0 \exp[-i\omega t + ik_1x_1] - u_-^0 \exp[-i\omega t - ik_1x_1]\}.$$

The transverse waves are given by relations

$$u_2 = B_+ \exp\left[-i\omega\left(t - \frac{x_1}{c_2}\right)\right] + B_- \exp\left[-i\omega\left(t + \frac{x_1}{c_2}\right)\right],$$

$$u_3 = C_+ \exp\left[-i\omega\left(t - \frac{x_1}{c_2}\right)\right] + C_- \exp\left[-i\omega\left(t + \frac{x_1}{c_2}\right)\right]. \qquad /7.6/$$

They move with the constant velocity $c_2 = \left(\frac{\mu}{\varrho}\right)^{1/2}$.

These waves do not cause volume change and do not produce the temperature field accompanying the wave motion.

The set of Eqs. /7.5/ will be called the equations of thermoelastic waves. The first Eq. /7.5/ presents a longitudinal wave, the second - the temperature accompanying to these waves. Denoting by v_β $(\beta=1,2)$ the phase velocity, and by ϑ_β the damping coefficient and combining them with the roots of Eq. /7.4/ by means of relations

$$v_\beta = \frac{\omega}{Re(k_\beta)}\,, \quad \vartheta_\beta = Im(k_\beta), \quad \beta = 1,2,$$

we transform Eq. /7.5/ into the form

$$\begin{aligned} v_1 = {} & u_+^\circ \exp\left[-i\omega\left(t-\frac{x_1}{v_1}\right)-\vartheta_1 x_1\right] + u_-^\circ \exp\left[-i\omega\left(t+\frac{x_1}{v_1}\right)+\vartheta_1 x_1\right] + \\ & + \frac{mik_2}{\sigma^2-k_2^2}\left\{\theta_+^\circ \exp\left[-i\omega\left(t-\frac{x_1}{v_2}\right)-\vartheta_2 x_1\right] - \right. \\ & \left. - \theta_-^\circ \exp\left[-i\omega\left(t+\frac{x_1}{v_2}\right)+\vartheta_2 x_1\right]\right\}, \end{aligned} \qquad /7.7/$$

$$\begin{aligned} \theta = {} & \theta_+^\circ \exp\left[-i\omega\left(t-\frac{x_1}{v_2}\right)-\vartheta_2 x_1\right] + \theta_-^\circ \exp\left[-i\omega\left(t+\frac{x_1}{v_2}\right)+\vartheta_2 x_1\right] + \\ & + \frac{\eta\varkappa q i k_1}{k_1^2-q}\left\{u_+^\circ \exp\left[-i\omega\left(t-\frac{x_1}{v_1}\right)-\vartheta_1 x_1\right] - \right. \\ & \left. - u_-^\circ \exp\left[-i\omega\left(t+\frac{x_1}{v_1}\right)+\vartheta_1 x_1\right]\right\}. \end{aligned}$$

It is seen that both waves are damped and subjected to dispersion because the phase velocities v_β depend on frequencies ω. The physical meaning of the waves /7.7/ is clear if we compare them with waves in a hypothetical medium characterized by the zero value of linear expansion α_t. For $\alpha_t = 0$, and then for $\eta_t = 0,\ m = 0$ the two first of Eqs. /7.2/ become

$$(\partial_1^2 + \sigma^2)\hat{u}_1^* = 0, \quad (\partial_1^2 + q)\hat{\theta}^* = 0. \qquad /7.8/$$

The solutions for these equations take the form

$$\hat{u}_1^* = u_+^\circ \exp\left[-i\omega\left(t - \frac{x_1}{c_1}\right)\right] + u_-^\circ \exp\left[-i\omega\left(t + \frac{x_1}{c_1}\right)\right], \qquad /7.9/$$

$$\hat{\theta}^* = \theta_+^\circ \exp\left[-i\omega\left(t - \frac{x_1}{\hat{v}_2}\right) - \hat{\vartheta}_2 x_1\right] + \theta_-^\circ \exp\left[-i\omega\left(t + \frac{x_1}{\hat{v}_2}\right) + \hat{\vartheta}_2 x_1\right],$$

where

$$\hat{v}_2 = (2\varkappa\omega)^{1/2}, \quad \hat{\vartheta}_2 = \left(\frac{\omega}{2\varkappa}\right)^{1/2}.$$
$$c_1 = \left(\frac{\lambda_T + 2\mu_T}{\rho}\right)^{1/2}.$$

Here $\hat{u}_1^*$ represents the wave purely elastic moving in the direction of the x_1 -axis or $-x_1$ -axis with the constant velocity $\hat{v}_1 = c_1$. These waves are subjected neither damping nor dispersion. The second one of Eqs. /7.9/ represents the wave purely thermal undergoing damping and dispersion. The damping is characterized by the coefficient $\hat{\vartheta}_2 = \operatorname{Im}(\lambda_2) = \left(\frac{\omega}{2\varkappa}\right)^{1/2}$.

Dispersion takes here place since the phase velocity

$\hat{v}_2 = \frac{\omega}{Re(\lambda_2)} = (2\varkappa\omega)^{1/2}$ is a function of the frequency ω.

Eqs. /7.7/ describe the modified longitudinal wave and the modified thermal wave. From comparing /7.7/ and /7.9/, it results that the root $k_1(\varepsilon)$ characterizes the quasi-elastic form of a thermoelastic wave, since $k_1(0) = \sigma = \omega/c_1$ refers to the wave purely elastic. Similarly, the root $k_2(\varepsilon)$ characterizes the form of quasi-thermal wave, whereas $k_2(0) = \lambda_2 = \sqrt{q}$ concerns the purely thermal waves in a hypothetic medium. It is interesting fact that in the modified elastic wave /the first equation of the group /7.9/ / there appear closely each other the quasi-elastic terms

$$u_r^c \exp\left[-i\omega\left(t - \frac{x_1}{v_1}\right) - v_1^q x_1\right], \quad u_-^c\left[-i\omega\left(t + \frac{x_1}{v_1}\right) + v_1^q x_1\right],$$

and the quasi-thermal terms

$$\theta_r^c \exp\left[-i\omega\left(t - \frac{x_1}{v_2^t}\right) - v_2^q x_1\right], \quad \theta_-^c \exp\left[-i\omega\left(t + \frac{x_1}{v_2^t}\right) + v_2^q x_1\right]$$

The similar situation exists in the modified thermal wave. Moreover, we should discuss the roots k_1, k_2 or the quantities v_β^t, v_β^q, $\beta = 1, 2$. Introducing new denotations

$$\zeta = \frac{c_1}{\omega^*} k, \quad \omega^* = \frac{c_1^2}{\varkappa}, \quad \chi = \frac{\omega}{\omega^*}$$

we lead the Eq. /7.4/ to the simple form

$$\xi^4-\xi^2[\chi^2+i\chi(1+\varepsilon)]+i\chi^3=0. \qquad /7.10/$$

The roots ξ_1, ξ_2 of this equation are the functions of parameters ε and $\chi=\frac{\omega}{\omega^*}$. The quantity $\varepsilon=\eta m\varkappa$ is a constant depending on thermal and mechanical properties of materials /whereas the χ changes together with a change of frequency ω). The quantity ω^* is a characteristic quantity for given material.

The frequency of forced vibrations ω is limited by the quantity

$$\omega_c = 2\pi(c_1)_s\left(\frac{3\varrho}{4\pi M}\right)^{1/2}$$

resulting from the Debye's spectrum for longitudinal waves [35]. In this formula, M denotes the atomic mass of a material constituting an elastic body, and $(c_1)_s = \left(\frac{\lambda_s+2\mu_s}{\varrho}\right)^{1/2}$ where λ_s, μ_s are the Lamé's constants for an adiabatic state.

The fundamental values for four metals are set in the table

	Aluminium	Copper	Steel	Lead
$(c_1)_s$ cm/sek.	$6{,}32\times10^5$	$4{,}36\times10^5$	$5{,}80\times10^5$	$2{,}14\times10^5$
ε	$3{,}56\times10^{-2}$	$1{,}68\times10^{-2}$	$2{,}97\times10^{-4}$	$7{,}33\times10^{-2}$
ω^* sek.$^{-1}$	$4{,}66\times10^{11}$	$1{,}73\times10^{11}$	$1{,}75\times10^{12}$	$1{,}91\times10^{11}$
ν_1^{ω} cm^{-1}	$1{,}31\times10^4$	$3{,}29\times10^3$	$4{,}48\times10^2$	$3{,}27\times10^4$
ω_c sek.$^{-1}$	$9{,}80\times10^{13}$	$7{,}55\times10^{13}$	$9{,}95\times10^{13}$	$3{,}69\times10^{13}$

In the table, there is also given the damping coefficient ϑ_1^∞ for $\chi = \infty$ where $\vartheta_1^\infty = \frac{1}{2}\frac{\varepsilon\omega^*}{(c_1)_T}$.

Let us note that ω_c is considerably greater than ω^*. In the laboratory tests performed with the help of ultrosonic vibration of very high frequency there is

$$\omega_c > \omega^* \gg \omega,$$

so that for mechanical vibration encountered in practice it can be assumed that $\chi = \frac{\omega}{\omega^*} \ll 1$.

The graphs of ratios $v_1/(c_1)_T$ and $\vartheta_1/\vartheta_1^\infty$ versus the variable $\chi = \omega/\omega^*$ for copper [33] are shown in Figs. 1 and 2.

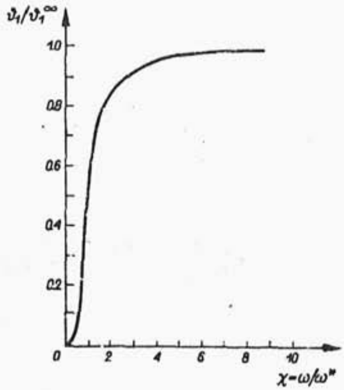

Fig. 1

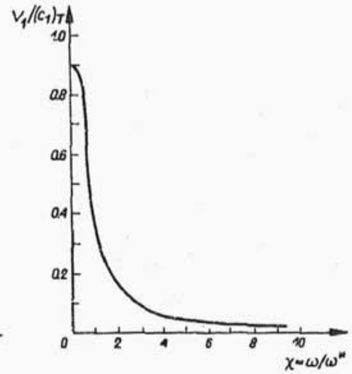

Fig. 2

It is seen from the Fig.1 that the phase velocity v_1 is greater than $(c_1)_T$ and tends to this value as $\chi \to \infty$. The damping coefficient ϑ_1 increases together with χ and at small frequencies it is proportional to χ^2 approaching the asymptotic value ϑ_1^∞. In the neighbourhood of abscissa $\chi=1$ $(\omega=\omega^*)$, the quantities change abruptly. But for the practical application of the theory, we take into account only a small region of variability of $\chi=\omega/\omega^*$. Therefore, for $\chi\ll 1$, the roots ξ_1, ξ_2 can be expanded into power series in χ and we can employ the relation

$$\xi_\beta = (c_1)_T\left(\frac{\chi}{v_\beta}+i\frac{v_\beta^*}{\omega^*}\right), \quad \beta=1,2.$$

In this way, we can obtain approximate values of phase velocities and damping coefficients. We present them according to P.Chadwick [36]

$$v_1 = c_1(1+\varepsilon)^{1/2}\left[1-\frac{\chi^2\varepsilon(4-3\varepsilon)}{8(1+\varepsilon)^4}+O(\chi^4)\right], \qquad /7.11/$$

$$v_1^* = \frac{\omega^*}{c_1(1+\varepsilon)^{1/2}}\left[\frac{\chi^2\varepsilon^2}{2(1+\varepsilon)}+O(\chi^4)\right],$$

$$v_2 = c_1\left(\frac{2\chi}{1+\varepsilon}\right)^{1/2}\left[1-\frac{\chi\varepsilon}{2(1+\varepsilon)^2}+\frac{\chi^2\varepsilon(4+\varepsilon)}{8(1+\varepsilon)^4}+\frac{\chi^3\varepsilon(8-20\varepsilon+\varepsilon^2)}{16(1+\varepsilon)^6}+O(\chi^4)\right],$$

$$v_2^* = \frac{\omega^*}{c_1}\left(\frac{\chi}{2}(1+\varepsilon)\right)^{1/2}\left[1-\frac{\chi\varepsilon}{2(1+\varepsilon)^2}+\frac{\chi^2\varepsilon^2(4+\varepsilon)}{8(1+\varepsilon)^4}+\frac{\chi^3\varepsilon(8-12\varepsilon+\varepsilon^2)}{16(1+\varepsilon)^6}+O(\chi^4)\right]$$

It seen that for $\chi\ll 1$, $v_1\approx c_1(1+\varepsilon)^{1/2}$ can be considered as a constant value slightly greater than $c_1=(c_1)_T$ and the quasi elastic longitudinal wave can be treated as damped but not subjected to dispersion.

We shall present below the solution of a very simple example of plane wave when plane heat source acts with the intensity Q_0 . This source changes harmonically in time and operates in the plane $x_1 = 0$. We get here

$$v_1 = \frac{m Q_0}{2\varkappa} Re\left\{\frac{1}{k_1^2-k_2^2}\left[\exp\left(-i\omega\left(t-\frac{x_1}{v_1}\right)-\vartheta_1 x_1\right) - \right.\right.$$
$$\left.\left. - \exp\left(-i\omega\left(t-\frac{x_1}{v_2}\right)-\vartheta_2 x_1\right)\right]\right\}, \quad x_1 > 0 \qquad /7.12/$$

$$\theta = \frac{Q_0}{2\varkappa} Re\left\{\frac{1}{k_1^2-k_2^2}\left[\frac{k_2^2-\sigma^2}{ik_2}\exp\left(-i\omega\left(t-\frac{x_1}{v_2}\right)-\vartheta_2 x_1\right) - \right.\right.$$
$$\left.\left. - \frac{k_1^2-\sigma^2}{ik_1}\exp\left(-i\omega\left(t-\frac{x_1}{v_1}\right)-\vartheta_1 x_1\right)\right]\right\}, \quad x_1 > 0.$$

The phase velocities v_β and the damping coefficients ϑ_β are taken from the formulae /7.11/.

If we neglect the coupling of deformation and temperature fields, i.e. if in the thermal conductivity equation we delete the term $\eta\dot{\varepsilon}_{kk}$, then inserting $k_1(0) = \sigma$, $k_2(0) = \sqrt{q}$ instead of $k_1(\varepsilon)$, $k_2(\varepsilon)$, we obtain from /7.12/ the approximate solution of the thermal stresses theory

$$\theta = \frac{m Q_0}{2\varkappa} Re\left\{\frac{i}{\sqrt{q}}\exp\left(-i\omega\left(t-\frac{x_1}{\sqrt{2\varkappa\omega}}\right)-x_1\sqrt{\frac{\omega}{2\varkappa}}\right)\right\} \qquad /7.13/$$

$$u_1 = \frac{m Q_0}{2\varkappa} Re\left\{\frac{1}{\sigma^2-q}\left[\exp\left(-i\omega\left(t-\frac{x_1}{c_1}\right)\right) - \right.\right.$$
$$\left.\left. - \exp\left[-i\omega\left(t-\frac{x_1}{\sqrt{2\varkappa\omega}}\right)-x_1\sqrt{\frac{\omega}{2\varkappa}}\right]\right]\right\}$$

The displacement u_1 is composed of two parts: of the undamped elastic wave moving with the velocity c_1 and of the diffusion wave damped and subjected to dispersion.

So far, a number of particular problems concerning the propagation of plane waves in elastic space and semi-space has been solved. Namely, I.N.Sneddon [37] studied the propagation of wave in a semi-infinite and infinite rod with the assumption of various boundary conditions, and thereby various causes inducing waves. The author considered forced vibration for a finite rod. W.Nowacki [38] considered the action of plane body forces in an unbounded space and the action of plane heat sources excitating vibration in the thermoelastic layer [38].

The interesting result is here that the phenomenon of resonance does not arise for forced vibration. It follows from the character of wave motion which is damped. For forced vibration we have the amplitudes with finite values. Namely, for the case of layer of the thickness ℓ which is free of stresses and temperature in the planes bounding the layer $x_1 = 0, \ell$ subjected to the action of heat sources $Q = Q^* \cos\omega t$ we obtain the following expression for the stress

$$\sigma_{11} = \frac{m\rho\omega^2}{\varkappa} \sum_{n=1}^{\infty} \frac{Q_n^* \{\alpha_n^2(\alpha_n^2 - \sigma^2)\cos\omega t - \xi[\alpha_n^2(1+\varepsilon) - \sigma^2]\sin\omega t\}}{\alpha_n^4(\alpha_n^2 - \sigma^2) + \xi^2[\alpha_n^2(1+\varepsilon) - \sigma^2]^2} \times \qquad /7.14/$$

$$\times \sin\alpha_n x_1$$

where

$$\xi = \frac{\omega}{\varkappa}, \quad \alpha_n = \frac{n\pi}{a}, \quad \Theta_n^* = \frac{2}{a}\int_0^a \Theta^*(x_1)\sin\alpha_n x_1\, dx_1$$

We shall not obtain here resonance, since the denominater under the sum sign is always positive. In the particular case $\alpha_r^2 = \sigma^2$ corresponding to the resonance for uncoupled problem, the r-th term of this series can be written as

$$\sigma_{11}^{(r)} = -\frac{\xi\omega m}{\varepsilon}\sin\omega t\,\frac{\Theta_r^*\sin\alpha_r x_1}{\alpha_r^2} \qquad /7.15/$$

This term possesses a finite value altough a magnitude of stress $\sigma_{11}^{(r)}$ will be considerable because the ε is for metals of the order of several percents.

8. Spherical and cylindrical waves

Let us consider the wave equation characterizing longitudinal thermoelastic waves which was derived in Sec. 3 /formulae /3.9/ and /3.11/

$$\Box_1^2\phi = m\vartheta, \qquad /8.1/$$

$$D\vartheta - \eta\nabla^2\dot{\phi} = 0. \qquad /8.2/$$

If we assume that the wave motion changes harmonically in time, then if

$$\phi(x,t) = \phi^*(x)e^{-i\omega t}, \qquad \vartheta(x,t) = \vartheta^*(x)e^{-i\omega t}$$

then from Eqs. /8.1/ and /8.2/, we obtain the following equations

$$\left(\nabla^2+k_1^2\right)\left(\nabla^2+k_2^2\right)\left(\Phi^*,\Theta^*\right)=0, \qquad /8.3/$$

where the quantities k_1, k_2 are the roots of Eq. /7.4/ discussed in the preceding section.

Let us consider these solutions for Eq. /8.3/ which are characterized by singularity at the point ξ and which depend on radius r, distance between the points x and ξ. These solutions which will be denoted by $\varphi_\alpha^*(r)$ satisfy the equations

$$\frac{d^2\varphi_\alpha^*}{dr^2}+\frac{n-1}{r}\frac{d\varphi_\alpha^*}{dr}+k_\alpha^2\varphi_\alpha^*=0, \qquad /8.4/$$
$$\alpha=1,2.$$

Here $n=3$ refers to a three-dimensional problem, $n=2$ to a two-dimensional problem. In Eq. /8.4/, the summation with respect to the index α should not be performed.

The general solution of Eq. /8.4/ takes the form

$$\varphi_\alpha^*(r)=\frac{1}{r^m}\left[AH_m^{(1)}(k_\alpha r)+BH_m^{(2)}(k_\alpha r)\right], \qquad /8.5/$$
$$m=\frac{n-2}{2}$$

Here $H_m^{(1)}$ and $H_m^{(2)}$ are the Hankel's functions of m-th order and of the first and the second kind.

For $n=3$ /then for $m=\frac{1}{2}$ / we have

$$H_{1/2}^{(1)}(k_\alpha r)=i\sqrt{\frac{2}{\pi k_\alpha}}\,\frac{e^{ik_\alpha r}}{r},$$

$$H_{1/2}^{(2)}(k_\alpha r)=-i\sqrt{\frac{2}{\pi k_\alpha}}\,\frac{e^{-ik_\alpha r}}{r}, \quad \alpha=1,2,$$

and the function

$$\varphi_\alpha^*(r) = A_1 \frac{e^{ik_\alpha r}}{r} + A_2 \frac{e^{-ik_\alpha r}}{r}, \quad r^2 = (x_j - \xi_j)(x_j - \xi_j), \; j = 1,2,3 \qquad /8.6/$$

becomes the solution of Eq. /8.4/.

In an unbounded thermoelastic space, we take into account only the first term of Eq. /8.6/, since the solution

$$Re\left[e^{-i\omega t}\frac{e^{ik_\alpha r}}{r}\right] = \frac{e^{-\vartheta_\alpha}}{r}\cos\omega\left(t - \frac{r}{v_\alpha}\right)$$

$$v_\alpha = \frac{\omega}{Re(k_\alpha)}, \quad \vartheta_\alpha = Im(k_\alpha),$$

represents a divergent wave propagating with the adopted phase from the system origin $r = 0$ to infinity. Only this solution has the physical sence. For a cylindrical wave for $n = 2$ and $m = 0$ we get

$$\varphi_\alpha^*(r) = A H_0^{(1)}(k_\alpha r) + B H_0^{(2)}(k_\alpha r), \quad r^2 = (x_j - \xi_j)(x_j - \xi_j), \qquad /8.7/$$
$$j = 1,2.$$

Here, we take into account only the first term of /8.7/ for un unbounded medium since for high values of the argument, we obtain

$$Re\left[e^{-i\omega t} H_0^{(1)}(k_\alpha r)\right] \to \sqrt{\frac{2}{\pi r k_\alpha}}\cos\left(k_\alpha r - \frac{\pi}{4} - \omega t\right)\left[1 + O(r^{-1})\right], \qquad /8.8/$$

representing a diverent wave propagating in the direction of increasing r.

In the expression /8.8/ the symbol $O(r^{-\alpha})$ denotes the value X which is such that the ratio X/r^{α} remains bounded as $r \to \infty$. The solutions here presented $\frac{e^{ik_\alpha r}}{r}$, $H_0^{(1)}(k_\alpha r)$ should satisfy in infinity the so-called emission conditions [38, 39 and 40]:

$$n=3: \ \frac{\partial}{\partial r}\left(\frac{e^{ik_\alpha r}}{r}\right) - ik_\alpha \frac{e^{ik_\alpha r}}{r} = e^{ik_\alpha r} O(r^{-2}), \ \vartheta_\alpha > 0, \qquad /8.9/$$

$$n=2: \ \frac{\partial}{\partial r}\left(H_0^{(1)}(k_\alpha r)\right) - ik_\alpha H_0^{(1)}(k_\alpha r) = e^{ik_\alpha r} O(r^{-3/2}), \ \vartheta_\alpha > 0,$$

$$\alpha = 1,2.$$

These formulae inform about the behaviour of fundamental solutions in the neighbourhood of a point removed infinitely.

If we consider such a class of solutions for Eqs. /8.3/ which behave in infinity similarly as the fundamental solutions $\frac{e^{ik_\alpha r}}{r}$, $H_0^{(1)}(k_\alpha r)$ then we should require the satisfaction, by the functions $\phi^* = \phi_1^* + \phi_2^*$ of the following conditions in infinity

$$n=3: \ \frac{\partial \phi_\alpha^*}{\partial r} - ik_\alpha \phi_\alpha^* = e^{ik_\alpha r} O(r^{-2}), \ \vartheta_\alpha \geq 0, \qquad /8.10/$$

$$n=2: \ \frac{\partial \phi_\alpha^*}{\partial r} - ik_\alpha \phi_\alpha^* = e^{ik_\alpha r} O(r^{-3/2}), \ \vartheta_\alpha > 0,$$

$$\alpha = 1,2.$$

To these solutions we should add the condition on a finite value of the function

$$\phi_\alpha^* = O(1) \qquad \text{for } r \to \infty$$

where the symbol $O(1)$ denotes a value arbitrarily small.

Longitudinal spherical waves are obtained only for a special choice of disturbances. They arise owing to the action of heat sources and body forces of the potential origin, both in unbounded and bounded medium with a spherical void with the boundary conditions characterized by the symmetry with respect to a point.

Let us consider one of those cases, namely, the action of the concentrated heat source $Q_0 e^{-i\omega t}\delta(r)$. We assume the following form of the solution for Eq. /8.3/

$$\phi^* = \frac{1}{r}\left(A_1 e^{ik_1 r} + A_2 e^{ik_2 r}\right) \qquad /8.11/$$

where constants A_1, A_2 will be determined from the condition of requirement in order that the heat flow through the surface of sphere $r \to 0$ be equal to the heat source intensity, and in order that $u_1^* = \frac{\partial \phi^*}{\partial r}$ be equal to zero for $r = 0$.

In consequence we obtain for the functions ϕ^*, θ^* the following formulae [41]

$$\phi^* = \frac{m Q_0}{4\pi\varkappa r(k_2^2 - k_1^2)}\left\{\exp\left[-i\omega\left(t - \frac{r}{v_1}\right) - \vartheta_1 r\right] - \exp\left[-i\omega\left(t - \frac{r}{v_2}\right) - \vartheta_2 r\right]\right\}, \qquad /8.12/$$

$$\theta^* = \frac{Q_0}{4\pi\varkappa r(k_2^2 - k_1^2)}\left\{(k_2^2 - \sigma^2)\exp\left[-i\omega\left(t - \frac{r}{v_2}\right) - \vartheta_2 r\right] - (k_1^2 - \sigma^2)\exp\left[-i\omega\left(t - \frac{r}{v_1}\right) - \vartheta_1 r\right]\right\}$$

Here ϑ_i is a damping coefficient, v_i a phase velocity of the wave. The functions Φ_1^*, θ^* are damped, subjected to dispersion, satisfy emission conditions and exhibit a singularity at the point $r=0$

Having known the function Φ^*, we are able to determine radial displacement $u_R = \frac{\partial \Phi}{\partial R}$. For $\theta_0 = 1$ the formulae /3.12/ become the Green's functions for the potential $\hat{\Phi}^*$ and temperature $\hat{\theta}^*$. If the distribution of sources $Q(x,t) = Q^*(x)e^{-i\omega t}$ is given in a bounded region V_1 , the potential is expressed by the formula

$$\Phi^*(x,\omega) = \int_{V_1} Q^*(\xi)\,\hat{\Phi}^*(x,\xi,\omega)\,dV(\xi). \qquad /8.13/$$

Till then, a number of particular cases has been solved referring to spherical waves. They concern the action of compression centre in an unbounded region and the space with void with the assumption of various boundary conditions characterized by a spherical symmetry [41] [38] .

A number of theorems has been developed for spherical waves. They can be thought of as an extension of the Helmholtz's theorem for elastokinetics and the analogous theorem of the thermal conductivity theory for the problems of thermoelasticity [31] . The idea of this theorem is the following. The system of equations is given

$$(\nabla^2+\sigma^2)u^* - m v^* = 0, \qquad (\nabla^2+q)v^* + \frac{q\varepsilon}{m}\nabla^2 u^* = 0, \qquad /8.14/$$

which is regular in the considered region B . Here u^* denotes the potential of thermoelastic displacement, v^* temperature. The elimination of functions u^* or v^* from Eqs. /8.14/ lead – to the equation of the type /8.3/.

It can be exhibited that if the function $u^*, v^*, \frac{\partial u^*}{\partial n}, \frac{\partial v^*}{\partial n}$ are given on the boundary A of the region B , then the function v^* at a point $x \in B$ can be written as

$$v^*(x) = \varkappa \int_A \left[\theta^*(\xi, x) \frac{\partial v^*(\xi)}{\partial n} - v^*(\xi) \frac{\partial \theta^*(\xi, x)}{\partial n}\right] dA(\xi) + \qquad /8.14'/$$

$$+ \frac{\varepsilon q \sigma^2 \varkappa}{m^2} \int_A \left[\phi^*(\xi, x) \frac{\partial u^*(\xi)}{\partial n} - u^*(\xi) \frac{\partial \phi^*(\xi, x)}{\partial n}\right] dA(\xi),$$
$$\xi \in B$$

In this case, the functions $\phi^*(x, \xi)$, $\theta^*(x, \xi)$ are the solutions of equations

$$(\nabla^2 + \sigma^2)\phi^* - m\theta^* = 0, \quad (\nabla^2 + q)\theta^* + \frac{q\varepsilon}{m}\nabla^2\phi^* = -\frac{1}{\varkappa}\delta(x - \xi), \qquad /8.15/$$
$$\xi \in B,$$

where

$$\phi^* = \frac{m}{4\pi\varkappa(k_2^2 - k_1^2)r}\left(e^{ik_1 r} - e^{-ik_2 r}\right),$$
$$\theta^* = \frac{1}{4\pi\varkappa(k_2^2 - k_1^2)r}\left(n_2 e^{ik_2 r} - n_1 e^{ik_1 r}\right), \qquad /8.16/$$
$$n_\alpha = k_\alpha^2 - \sigma^2, \quad \alpha = 1, 2$$

For $x = E - B$, where E is a whole space, there is $v^*(x) = 0$. For an uncoupled problem $(\varepsilon = 0)$ i.e. for the theory of thermal stresses, the second integral of Eq. /8.14/ disappears. In consequence, we obtain the equation

$$v^*(x) = \frac{1}{4\pi}\int_A \left[v^*(\xi)\frac{\partial}{\partial n}\left(\frac{e^{ir\sqrt{q}}}{r}\right) - \frac{e^{ir\sqrt{q}}}{r}\frac{\partial v^*(\xi)}{\partial n}\right] dA(\xi), \quad r = r(x,\xi); \qquad /8.17/$$

so it is the theorem familiar in the thermal conductivity theory. For the function $u^*(x)$ we obtain the following formula

$$u^*(x) = \varkappa\int_A \left[\phi^*(\xi,x)\frac{\partial v^*(\xi)}{\partial n} - v^*(\xi)\frac{\partial \phi^*(\xi,x)}{\partial n}\right] dA(\xi) + \qquad /8.18/$$

$$+ \frac{\varkappa}{m}\int_A \left[\Box_k^2\phi^*(\xi,x)\frac{\partial u^*(\xi)}{\partial n} - u^*(\xi)\frac{\partial}{\partial n}\left(\Box_k^2\phi^*(\xi,x)\right)\right] dA(\xi), \quad x\in B,$$

$$u^*(x) = 0, \quad x \in \xi - B$$

The symbol $\Box_k^2 = \nabla^2 + \sigma_1^2 + k_0^2$ is introduced in this formula. The formula /8.18/ is expressed through the function $u^*(x)$ inside the region B by means of the function

$$u^*(\xi), \quad \frac{\partial u^*(\xi)}{\partial n}, \quad v^*(\xi), \quad \frac{\partial v^*(\xi)}{\partial n},$$

on the surface A. When going on from thermoelasticity to elastokinetics, we obtain from /8.18/ after a number of transformations, the Helmholtz's familiar theorem [42]

$$u^*(x) = \begin{cases} \frac{1}{4\pi}\int_A \left[\frac{e^{i\sigma_1 r}}{r}\frac{\partial u^*}{\partial n} - u^*\frac{\partial}{\partial n}\left(\frac{e^{i\sigma_1 r}}{r}\right)\right] dA(\xi) & x\in B \\ 0 \quad \text{if} & x \in \xi - B. \end{cases} \qquad /8.19/$$

Here

$$q_1 = \frac{\omega}{(c_1)_s} \qquad \text{where} \quad (c_1)_s = \left(\frac{\lambda_s + 2\mu_s}{\rho}\right)^{1/2}.$$

Spherical waves can arise in the case of linear heat source or linear compression centre, or in an unbounded thermoelastic medium with a cylindrical void on the boundary of which heating, pressure or deformation takes place and it is distributed in a axial-symmetrical way.

Of numerous solutions [38, 41, 43], we referring to a linear heat source $Q(r,t) = Q_0 e^{-i\omega t}\frac{\delta(r)}{2\pi r}$, $r = (x_1^2 + x_2^2)^{1/2}$

For the amplitudes of displacement thermoelastic potential and for temperature we obtain the following formulae / 41 / .

$$\phi^* = \frac{Q_0 m i}{4\varkappa(k_2^2 - k_1^2)}\left[H_0^{(1)}(k_1 r) - H_0^{(1)}(k_2 r)\right], \qquad /8.20/$$

$$\theta^* = \frac{Q_0 i}{4\varkappa(k_2^2 - k_1^2)}\left[(\sigma^2 - k_1^2)H_0^{(1)}(k_1 r) - (\sigma^2 - k_2^2)H_0^{(1)}(k_2 r)\right]$$

These functions satisfy the emission conditions. They are damped and subjected to dispersion.

9. The Green's functions for an unbounded thermoelastic medium. The singular integral equation of thermoelasticity

In the preceding section, we have presented the Green's functions for point linear heat source. They satisfy the equations

$$\hat{\sigma}_{ji,j} = -\omega^2 \rho \hat{u}_i ,$$
$$\hat{\theta}_{,kk} + h_3^2 \hat{\theta} + \frac{\mu}{\alpha} \hat{u}_{k,k} = -\frac{1}{\varkappa} \delta(x-\xi) \qquad /9.1/$$
$$\alpha = \frac{m\mu}{\rho h_3^2}, \quad h_3 = i\left(\frac{i\omega}{\varkappa}\right)^{1/2}$$

By $\hat{u}, \hat{\theta}$ we denote here the amplitudes of displacements and temperature. In turn we should determine the Green's functions for a concentrated force. Let, at the point ξ of an unbounded region, the concentrated force $X_i = \delta(x-\xi)\delta_{i1} e^{-i\omega t}$ act which is directed towards the x_1-axis. The action of this forces produces both longitudinal and transverse waves. We should solve the system of equations

$$\sigma_{ji,j}^{(1)} = -\omega^2 \rho u_i^{(1)} - \delta(x-\xi)\delta_{i1}, \qquad /9.2/$$
$$\theta_{,kk}^{(1)} + h_3^2 \theta^{(1)} + \frac{\mu}{\alpha} u_{k,k}^{(1)} = 0,$$

in which we have denoted by $\sigma_{ji}^{(1)}, u_i^{(1)}, \theta^{(1)}$ the amplitudes of stresses, displacements and the temperature caused by the action of the concentrated force applied at the point ξ and directed towards the x_1-axis. The system of Eq. /9.2/ can be replaced by the system of wave equations

$$(\nabla^2 + k_1^2)(\nabla^2 + k_2^2)\Phi^{(1)} = -\frac{1}{c_1^2}(\nabla^2 + q)\vartheta, \qquad /9.3/$$

$$(\nabla^2 + \tau^2)\psi_i^{(1)} = -\frac{1}{c_2^2}\chi_i, \quad i = 1,2,3 \qquad /9.4/$$

These equations follow from Eqs. /9.2/ under the assumption that

$$u^{(1)} = \operatorname{grad}\Phi + \operatorname{rot}\psi^{(1)}, \quad X = \rho(\operatorname{grad}\vartheta + \operatorname{rot}\chi). \qquad /9.5/$$

The amplitudes of body forces are determined from the formulae [44]

$$\vartheta(x) = -\frac{1}{4\pi\varrho}\int_V X(x')\,\mathrm{grad}_x\left(\frac{1}{r(x',x)}\right)dV(x'),$$

$$\chi(x) = -\frac{1}{4\pi\varrho}\int_V X(x')\times\mathrm{grad}_x\left(\frac{1}{r(x',x)}\right)dV(x'). \qquad /9.6/$$

For the considered here case of the concentrated force directed towards the x_1 -axis, we find

$$\vartheta = -\frac{1}{4\pi\varrho}\partial_1\left(\frac{1}{r}\right), \quad \chi_1 = 0$$

$$\chi_2 = \frac{1}{4\pi\varrho}\partial_3\left(\frac{1}{r}\right), \quad \chi_3 = -\frac{1}{4\pi\varrho}\partial_2\left(\frac{1}{r}\right).$$

From the solutions of Eqs. /9.4/ we obtain

$$\psi_1 = 0, \quad \psi_2 = \frac{1}{4\pi\varrho\omega^2}\partial_3 F_0(r,\omega), \quad \psi_3 = -\frac{1}{4\pi\varrho\omega^2}\partial_2 F_0(r,\omega) \qquad /9.6/$$

where

$$F_0(r,\omega) = \frac{1}{r}\left(e^{i\varepsilon r} - 1\right), \quad r^2 = (x_i - \xi_i)(x_i - \xi_i), \quad i = 1,2,3.$$

From the solution of Eq. /9.3/ with taking into account the fact that the function $\phi^{(1)}$ is characterized by an axial symmetry with respect to the x_1 -axis, we obtain [44] [45]

$$\phi^{(1)} = -\frac{1}{4\pi\varrho\omega^2}\partial_1 F(r,\omega), \qquad /9,7/$$

where

$$F(r,\omega) = A_1 I_1 - A_2 I_2 - I_0$$

$$I_0 = \frac{1}{r}, \quad I_\beta = \frac{e^{ik_\beta r}}{r}, \quad \beta = 1,2$$

$$A_1 = \frac{(k_1^2-q)\sigma^2}{k_1^2(k_1^2-k_2^2)}\,, \qquad A_2 = \frac{(k_2^2-q)\sigma^2}{k_2^2(k_1^2-k_2^2)}\,.$$

The temperature $\theta^{(1)}$ is determined from the formula

$$\theta^{(1)} = \frac{1}{m}(\nabla^2+\sigma^2)\phi^{(1)} + \frac{1}{c_1^2 m}\,\nu\delta. \qquad /9.8/$$

Employing the formulae /9.5/ and /9.8/ we get

$$u_j^{(1)} = -\frac{1}{4\pi\rho\omega^2}\,\partial_1\partial_j\left[F(r,\omega)-F_0(r,\omega)\right] + \frac{1}{4\pi\rho c_2^2}\,\delta_{1j}\,\frac{e^{i\tau r}}{r}\,, \qquad /9.9/$$

$$\theta^{(1)} = \frac{q\varepsilon}{4\pi\rho m c_1^2(k_1^2-k_2^2)}\,\partial_1\left[I_1(r,\omega)-I_2(r,\omega)\right]. \qquad /9.10/$$

These functions have a singularity at the point ξ and satisfy in infinity the emission conditions. If a concentrated force acts in the direction of the x_s -axis, we have the following expression for the Green's displacement tensor u_j^s and for temperature θ^s

$$u_j^s = -\frac{1}{4\pi\rho\omega^2}\left\{\partial_j\partial_s\left[F(r,\omega)-F_0(r,\omega)\right] - \tau^2\delta_{js}\,e^{i\omega\tau}\right\}, \qquad /9.11/$$

$$\theta^s = \frac{q\varepsilon}{4\pi\rho m c_1^2(k_1^2-k_2^2)}\,\partial_s\left[I_1(r,\omega)-I_2(r,\omega)\right],\ j,s=1,2,3 \qquad /9.12/$$

From the found solutions for a concentrated force, we can obtain further singularities, expressions u_i^s, θ^s for a double force, for a concentrated moment and for a centre of compression.

For a two-dimensional problem, we obtain for the force concentrated and directed towards the x_s-axis, the following

Green's functions [46] :

$$v_j^s = -\frac{i}{4\varrho\omega^2}\left\{\partial_j\partial_s\left[A_1 H_0^{(1)}(k_1 r) - A_2 H_0^{(1)}(k_2 r) - H_0^{(1)}(\tau r)\right] - \tau^2\delta_{js} H_0^{(1)}(\tau r)\right\} \quad /9.13/$$

$$\vartheta^s = \frac{q\varepsilon i}{4\varrho m\, a^2(k_1^2 - k_2^2)}\,\partial_s\left[H_0^{(1)}(k_1 r) - H_0^{(1)}(k_2 r)\right], \quad /9.14/$$

$$r^2 = (x_j - \xi_j)(x_j - \xi_j)\,, \quad j, s = 1, 2.$$

Having known displacement functions and temperature for the action of a concentrated heat source and a concentrated force, we are able to construct methods for integrating the thermoelasticity equations for a bounded body [31] .

We introduce the thermoelastic surface potentials analogous to the elastokinetics potentials / 39 /

$$V_s(x) = 2\int_\Sigma d\Sigma(\xi)\,\varphi_k(\xi)\,v_s^k(\xi,x) + 2\alpha\int_\Sigma d\Sigma(\xi)\,\psi(\xi)\,\vartheta^s(\xi,x),$$

$$V(x) = 2\int_\Sigma d\Sigma(\xi)\,\psi(\xi)\,\vartheta(\xi) + \frac{2}{\alpha}\int_\Sigma d\Sigma(\xi)\,\varphi_k(\xi)\,\hat{v}_k(\xi,x). \quad /9.15/$$

Here $\varphi_k(\xi)$, $\psi(\xi)$ are the unknown densities of the corresponding regularity. The functions $\hat{v}_k$, $\hat{\vartheta}$, v_k^s, ϑ^s are the Green's functions satisfying the Eqs. /9.1/ and /9.2/, i.e. they are the known functions. The following system is called the thermoelastic potential of a double layer

$$W_s(x) = 2\int_\Sigma d\Sigma(\xi)\,\varphi_k(\xi)\,p_k^s(\xi,x) + 2\alpha\int_\Sigma d\Sigma(\xi)\,\psi(\xi)\,\frac{\partial\vartheta^s(\xi,x)}{\partial n},$$

$$W(x) = 2\int_\Sigma d\Sigma(\xi)\,\psi(\xi)\,\frac{\partial\hat{\vartheta}(\xi,x)}{\partial n} + \frac{2}{\alpha}\int_\Sigma d\Sigma(\xi)\,\varphi_k(\xi)\,\hat{p}_k(\xi,x). \quad /9.16/$$

The following denotations are introduced here

$$p_k^1(\xi,x) = [2\mu\varepsilon_{kj}^1 + (\lambda u_{p,p}^1 - \mu t^1)\delta_{kj}]n_j,$$

$$\hat{p}_k^1(\xi,x) = [2\mu\hat{\varepsilon}_{kj} + (\lambda\hat{u}_{p,p} - \mu\hat{\theta}^1)\delta_{kj}]n_j.$$

Lastly, we can utilize the thermoelastic potential being the combination of potential of single and double layer

$$W_1(x) = 2\int_\Sigma d\Sigma(\xi)\varphi_k(\xi)\,p_k^1(\xi,x) + 2\alpha\int_\Sigma d\Sigma(\xi)\psi(\xi)\,\theta^0(\xi,x),$$
$$W(x) = 2\int_\Sigma d\Sigma(\xi)\psi(\xi)\,\hat{\theta}(\xi,x) + \frac{2}{\alpha}\int_\Sigma d\Sigma(\xi)\varphi_k(\xi)\,\hat{p}_k(\xi,x). \qquad /9.17/$$

It is exhibited that the potentials $V_1(x)$, $V(x)$ are the continuous functions of the points $x \in \Sigma$. But the potentials of double layer $W_1(x)$, $W(x)$ exhibit discontinuity on this surface. For we have

$$W_1^{(i)}(\xi_0) = -\varphi_1(\xi_0) + W_1(\xi_0), \quad W^{(i)} = -\psi(\xi_0) + W(\xi_0),$$
$$W_1^{(e)}(\xi_0) = \varphi_1(\xi_0) + W_1(\xi_0), \quad W^{(e)}(\xi_0) = \psi(\xi_0) + W(\xi_0). \qquad /9.18/$$

The functions $W_1(\xi_0)$, $W_1^{(i)}(\xi_0)$ and $W_1^{(e)}(\xi_0)$ denote correspondingly the limit of vector $W_1(\xi)$ as $\xi \to \xi_0 \in \Sigma$ on the surface Σ, $W_1(\xi)$ as $\xi \to \xi_0 \in \Sigma$ from the interior of the region V and $W_1(\xi)$ as $\xi \to \xi_0 \in \Sigma$ for $\xi \in \mathcal{E} - V$. It is exhibited that the first surface integral in the formulae /9.16/ represent a discontinuous function, the second integral - a continuous function.

We next introduce the denotations

$$\check{p}_k(x) = [2\mu V_{(i,j)} + \lambda (V_{,kk} - \mu V)\delta_{ij}] n_j(x),$$
$$\check{\vartheta}(x) = V_{,k}\, n_k(x), \qquad /9.19/$$

where V_k, V are defined by the formulae /9.15/. It may be shown that

$$\check{p}_k^{(i)}(\xi_0) = \varphi_k(\xi_0) + \check{p}_k(\xi_0), \quad \check{\vartheta}^{(i)}(\xi_0) = \psi(\xi_0) + \check{\vartheta}(\xi_0); \qquad /9.20/$$
$$\check{p}_k^{(e)}(\xi_0) = -\varphi_k(\xi_0) + \check{p}_k(\xi_0), \quad \check{\vartheta}^{(e)}(\xi_0) = -\psi(\xi_0) + \check{\vartheta}(\xi_0).$$

The thermoelastic potentials /9.15/ - /9.17/ and the relations concerning discontinuities of these potentials allow to reduce the fundamental boundary problems to solving a system of singular integral equations.

Let us consider the case when displacements $u_s(\xi_0) = f_s(\xi_0)$ and temperature $\theta(\xi_0) = g(\xi_0)$ are given on the boundary Σ, then we look for solutions in the form

$$U_s(x) = W_s(x), \quad \theta(x) = W(x).$$

where the functions $U_s(x)$, $\theta(x)$ are given by the formulae /9.16/. We can easily verify that inside the region V, the Eqs.

$$L_{sk} U_k - \mu \partial_s \theta = 0, \quad (\nabla^2 + q)\theta + \frac{\varepsilon}{m} \partial_k U_k = 0, \qquad /9.21/$$
$$x \in V$$

are satisfied, where

$$L_{sk} = (\mu \partial_p \partial_p + \omega^2 \varsigma)\delta_{sk} + (\lambda+\mu)\partial_s \partial_k .$$

Taking into account the relations /9.18/ for the functions $\varphi_k(\xi)$, $\psi(\xi)$ we arrive at the following system of coupled integral equations

$$\varphi_s(\xi_0) - 2\int_\Sigma d\Sigma(\xi)\varphi_k(\xi) p_k^s(\xi,\xi_0) - 2\alpha\int_\Sigma d\Sigma(\xi)\psi(\xi)\frac{\partial\theta^s(\xi,\xi_0)}{\partial n} = -f_s(\xi_0) \qquad /9.22/$$

$$\psi(\xi_0) - 2\int_\Sigma d\Sigma(\xi)\psi(\xi)\frac{\partial\hat\theta(\xi,\xi_0)}{\partial n} - \frac{2}{\alpha}\int_\Sigma d\Sigma(\xi)\varphi_k(\xi)\check{p}_k(\xi,\xi_0) = -g(\xi_0)$$

These equations have the form of singular integral equations of the second kind, and the integrals occurring in them should be thought of in the sense of major values. If, on the boundary Σ , displacements $u_i(\xi_0) = f_i(\xi_0)$ and heat flow $\frac{\partial\theta}{\partial n}\Big|_{\xi=\xi_0} = S(\xi_0)$ are given, then we look for solutions in the form

$$U_s(x) = M_s(x), \quad \theta(x) = M(x), \quad x \in V,$$

where the functions M_s, M are given by the formulae /9.17/. We can easily verify that inside the region V , the Eqs. /9.21/ are satisfied, and the unknown densities fulfill the system of integral singular equations

$$\varphi_s(\xi_0) - 2\int_\Sigma d\Sigma(\xi)\varphi_k(\xi) p_k^s(\xi,\xi_0) - 2\alpha\int_\Sigma d\Sigma(\xi)\theta^s(\xi,\xi_0) = -f_s(\xi_0) \qquad /9.23/$$

$$\psi(\xi_0) + 2\int_\Sigma d\Sigma(\xi)\psi(\xi)\frac{\partial\hat\theta(\xi,\xi_0)}{\partial n_0} + \frac{2}{\alpha}\int_\Sigma d\Sigma(\xi)\varphi_k(\xi)\frac{\partial\hat{p}_k(\xi,\xi_0)}{\partial n_0} = S(\xi_0)$$

where

$$\frac{\partial \hat{\theta}(\xi,\xi_0)}{\partial m_0} = \lim_{x \to \xi_0} \frac{\partial}{\partial m_x} \hat{\theta}(\xi,x) \quad x \in \Sigma.$$

The quantity $\frac{\partial \beta(\xi,\xi_0)}{\partial m_0}$ is defined analogously. Let us note lastly that if loading $p_i = p_i(\xi_0)$ and heat flow $S = S(\xi_0)$ are given on Σ, then the solution should be sought for using the potentials of a single layer $V_i(x)$, $V(x)$. The investigation of existence and uniqueness of the obtained singular equations is carried out in the similar way as it takes places in elastodynamics. The systems of singular integral equations presented here comprise particular cases related to thermal stresses theory, thermal conductivity theory and elastodynamics.

When developing the general theory of propagation of thermoelastic waves changing harmonically in time, there was solved simultaneously a number of particular problems, leading them to the form useful for discussion. They are mostly the problems typical for classical elastokinetics which in the framework of thermoelasticity were extended and generalized. A great deal of attention was devoted to surface waves. This problem was first discussed in the work by F.J.Lockett [47] and then, in broader and more thorough manner, by P.Chadwick and D.W.Windle [49].

When deriving surface waves in a plane state of deformation, we start from the wave equations /for longitudinal and transverse wave/ and from the thermal conductivity equation. The wave travels parallelly to the plane bounding the semi-space and vanishes in greater depth. It is assumed that stresses and temperature, or stresses and heat flow disappear in the plane bounding the space. An algebraic equation of the third order with complex coefficients is obtained from the determinant of the system of equations expressing homogeneous boundary conditions. One of roots of this equation satisfying prescribed inequalities provides a phase velocity of surface wave. It is found that the surface wave undergoes damping and dispersion, its velocity is smaller than velocity of longitudinal and transverse waves.

W.Nowacki and M.Sokołowski have investigated, in the similar way, the propagation of harmonic wave in a thermoelastic layer. The authors considered there both symmetric and antisymmetric /elastic wave/ form of wave for two thermal conditions on the boundary: $\vartheta = 0$ and $\vartheta_{,n} = 0$. Owing to a small value of the parameter characterizing the thermoelastic medium, the approximate solution of transcedental equation has been presented using the perturbation method.

The propagation of harmonic waves in an infinite circular cylinder and thick-walled pipe was studied by F.J.Lockett [50] giving the transcedental equations relevant to this problem. J.Ignaczak and W.Nowacki [52] have considered the forced vibration of an infinite cylinder with rectangular cross-section. Heating the cylinder surface and action of heat sources were here the cause exciting vibration. The same authours presented in [53] the method for solving and solution itself of the problem of the forced longitudinal vibration in discs and of the flexural vibrations produced by loadings and heating in plates. The work by P.Chadwick [54] is devoted for the analogous problems.

The propagation of thermoelastic plane wave in an unbounded medium in spherical and cylindrical wave [40] is the next problem solved. The idea is following. A plane wave induced by the action of plane heat source moves in an unbounded space and encounters spherical or cylindrical void. Flowing around this void the temperature field undergoes a disturbance, the concentration of temperature and stresses takes place in the neighbourhood of the void. The partial solution obtained here is in a closed form and the residual solution is expressed as an infinite system of algebraic equations with complex coefficients.

A pretty big group of solutions corresponds to the so-called Lamb's problem of classical elastokinetics. The question consists in considering the influence of loadings and heatings acting on a thermoelastic semi-space. Two typical problems have been solved here, namely, when loading or heating is axially symmetric and when loading and heating produces a plane state of deformation [43]. Further problems concerning the action of sources of heat /concentrated or linear/ in an elastic semi-space [41] have something in common with those above subjects. However, the solutions of this group are only of formal character - till then scientists failed in obtaining even approximate solutions suitable for discussion.

10. The aperiodic problems of thermoelasticity

The domain of investigation discussed here is a branch of thermoelasticity developed most weakly. This is owing to great mathematic troubles encountered in obtaining solutions.

In general, three ways are used for solving the aperiodic problems of thermoelasticity. The first one consists in eliminating the time t from the differential equations of thermoelasticity

$$\begin{aligned} &\mu u_{i,jj} + (\lambda+\mu) u_{j,ji} + X_i = \rho \ddot{u}_i + \gamma \theta_{,i} , \\ &\theta_{,jj} - \frac{1}{\varkappa}\dot{\theta} - \eta \dot{e} = -\frac{Q}{\varkappa} , \end{aligned} \qquad /10.1/$$

by subjecting these equations to the Laplace's transformations or the Fourier's transformation with respect to time. The former transformation is most frequently applied owing to numerous inverse transformations.

Subjecting then /10.1/ to the Laplace'a transformation defined by relation

$$\mathcal{L}(u_i,\theta) = (\bar{u}_i,\bar{\theta}) = \int_0^\infty (u_i,\theta)\, e^{-pt}\, dt , \quad p>0 ,$$

and assuming the homogeneity of initial conditions, we obtain from /10.1/ the following transformed solutions

$$\mu \bar{v}_{i,jj} + (\lambda+\mu)\bar{v}_{j,ji} + \bar{X}_i = \rho p^2 \bar{v}_i + \gamma \bar{\theta}_{,i} , \qquad /10.2/$$
$$\bar{\theta}_{,jj} - \frac{p}{\varkappa}\bar{\theta} - \eta p \bar{v}_{j,j} = -\frac{\bar{Q}}{\varkappa}$$

Here, the unknown functions $\bar{v}_i, \bar{\theta}$ depend on position and transformation parameter p . Solving the Eqs. /10.2/ is not very difficult for many particular problems ; the troubles are of the same order as in the problems of vibration harmonically changing in time. The essential difficulty consists in performing the Laplace's inverse transformation for the obtained solutions $\bar{v}_i(x,p), \bar{\theta}_i(x,p)$.

The second way of solving consists in subjecting the Eqs. /10.1/ to the Fourier's triple integral transformation with respect to the variables x_i . Thus, the Eqs. /10.1/ are led to a system of ordinary differential equations in which time appears as an independent variable. After solving this equation, the Fourier's triple inverse transformation is accomplished [56] .

The third way eagerly used for thermoelastic space and semi-space consists in applying the Fourier's quadruple transformation. The system of equations /10.1/ is led to a system of four algebraic equations for the transforms $\bar{v}_i$ $\bar{\theta}$. The quadruple inverse transformation provides final result [65] , [66] .

Each of these ways is accompanied by large mathematical troubles; they are so immense that so far no solution is obtained in a closed form.

We shall consider more exactly the wave equation /3.9/ and /3.11/ obtainable from Eqs. /10.1/. If we use the first way of investigation and apply the Laplace's transformation for the wave equations with assumption of homogeneous boundary conditions, then we get the system of equations

$$\left[\left(\nabla^2-\frac{p^2}{c_1^2}\right)\left(\nabla^2-\frac{p}{\varkappa}\right)-\frac{\varepsilon p}{\varkappa}\nabla^2\right]\bar{\varphi}=-\frac{m}{\varkappa}\bar{Q}-\frac{1}{c_1^2}\left(\nabla^2-\frac{p}{\varkappa}\right)\bar{S}, \qquad /10.3/$$

$$\left(\nabla^2-\frac{p^2}{c_2^2}\right)\bar{\psi}_i=-\frac{1}{c_2^2}\bar{X}_i,$$

$$\bar{\theta}=\frac{1}{m}\left(\nabla^2-\frac{p^2}{c_1^2}\right)\bar{\varphi}, \quad \varepsilon=\eta m\varkappa.$$

The equation of longitudinal wave for $Q=0$, $\vartheta=0$ can be presented in the form

$$\left(\nabla^2-\lambda_1^2\right)\left(\nabla^2-\lambda_2^2\right)\bar{\varphi}=0, \qquad /10.4/$$

where λ_1, λ_2 are the roots of bi-quadratic equation

$$\lambda^4-\lambda^2 p\left(\frac{p}{c_1^2}+\frac{1}{\varkappa}(1+\varepsilon)\right)+\frac{p^3}{\varkappa c_1^2}=0.$$

Since the roots of this equation

$$\lambda_{1,2}^2=\frac{1}{2}\left(\frac{p}{\varkappa}(1+\varepsilon)+\frac{p^2}{c_1^2}\pm\left[\left(\frac{p}{\varkappa}(1+\varepsilon)+\frac{p^2}{c_1^2}\right)^2-\frac{4p^3}{\varkappa c_1^2}\right]^{1/2}\right)$$

are expressed in a greatly complicated manner as functions of the parameter [illegible], it is clear that applying the Laplace's inverse transformation for the functions φ, [illegible] encounters great difficulties. Therefore, we are forced to employ [illegible] approximate solutions. In general, two ways of approximate solutions are used. The first consists in taking advantage of the fact that the quantity $\varepsilon = \eta m \varkappa$ is small parameter [36]. Writting then the functions ϕ and θ as a power series in

$$\phi = \phi_0 + \varepsilon\phi_1 + \varepsilon^2\phi_2 + \dots, \quad \theta = \theta_0 + \varepsilon\theta_1 + \varepsilon^2\theta_2 + \dots \qquad /10.5/$$

we lead the Eq. /10.3/ to the system of equations

$$\begin{aligned} D_1 D_2 \bar{\phi}_0 &= -\frac{m}{\varkappa} Q - \frac{1}{c_1^2} D_2 \bar{\vartheta}, \\ D_1 D_2 \bar{\phi}_1 &= \frac{p}{\varkappa} \nabla^2 \bar{\phi}_1 \\ &\dots \end{aligned} \qquad /10.6/$$

where

$$D_1 = \nabla^2 - \frac{p^2}{c_1^2}, \qquad D_2 = \nabla^2 - \frac{p}{\varkappa}.$$

For the temperature $\bar{\theta}$ we obtain

$$\bar{\theta} = \frac{1}{m} D_1 \left(\bar{\phi}_0 + \varepsilon\bar{\phi}_1 + \varepsilon^2\bar{\phi}_2 + \dots\right). \qquad /10.7/$$

When we use the perturbation method, it is satisfactory for practical purposes to retain only two terms of series/10.5/. Let us note, moreover, that the functions (ϕ_0, θ_0) concern uncoupled problem.

Other variant of the perturbation method consists in solving Eqs. /10.3/ and next expanding the functions containing the quantities $k_1(\varepsilon, p)$, $k_2(\varepsilon, p)$ into a power series in the parameter ε . This variant was successfully applied by R.B.Hetnarski [55], [61] for solving a number problems reffered to a thermoelastic space and semi-space.

The second way for approximate solution consists in determining the functions ϕ, t for small times. The solutions of this type are very useful since an essential difference between dynamic and quasi-static problem exists for small times t . This difference vanishes as time flows.

According the Abel's theorem

$$\lim_{t \to 0} f(t) = \lim_{p \to \infty} p \mathcal{L}[f(t)],$$

to small times there correspond large values of the parameter p in the Laplace's transforms. Therefore, in the solutions for Eqs. /10.2/ or Eqs. /10.3/ the term containing the quantities $k_1(\varepsilon,p)$, $k_2(\varepsilon,p)$ should be expanded in powers of $\frac{1}{p}$ and several terms of this expansion should be retained. Performing the Laplace's inverse transformation provides finally the approximate solution of the problem.

The works on the propagation of aperiodic waves are not numerous and deal with the simplest systems, they refer to an elastic space and semi-space. Namely, the problem of the action of instantaneous and continuous concentrated source of heat in an unbounded thermoelastic space was investigated by R.B.Hetnarski [55, 61] who applied the method of perturbation and small times. The problem of the action of instantaneous and concentrated force in a space was considered by E.Soós [17] . W.Nowacki studied the influence of initial conditions on the propagation of thermoelastic waves in an unbounded space [57] .

The problem of determining the field of deformation and temperature around spherical void in unbounded space is allied with problems presented here.

The problem of sudden loading the body boundary was the subject of two works. In the first, M.Lessen [58] applied the perturbation method, in the second P.Chadwick [36] presents the application of the asymptotic method for small times.

The problem of sudden heating boundary of a body with a spherical void with the application of the perturbation was investigated by G.A.Nariboli [59] . It results from the obtained approximate solutions that thermoelastic waves are damped and dispersed. The influence of coupling deformation and temperature fields is small. The solutions quantatively differ slightly from the solutions obtained within framework of the theory of thermal stresses.

The second important problem to which several works are devoted is the propagation of plane wave in a thermoelastic semi-space caused by sudden heating of the plane bounding a space. The question consists in the generalization of the "Daniłowski's problem" familiar in the theory of thermal stresses. This subject was undertaken by R.B.Hetnarski [66] [61] with the application of the perturbation method and making use of the Abel's theorem for small times. The same problem was investigated by B.A.Boley and I.S.Tolins [62] as well as by R.Muki and S.Breuer [63]. The action of the point heating of thermodynamic semi-space was the subject of work by G.Pari [64].

The works by I.N.Sneddon [37] and J.Ignaczak [56] were devoted to the propagation of longitudinal wave in an elastic semi-space and in aninfinite and semi-infinite space rod. In this publication, the Fourier's transformation with respect to the position variable was first applied and then an ordinary differential equation of the second order with respect to time has been solved. Solving this equation and performing the Fourier's inverse transformation led to final result.

At the end of this survey, we should present the further developing directions of thermoelasticity.

It seems that further general theorems will be obtained which will constitute the generalization of the theorems familiar in elastodynamics. We mean the generalization of Kirchoff's, Weber's and Volterra's theorems. The attempts are being made [72] to obtain further and wider variational theorems. The next efforts will be directed towards rejecting the restrictions on small deformations, and thereby towards developing thermoelasticity nonlinear geometrically. Other direction intends to removing the restriction $\frac{\theta}{T_0} \ll 1$, i.e. to investigating bodies with higher temperatures when thermal and mechanical coefficients are the functions of temperature.

Recently, investigations have been initiated in the field of combining the fields of deformation, temperature and electric fields in piezoelectric materials [73, 74, 75]. The initiated direction of magneto-thermoelasticity is also interesting [76 - 81].

The question consists in investigating deformation field, temperature field in electrical conductors in the presence of strong primary magnetic field.

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