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MICROPOLAR ELASTICITY

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THE LINEAR THEORY OF MICROPOLAR ELASTICITY

1. Introduction

The classical theory of elasticity describes well the behaviour of construction materials (various sorts of steel, aluminium, concrete) provided the stresses do not exceed the elastic limit and no stress concentration occurs.

The discrepancy between the results of the classical theory of elasticity and the experiments appears in all the cases when the microstructure of the body is significant, i. e. in the neighbourhood of the cracks and notches where the stress gradients are considerable. The discrepancies also appear in granular media and multimolecular bodies such as polimers.

The influence of the microstructure is particularly evident in the case of elastic vibrations of high frequency and small wave length.

W. Voigt tried to remove the shortcomings of the classical theory of elasticity [1] by the assumption that the interaction of two parts of the body is transmitted through an area element dA by means not only of the force vector $\underline{p}dA$ but also by the moment vector $\underline{m}dA$. Thus, besides the force stresses σ_{ji} also the moment stresses have been defined.

However, the complete theory of asymmetric elasticity was developed by the brothers François and Eugène Cosserat [2] who published it in 1909 in the work "Théorie des corps deformables".

They assumed that the body consists of interconnected particles in the form of small rigid bodies. During the deformation each particle is displaced by $\underline{u}(\underline{x},t)$ and rotated by $\underline{\varphi}(\underline{x},t)$, the functions of the position $\underline{x}$ and time t.

Thus an elastic continuum has been described such that its points possess the orientation (polar media) and for which we can speak of the rotation of a point. The vectors $\underline{u}$ and $\underline{\varphi}$ are mutually independent and determine the deformation of the body. The introduction of the vectors $\underline{u}$ and $\underline{\varphi}$ and the assumption that the transmission of forces through an area element dA is carried out by means of the force vector $\underline{p}$ and the moment vector $\underline{m}$ leads in the consequence to asymmetric stress tensors σ_{ji} and μ_{ji}.

The theory of the brothers E. and F. Cosserat remained unnoticed and was not duly appreciated during their lifetime. This was so because the presentation was very general (the theory was non-linear, including large deformations) and because its frames exceeded the frames of the theory of elasticity. They attempted to construct the unified field theory, containing mechanics, optics and electrodynamics and combined by a general principle of the least action.

The research in the field of the general theories of continuous media conducted in the last fifteen years, drew the attention of the scientists to Cosserats' work. Looking for the new models, describing more precisely the behaviour of the real elastic media, the models similar to, or identical with that of Cosserats' have been encountered. Here, we mention, first of all, the papers by C. Truesdell and R. A. Toupin [3], G. Grioli [4], R. D. Mindlin and H. F. Tiersten [5]. At the beginning the author's attention was concentrated on the simplified theory of elasticity, so called the Cosserat pseudo-continuum. By this name we understand a continuum for which the asymmetric force stresses and moment stresses occur, however, the deformation is determined by the displacement vector $\underline{u}$ only. Here we assume, as in the classical theory of elasticity, that $\underset{\sim}{\varphi} = \frac{1}{2}\operatorname{curl}\underline{u}$. It is interesting to notice that this model was also considered by the Cosserats who called it the model with the latent trihedron.

A number of German authors, W. Günther, H. Schäfer [7], H. Neuber [8] referred directly to the general theory of Cosserats supplementing it with constitutive equations. The general relations and equations of the Cosserats' theory have also been derived by E. V. Kuvshinskii and A. L. Aero [9] and N. A. Palmov [10]. Here one should also mention the generalizing work by A. C. Eringen and E. S. Suhubi [11].

At the present moment the theory of Cosserats is in the full development. The literature on the subject increases, and the problems of the asymmetric theory of elasticity were discussed in two symposia, namely IUTAM Symposium in Freudenstadt in 1968 and in this Symposium organized by CISM. Likewise the first monographs devoted to the micropolar elasticity, by R. Stojanovic [12] and W. Nowacki [13] appeared, both were published in 1970.

The discussion in the present work is confined to the linear theory of the micropolar elasticity. We begin with the dynamic problems, then we consider the statical ones.

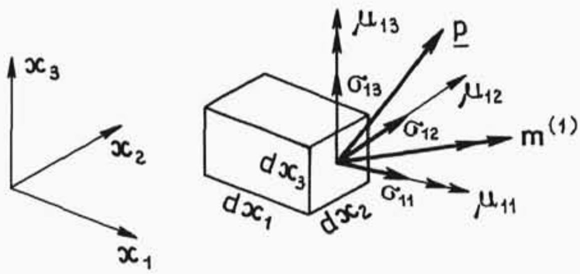
2. The Dynamical Problems of the Micropolar Elasticity

Let us consider a regular region $V+A$ bounded by a smooth surface A, containing a homogeneous, isotropic, centrosymmetric and micropolar continuum of the density ϱ and the rotational inertia $\mathfrak{J}$.

The body is deformed by the external loading. Let on part A_σ of the bounding surface of the body the forces $\underline{p}$ and the moments $\underline{m}$ act, while on part A_u the rotations $\underline{\varphi}$

and displacements $\underline{u}$ be prescribed. The body forces $\underline{X}$ and the body moments $\underline{Y}$ act inside the body. The loadings generate the deformation of the body described by the displacement vector $\underline{u}(\underline{x},t)$ and the rotation vector $\varphi(\underline{x},t)$. Consequently, in the body there develop the force stresses $\sigma_{ji}(\underline{x},t)$ and the moment stresses $\mu_{ji}(\underline{x},t)$. The components σ_{1i}, μ_{1i} of these stresses are presented in Fig. 1. The stresses σ_{ji}, μ_{ji} are connected with the asymmetric tensor of deformation γ_{ji} and the torsion flexure tensor $\varkappa_{ji}$.

The dynamic problem of the micropolar theory of elasticity consists in determining the stresses σ_{ji}, μ_{ji} the deformations γ_{ji}, $\varkappa_{ji}$ the displacement $\underline{u}$, and the rotation $\underline{\varphi}$. These functions should satisfy the equations of motion, the constitutive equations, the boundary conditions, and the initial conditions.



$$\underline{p}^{(1)} = (\sigma_{11}, \sigma_{12}, \sigma_{13})$$
$$m^{(1)} = (\mu_{11}, \mu_{12}, \mu_{13})$$

Fig. 1

The equations of motion take the form

$$\begin{aligned} &\sigma_{ji,j} + X_i = \rho \ddot{u}_i , \\ &\epsilon_{ijk}\sigma_{jk} + \mu_{ji,j} + Y_i = \mathcal{J}\ddot{\varphi}_i . \end{aligned} \tag{2.1}$$

In these equations, written in the cartesian orthogonal coordinate system, ϵ_{ijk} is Ricci's alternator, ρ denotes the density, while $\mathcal{J}$ is the rotational inertia.

The constitutive equations can be obtained from the following discussion. We have, from the principle of the energy conservation, under the assumption of an adiabatic process that

$$(2.2)\quad \frac{d}{dt}(\mathcal{U}+\mathcal{K}) = \int_V (X_i v_i + Y_i w_i)\,dV + \int_A (p_i v_i + m_i w_i)\,dA\,,\quad v_i = \dot{u}_i\,,\ w_i = \dot{\varphi}_i\,.$$

Here $\mathcal{U}$ is the internal energy, $\mathcal{K}$ is the kinetic energy where

$$(2.3)\qquad \mathcal{K} = \frac{1}{2}\int_V (\rho\, v_i v_i + \mathcal{J} w_i w_i)\,dV\ .$$

The right hand side of Eq. 2.2 represents the power of the external forces. Taking into account the equations of motion (1.1) we obtain

$$(2.4)\quad \dot{U} = \sigma_{ji}\dot{\gamma}_{ji} + \mu_{ji}\dot{\varkappa}_{ji}\ ,\quad \mathcal{U} = \int_V U\,dV\ ,\quad U = U(\gamma_{ji}, \varkappa_{ji})\ .$$

Hence we obtain the definition of the deformation tensors

$$(2.5)\qquad \gamma_{ji} = u_{i,j} - \epsilon_{kji}\varphi_k\ ,\qquad \varkappa_{ji} = \varphi_{i,j}\ .$$

The internal energy U is the function of independent variables γ_{ji}, $\varkappa_{ji}$ and is the function of state. Thus we have

$$(2.6)\qquad \dot{U} = \frac{\partial U}{\partial \gamma_{ji}}\dot{\gamma}_{ji} + \frac{\partial U}{\partial \varkappa_{ji}}\dot{\varkappa}_{ji}\ .$$

We assume that the functions σ_{ji}, μ_{ji} do not depend explicitly on the time derivatives of the functions γ_{ji}, $\varkappa_{ji}$. We have

$$\sigma_{ji} = \frac{\partial U}{\partial \gamma_{ji}} \quad , \quad \mu_{ji} = \frac{\partial U}{\partial \varkappa_{ji}} \quad . \tag{2.7}$$

The internal energy can be represented in the following form

$$\begin{aligned} U = {} & \frac{\mu+\alpha}{2}\gamma_{ji}\gamma_{ji} + \frac{\mu-\alpha}{2}\gamma_{ji}\gamma_{ij} + \frac{1}{2}\lambda\gamma_{kk}\gamma_{nn} + \\ & + \frac{\gamma+\varepsilon}{2}\varkappa_{ji}\varkappa_{ji} + \frac{\gamma-\varepsilon}{2}\varkappa_{ji}\varkappa_{ij} + \frac{\beta}{2}\varkappa_{kk}\varkappa_{nn} \quad . \end{aligned} \tag{2.8}$$

The form of the energy, presented here, can be justified in the following way. Since the internal energy is scalar, then each term on the right hand side of the equation must also be a scalar. By means of the components of the tensor γ_{ji} one can construct three independent square invariants, namely $\gamma_{ji}\gamma_{ji}$, $\gamma_{ji}\gamma_{ij}$ and $\gamma_{kk}\gamma_{nn}$. The same thing refers to the tensor $\varkappa_{ji}$. The terms $\gamma_{ji}\varkappa_{ji}$, $\gamma_{ji}\varkappa_{ij}$ and $\gamma_{kk}\varkappa_{nn}$ do not enter the expression (2.8) since this would contradict the postulate of the centrosymmetry. Thus, we have six material constants μ, λ, α, β, measured in the adiabatic conditions. These constants should satisfy the following inequalities

$$3\lambda + 2\mu > 0 \ , \mu > 0, 3\beta + 2\gamma > 0 \ , \ \gamma > 0 \ ,$$

$$\mu + \alpha > 0 \ , \ \gamma + \varepsilon > 0 \ , \alpha > 0 \ , \ \varepsilon > 0 \ .$$

These limitations result from the fact that U is a quadratic, positively defined form. Taking into account 2.7 we obtain the following constitutive equations

$$
\begin{aligned}
\sigma_{ji} &= (\mu+\alpha)\gamma_{ji} + (\mu-\alpha)\gamma_{ij} + \lambda\delta_{ij}\gamma_{kk}\,, \\
\mu_{ji} &= (\gamma+\varepsilon)\varkappa_{ji} + (\gamma-\varepsilon)\varkappa_{ij} + \beta\delta_{ij}\varkappa_{kk}\,.
\end{aligned} \tag{2.9}
$$

Now if we eliminate the stresses from the equations of motion by means of the constitutive equations and then we make use of the defining relations for the tensors γ_{ji}, $\varkappa_{ji}$ we obtain the system of six equations in terms of the displacement $\underline{u}$ and rotation $\underline{\varphi}$.

In the vector form the equations are the following

$$
\begin{aligned}
&\Box_2 \underline{u} + (\lambda+\mu-\alpha)\,\mathrm{grad\,div}\,\underline{u} + 2\alpha\,\mathrm{curl}\,\underline{\varphi} + \underline{X} = 0\,, \\
&\Box_4 \underline{\varphi} + (\beta+\gamma-\varepsilon)\,\mathrm{grad\,div}\,\underline{\varphi} + 2\alpha\,\mathrm{curl}\,\underline{u} + \underline{Y} = 0\,.
\end{aligned} \tag{2.10}
$$

Here the following differential operators have been introduced

$$
\Box_2 = (\mu+\alpha)\Delta - \rho\partial_t^2\,, \qquad \Box_4 = (\gamma+\varepsilon)\Delta - 4\alpha J\partial_t^2\,.
$$

The first of these operators is d'Alembert operator, the second one Klein-Gordon's operator.

We have obtained the complex system of hyperbolic, coupled differential equations. The boundary and initial conditions should be added to the system. According to the assumption the boundary conditions have the form

$$
\begin{aligned}
&\sigma_{ji}(\underline{x},t)n_j(\underline{x}) = p_i(\underline{x},t)\,,\ \mu_{ji}(\underline{x},t)n_j(\underline{x}) = m_i(\underline{x},t)\,,\ \underline{x}\in A_\sigma\,, t>0\,, \\
&u_i(\underline{x},t) = f_i(\underline{x},t)\,,\ \varphi_i(\underline{x},t) = g_i(\underline{x},t)\,,\quad \underline{x}\in A_u\,,\ t>0\,.
\end{aligned} \tag{2.11}
$$

Here $\underline{n}$ is the unit vector normal to the boundary while p_i, m_i, f_i, and g_i are the given functions.

The initial conditions have the form

$$u_i(\underline{x},0) = k_i(\underline{x}), \quad \varphi_i(\underline{x},0) = t_i(\underline{x}),$$
$$\dot{u}_i(\underline{x},0) = h_i(\underline{x}), \quad \dot{\varphi}_i(\underline{x},0) = j_i(\underline{x}), \quad \underline{x} \in V, \quad t = 0. \tag{2.12}$$

The coupled system of differential equations in displacements and rotations is very complicated and inconvenient to deal with, therefore our prime objective will be to uncouple it.

There are two possibilities to uncouple the equations. The first one is analogous to the method used by Lamé in the classical elastokinetics. Let us decompose the vectors $\underline{u}$ and $\underline{\varphi}$ into the potential and solenoidal parts

$$\underline{u} = \operatorname{grad} \Phi + \operatorname{curl} \underline{\Psi}, \quad \operatorname{div} \underline{\Psi} = 0,$$
$$\underline{\varphi} = \operatorname{grad} \Gamma + \operatorname{curl} \underline{H}, \quad \operatorname{div} \underline{H} = 0. \tag{2.13 a}$$

We apply the same procedure to the body forces and moments

$$\underline{X} = \rho\,(\operatorname{grad} \vartheta + \operatorname{curl} \underline{\chi}), \quad \operatorname{div} \underline{\chi} = 0,$$
$$Y = \mathfrak{J}\,(\operatorname{grad} \sigma + \operatorname{curl} \underline{\eta}), \quad \operatorname{div} \underline{\eta} = 0. \tag{2.13 b}$$

Substituting the above relations into Eqs. (1.10) we obtain the following simple wave equations

$$
\begin{aligned}
&\Box_1 \Phi + \varrho \vartheta = 0 , \qquad \Box_3 \Gamma + \mathfrak{J} \sigma = 0 \\
(2.14) \qquad &\Box_2 \underline{H} + 2\alpha \operatorname{curl} \underline{H} + \varrho \underline{\chi} = 0, \\
&\Box_4 \underline{H} + 2\alpha \operatorname{curl} \underline{\Psi} + \mathfrak{J} \underline{\eta} = 0,
\end{aligned}
$$

where we have introduced the following operators

$$
\Box_1 = (\lambda + 2\mu)\Delta - \varrho \partial_t^2 , \qquad \Box_3 = (\beta + 2\gamma)\Delta - 4\alpha - \mathfrak{J}\partial_t^2 .
$$

The first of the equations represents the equation of the longitudinal wave, identical in the form to the longitudinal wave equation in the classical elastokinetics. The second equation is a new type of equation, namely the equation of the longitudinal microrotational wave. The third and fourth equations describe the propagation of the displacement shear wave and the microrotational shear wave respectively.

The longitudinal wave is well known in the classical elastokinetics. The displacement microrotational wave was investigated by N. A. Palmov [10] and W. Nowacki [14]. The last two equations of 2.14 after the elimination of $\underline{\Psi}$ and $\underline{H}$ assume the following form

$$
\begin{aligned}
&(\Box_2 \Box_4 + 4\alpha^2 \Delta) \underline{\Psi} = 2\alpha \mathfrak{J} \operatorname{curl} \underline{\eta} - \varrho \Box_4 \underline{\chi} , \\
(2.15) \qquad &(\Box_2 \Box_4 + 4\alpha^2 \Delta) \underline{H} = 2\alpha \varrho \operatorname{curl} \underline{\chi} - \mathfrak{J} \Box_2 \underline{\eta} .
\end{aligned}
$$

This type of equations has been investigated by J. Ignaczak [15]. He likewise has given the "radiation conditions" similar to

Sommerfeld's conditions. It is evident that the displacement wave Γ and the shear waves $\underline{\Psi}$ and $\underline{H}$ disperse. The system of wave equations (2.14) is very useful for the determination of the singular solutions (Green functions) in an infinite space. Such solutions have been obtained, in a closed form, for the case of concentrated forces and moments harmonically varying in time by W. Nowacki and W.K. Nowacki [16]. Finally it has been shown that the assumed method of solution by means of the potentials Φ, Γ, $\underline{\Psi}$, $\underline{H}$ leads to the complete solutions (W. Nowacki [17]).

The second method of resolution of eqs. (2.10) follows that of B. Galerkin [18] in the classical elastostatics, and M. Iacovache [19] in the classical elastodynamics. The functions of this type for the dynamical problems of the micropolar elasticity have been given by N. Sandru [20], and later, in a different way, by J. Stefaniak [21]. The representation of N. Sandru has the form

$$\begin{aligned} \underline{u} &= \Box_1 \Box_4 \underline{F} - \operatorname{grad} \operatorname{div} \Xi \underline{F} - 2\alpha \operatorname{curl} \Box_3 \underline{G}, \\ \underline{\varphi} &= \Box_2 \Box_3 \underline{G} - \operatorname{grad} \operatorname{div} \Theta \underline{G} - 2\alpha \operatorname{curl} \Box_1 \underline{F}, \end{aligned} \tag{2.16}$$

where

$$\Xi = (\lambda + \mu - \alpha)\Box_4 - 4\alpha^2, \qquad \Theta = (\beta + \gamma - \varepsilon)\Box_2 - 4\alpha^2.$$

Here the displacements $\underline{u}$ and the rotations $\underline{\varphi}$ are represented by two vector functions $\underline{F}$ and $\underline{G}$. Substituting eqs. (2.16) into eqs. (2.10) we obtain two repeated wave equations for the functions $\underline{F}$ and $\underline{G}$.

$$(2.17)\qquad \begin{aligned} &\Box_1 \left(\Box_2 \Box_4 + 4\alpha^2 \right) \underline{F} + \underline{X} = 0 \, , \\ &\Box_3 \left(\Box_2 \Box_4 + 4\alpha^2 \right) \underline{G} + \underline{Y} = 0 \, . \end{aligned}$$

These equations are particularly useful for the determination of the displacements and the rotations generated in an infinite space by the concentrated forces and moments. So far only the singular solutions for the concentrated forces and moments varying harmonically in time have been obtained. In this case the system of equations (2.17) reduces to the system of simple elliptic equations

$$(2.18)\qquad \begin{aligned} &(\Delta + \mu_1^2)(\Delta + k_1^2)(\Delta + k_2^2)\, \underline{F}^* + \underline{X}^* = 0 \, , \\ &(\Delta + \mu_3^2)(\Delta + k_1^2)(\Delta + k_2^2)\, \underline{G}^* + \underline{Y}^* = 0 \, , \end{aligned}$$

where $\underline{X}(\underline{x},t)=\underline{X}(\underline{x})e^{-i\omega t}$ and so on.

There exists the second way of obtaining the fundamental equations of micropolar elasticity. It consists in the utilization of the compatibility equations

$$(2.19)\qquad \begin{aligned} &\gamma_{li,h} - \gamma_{hi,l} - \varepsilon_{khi}\, \varkappa_{lk} + \varepsilon_{kli}\, \varkappa_{hk} = 0 , \\ &\varkappa_{li,h} = \varkappa_{hi,l} \, , \end{aligned}$$

and expressing the functions $\gamma_{ji}, \varkappa_{ji}$ by the stresses σ_{ji}, μ_{ji}.

The system of stress equations constitutes a generalization of the Beltrami-Michell equations known in the classical theory of elasticity, and has been derived for the dynamical problems by Z. Olesiak [22] and for the statical problems by N. Sandru [20]. These equations may have a practical meaning in the two-dimensional problems.

Let us consider particular cases referring to the wave propagation. Many papers have been devoted to interesting problems concerning the one-dimensional waves, dependent on x_1 and t, next dependent on $r=(x_1^2+x_2^2)^{\frac{1}{2}}$ and t, and dependent on $R=(x_1^2+x_2^2+x_3^2)^{\frac{1}{2}}$ and t. Here we should mention the papers by A. C. Eringen [23] N. A. Palmov [10] and A. C. Smith [24].

Consider two-dimensional problems. Let us assume that we have to deal with the problem for which the displacements are independent of x_3. In such a case the system of equations (2.10) can be decomposed into two mutually independent systems of equations. In the first system of equations the following vectors occur

$$\underline{u}=(u_1,u_2,0),\quad \underline{\varphi}=(0,0,\varphi_3),\quad \underline{X}=(X_1,X_2,0),\quad \underline{Y}=(0,0,Y_3) \qquad (2.20)$$

Now the system of equations takes the form

$$\begin{aligned}
(\mu+\alpha)\nabla_1^2 u_1-\varrho\ddot{u}_1+(\mu+\lambda-\alpha)\partial_1 e+2\alpha\partial_2\varphi_3+X_1&=0,\\
(\mu+\alpha)\nabla_1^2 u_2-\varrho\ddot{u}_2+(\mu+\lambda-\alpha)\partial_2 e-2\alpha\partial_1\varphi_3+X_2&=0,\\
\left[(\gamma+\varepsilon)\nabla_1^2-4\alpha-\mathfrak{J}\partial_t^2\right]\varphi_3+2\alpha(\partial_1 u_2-\partial_2 u_1)+Y_3&=0,
\end{aligned} \qquad (2.21)$$

where $\nabla_1^2=\partial_1^2+\partial_2^2$, $e=\partial_1 u_1+\partial_2 u_2$.

The field of displacements $(u_1,u_2,0)$ and rotations $(0,0,\varphi_3)$ generates in the body the state of stresses described by the following matrices

$$(2.22)\qquad \underline{\sigma} = \begin{Vmatrix} \sigma_{11}, & \sigma_{21}, & 0 \\ \sigma_{21}, & \sigma_{22}, & 0 \\ 0, & 0, & \sigma_{33} \end{Vmatrix}, \quad \underline{\mu} = \begin{Vmatrix} 0, & 0, & \mu_{13} \\ 0, & 0, & \mu_{23} \\ \mu_{31}, & \mu_{32}, & 0 \end{Vmatrix}.$$

In the second system, determined by the vectors

$$(2.23)\qquad \underline{u} = (0, 0, u_3), \ \underline{\varphi} = (\varphi_1, \varphi_2, 0), \ \underline{X} = (0, 0, X_3), \ Y = (Y_1, Y_2, 0)$$

we have to deal with the system of equations

$$[(\gamma+\varepsilon)\nabla_1^2 - 4\alpha - J\partial_t^2]\varphi_1 + (\gamma+\beta-\varepsilon)\partial_1\varkappa + 2\alpha\partial_2 u_3 + Y_1 = 0,$$

$$(2.24)\qquad [(\gamma+\varepsilon)\nabla_1^2 - 4\alpha - J\partial_t^2]\varphi_2 + (\gamma+\beta-\varepsilon)\partial_2\varkappa - 2\alpha\partial_1 u_3 + Y_2 = 0,$$

$$(\mu+\alpha)\nabla_1^2 u_3 - \rho\ddot{u}_3 + 2\alpha(\partial_1\varphi_2 - \partial_2\varphi_1) + X_3 = 0,$$

where

$$\varkappa = \partial_1\varphi_1 + \partial_2\varphi_2 .$$

It is easy to verify that the matrices

$$(2.25)\qquad \underline{\sigma} = \begin{Vmatrix} 0, & 0, & \sigma_{13} \\ 0, & 0, & \sigma_{23} \\ \sigma_{31}, & \sigma_{32}, & 0 \end{Vmatrix}, \quad \mu = \begin{Vmatrix} \mu_{11}, & \mu_{12}, & 0 \\ \mu_{21}, & \mu_{22}, & 0 \\ 0, & 0, & \mu_{23} \end{Vmatrix},$$

correspond to the field of displacement $\underline{u} = (0, 0, u_3)$ and rotation $\underline{\varphi} = (\varphi_1, \varphi_2, 0)$.

Let us dwell our attention on the first system of equations. Introducing the potentials Φ and Ψ, where

$$u_1 = \partial_1\Phi - \partial_2\Psi \quad , \quad u_2 = \partial_2\Phi - \partial_1\Psi \quad , \quad \varphi_3 = \varphi \quad , \tag{2.26}$$

the system of equations 2.21 is reduced to simple wave equations (for $X = Y = 0$)

$$\begin{gathered}[(\lambda + 2\mu)\nabla_1^2 - \rho\partial_t^2]\,\Phi = 0 \quad , \\ \{[(\mu+\alpha)\nabla_1^2 - \rho\partial_t^2][(\gamma+\varepsilon)\nabla_1^2 - 4\alpha - J\partial_t^2] + 4\alpha^2\nabla^2\}(\Psi,\varphi) = 0 \quad .\end{gathered} \tag{2.27}$$

Many authors have investigated the above system of equations. V. R. Parfitt and A. C. Eringen [25] and J. Stefaniak [26] have investigated the reflection of a plane wave from the free boundary of an infinite space. A. C. Eringen and E. S. Suhubi [11] investigated the Rayleigh wave, generalized in the micropolar continuum. The same problem is discussed in the extensive paper by S. Kaliski, J. Kupelewski and C. Rymarz [27] . The wave propagation in a plate (the generalized Lamb's problem) has been considered by W. Nowacki and W. K. Nowacki [28] . Also a number of boundary value problems have been solved for the case when the loadings harmonically varying in time act on the boundary of an elastic semi-space (W. Nowacki and W. K. Nowacki [29] , G. Eason [30]). Finally we notice the trends to solve the approximate wave equations (G. Eason [31], J. D. Achenbach [32]) and the interesting results obtained in this way.

Let us return to the second system of two-dimensional equations for which the deformation is determined by the vectors $\underline{u} = (0, 0, u_3)$, $\varphi = (\varphi_1, \varphi_2, 0)$. By means of the potentials Γ , H the system of equations (2.24) is reduced to simple wave equations

$$\varphi_1 = \partial_1 \Gamma - \partial_2 H \ , \quad \varphi_2 = \partial_2 \Gamma + \partial_1 H \ , \tag{2.28}$$

The equations take the following form

$$\begin{gathered} [(\beta + 2\gamma)\nabla_1^2 - 4\alpha - J\partial_t^2]\Gamma = 0 \ , \\ \{[(\mu+\alpha)\nabla_1^2 - \rho\partial_t^2][(\gamma+\varepsilon)\nabla_1^2 - 4\alpha - J\partial_t^2] + 4\alpha^2\nabla_1^2\}(H, u_3) = 0 \ . \end{gathered} \tag{2.29}$$

The first equation corresponds to the longitudinal microrotational wave, the second one to the shear wave. If we assume that

$$(\varphi_1, \varphi_2, u_3) = \left(\varphi_1^*(x_1), \varphi_2^*(x_1), u_3^*(x_1)\right)e^{i(kx_2 - \omega t)}, \tag{2.30}$$

and the boundary of the elastic semi-space $x_1 = 0$ is free from stresses, the above functions lead to the Love surface waves. The propagation of these waves have been investigated in the paper [27] . It is interesting to note that within the frames of the classical elasticity Love's waves do not exist in the case of the homogeneous elastic semi space, the propagation of such waves is possible only for a layered semi-space and different densities and Lame's constants of both media.

Let us consider the second type of the two-dimensional problems, namely the problems of the axially symmetric deformations. In this case the system of equations (2.10) can be decomposed into two mutually independent systems of equations. The following vectors enter the first system of equations

$$\underline{u} = (u_r, 0, u_z), \ \underline{\varphi} = (0, \varphi_\theta, 0), \ \underline{X} = (X_r, 0, X_z), \ \underline{Y} = (0, Y_\theta, \theta) \quad (2.31)$$

The system of equations takes the form

$$\begin{aligned}
&\left[(\mu+\alpha)\left(\nabla^2 - \frac{1}{r^2}\right)u_r - \rho\ddot{u}_r\right] + (\lambda+\mu-\alpha)\frac{\partial e}{\partial r} - 2\alpha\frac{\partial\varphi_\theta}{\partial r} + X_r = 0, \\
&[(\mu+\alpha)\nabla^2 - \rho\partial_t^2]u_z + (\lambda+\mu-\alpha)\frac{\partial e}{\partial z} + 2\alpha\frac{1}{r}\frac{\partial}{\partial r}(r\varphi_\theta) + X_z = 0, \quad (2.32) \\
&\left[(\gamma+\varepsilon)\left(\nabla^2 - \frac{1}{r^2}\right) - 4\alpha - J\partial_t^2\right]\varphi_\theta + 2\alpha\left(\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}\right) + Y_\theta = 0,
\end{aligned}$$

here

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}, \quad e = \frac{1}{r}\frac{\partial}{\partial r}(r u_r) + \frac{\partial u_z}{\partial z}.$$

The following stress matrices correspond to the deformation presented here

$$\underline{\sigma} = \left\| \begin{matrix} \sigma_{rr}, & 0, & \sigma_{rz} \\ 0, & \sigma_{\theta\theta}, & 0 \\ \sigma_{zr}, & 0, & \sigma_{zz} \end{matrix} \right\|, \quad \underline{\mu} = \left\| \begin{matrix} 0, & \mu_{r\theta}, & 0 \\ \mu_{\theta\gamma} & 0, & \mu_{\theta z} \\ 0, & \mu_{z\theta}, & 0 \end{matrix} \right\|. \quad (2.33)$$

The following vectors occur in the second system of equations

$$\underline{u} = (0, u_\theta, 0), \ \varphi = (\varphi_r, 0, \varphi_z), \ \underline{X} = (0, X_\theta, 0), \ \underline{Y} = (Y_r, 0, Y_z), \quad (2.34)$$

and the stress matrices

$$(2.35) \qquad \sigma = \begin{Vmatrix} 0 & \sigma_{r\theta} & 0 \\ \sigma_{\theta r} & 0 & \sigma_{\theta z} \\ 0 & \sigma_{z\theta} & 0 \end{Vmatrix}, \qquad \mu = \begin{Vmatrix} \mu_{rr} & 0 & \mu_{rz} \\ 0 & \mu_{\theta\theta} & 0 \\ \mu_{zr} & 0 & \mu_{zz} \end{Vmatrix}.$$

Now the system of equations assumes the form

$$\left[(\gamma+\varepsilon)\left(\nabla^2-\frac{1}{r^2}\right)-4\alpha-J\partial_t^2\right]\varphi_r+(\beta+\gamma-\varepsilon)\frac{\partial\varkappa}{\partial r}-2\alpha\frac{\partial u_\theta}{\partial z}+Y_r=0,$$

$$(2.36) \qquad [(\gamma+\varepsilon)\nabla^2-4\alpha-J\partial_t^2]\varphi_z+(\beta+\gamma-\varepsilon)\frac{\partial\varkappa}{\partial z}+\frac{2\alpha}{r}\frac{\partial}{\partial r}(r u_\theta)+Y_z=0,$$

$$\left[(\mu+\alpha)\left(\nabla^2-\frac{1}{r^2}\right)-\rho\partial_t^2\right]u_\theta+2\alpha\left(\frac{\partial\varphi_r}{\partial z}-\frac{\partial\varphi_z}{\partial r}\right)+X_\theta=0,$$

where

$$\varkappa=\frac{1}{r}\partial/\partial r\,(r\varphi_r)+\partial\varphi_z/\partial z .$$

Both the systems of equations can be reduced to simple wave equations. These equations served in the investigation of longitudinal and torsional waves in an infinite cylinder of circular cross-section, and in solving two generalized axially symmetric Lamb's problems (W. Nowacki and W. K. Nowacki [33], [34]).

Concluding this review of the dynamical problems we should mention the general theorems of the micropolar elastokinetics. These theorems have been presented and derived by a number of authors.

The theorem on the reciprocity of work has the form (N. Sandru [20], D. Iesan [35])

$$\int_V (X_i * u_i' + Y_i * \varphi_i') dV + \int_A (p_i * u_i' + m_i * \varphi_i') dA = \int_V (X_i * u_i + Y_i' * \varphi_i) dV + \int_A (p_i' * u_i + m_i' * \varphi_i) dA , \tag{2.37}$$

where

$X_i * u_i' = \int_0^t X_i(\underline{x}, t - \tau) u_i'(\underline{x}, \tau) d\tau$ and so on are the convolutions. This equation constitutes a generalization of Graffi's theorem [36] known in the classical elastokinetics. The theorem on the reciprocity constitutes one of the most interesting theorems in the micropolar theory of elasticity. The theorem is extremely general and includes the possibility of derivating the method of integration of the equation of elastokinematics by means of Green's function.

The principle of virtual work is of considerable importance

$$\int_V [(X_i - \rho \ddot{u}_i) \delta u_i + (Y_i - J \ddot{\varphi}_i) \delta \varphi_i] dV + \int_A (p_i \delta u_i + m_i \delta \varphi_i) dA = \int_V (\sigma_{ji} \delta \gamma_{ji} + \mu_{ji} \delta \varkappa_{ji}) dV . \tag{2.38}$$

Here δu_i and $\delta \varphi_i$ denote the virtual displacements and rotations. The principle of virtual work may serve for the derivation of the equation of plate and shell bending under the suitable approximations, for the approximate solution of the equations of elastokinetics and finally for the derivation of the uniqueness theorem.

An important role is played by the extended

Hamilton's principle

$$\delta \int_{t_1}^{t_2} (W - \mathcal{K})\,dt = \int_{t_1}^{t_2} \delta \mathcal{L} \,. \tag{2.39}$$

Here we assume that $\delta \underline{u}(\underline{x}, t_1) = \delta u(\underline{x}, t_2) = \delta \underline{\varphi}(\underline{x}, t_1) = \delta \underline{\varphi}(\underline{x}, t_2) = 0$.

$\delta \mathcal{L}$ denotes the virtual work of external forces

$$\delta \mathcal{L} = \int_V (X_i \delta u_i + Y_i \delta \varphi_i)\,dV + \int_A (p_i \delta u_i + m_i \delta \varphi_i)\,dA \,,$$

and $\mathcal{K}$ is the kinetic energy. W is the work of deformation which, in our case of the adiabatic process, is identical to the internal energy $\mathcal{U}$.

In the present review we have only shown the most important, in our opinion, results of the micropolar elasticity. Let us note that the fundamental results have been obtained only in the case of propagation of the monochromatic waves. The investigation concerning the problems of the waves generated by aperiodic causes or by the causes moving with constant or varying speed have been hardly initiated. The contemporary investigations of the dynamical problems tend to include also the other physical fields. The research in the domain of micropolar thermoelasticity and micropolar magnetoelasticity is already developing.

3. The Micropolar Elastostatics

The substitution of the constitutive equations into the equilibrium equations, together with the definition of the deformations taken into account, leads to the system of six differential equations in terms of displacements and rotations. In the compact vector form the equations read

$$
\begin{gathered}
(\mu+\alpha)\nabla^2\underline{u} + (\lambda+\mu-\alpha)\operatorname{grad}\operatorname{div}\underline{u} + 2\alpha\operatorname{curl}\underline{\varphi} + \underline{X} = 0, \\
[(\gamma+\varepsilon)\nabla^2 - 4\alpha]\underline{\varphi} + (\beta+\gamma-\varepsilon)\operatorname{grad}\operatorname{div}\underline{\varphi} + 2\alpha\operatorname{curl}\underline{u} + \underline{Y} = 0 .
\end{gathered}
\tag{3.1}
$$

The system is coupled, of elliptic type. Let us note that the material constants μ, λ, α, β, γ, ε, occurring in the equations refer to the isothermal process. The system of equations can be decomposed into two independent systems of equations only in the particular case $\alpha = 0$. We can put the following question: is it possible to compose the solution to the system of equations of two parts, the first of which has exactly the same form as the solution of the classical elastostatics. An affirmative answer to this question has been given by H. Schaefer [37].

Introducing the vector

$$
\underline{\zeta} = \frac{1}{2}\operatorname{curl}\underline{u} - \underline{\varphi} \tag{3.2}
$$

and eliminating the function $\underline{\varphi}$ from the system of homogeneous equations (3.1) we obtain

$$
(3.3) \quad
\begin{aligned}
&\mu \nabla^2 \underline{u} + (\lambda + \mu) \operatorname{grad} \operatorname{div} \underline{u} = 2\alpha \operatorname{curl} \underline{\zeta} , \\
&[(\gamma + \varepsilon)\nabla^2 - 4\alpha]\underline{\zeta} + (\beta + \gamma - \varepsilon) \operatorname{grad} \operatorname{div} \underline{\zeta} = \frac{1}{2}(\gamma + \varepsilon)\nabla^2 \operatorname{curl} \underline{u} .
\end{aligned}
$$

We assume the solution to this system in the form

$$
(3.4) \qquad \underline{u} = \underline{u}' + \underline{u}'' , \; \underline{\zeta} = \underline{\zeta}' + \underline{\zeta}'' , \qquad \text{where} \qquad \underline{\zeta}' = 0 .
$$

The above representation allows us to split the system of equations (3. 3) into two independent systems of equations

$$
(3.5) \qquad \mu \nabla^2 \underline{u}' + (\lambda + \mu) \operatorname{grad} \operatorname{div} \underline{u}' = 0 \quad , \quad \nabla^2 \operatorname{curl} \underline{u}' = 0 \quad ,
$$

and

$$
(3.6) \quad
\begin{aligned}
&\mu \nabla^2 \underline{u}'' + (\lambda + \mu) \operatorname{grad} \operatorname{div} \underline{u}'' = 2\alpha \operatorname{curl} \underline{\zeta}'' , \\
&[(\gamma + \varepsilon)\nabla^2 - 4\alpha]\underline{\zeta}'' + (\beta + \gamma - \varepsilon) \operatorname{grad} \operatorname{div} \underline{\zeta}'' = \frac{1}{2}(\gamma + \varepsilon)\nabla^2 \operatorname{curl} \underline{u}'' .
\end{aligned}
$$

Let us point out that the system of equations (3. 5) in its form is identical with the corresponding system of equations of the classical theory of elasticity.

Let us assume that on the boundary of the body the loadings $\underline{p}$ and moments $\underline{m}$ are prescribed

$$
(3.7) \qquad p_i = \sigma_{ji} n_j \quad , \qquad m_i = \mu_{ji} n_j \quad .
$$

The system of equations (3. 6) is satisfied with the boundary conditions $p_i = \sigma'_{ji} n_j$. The assumption $\underline{\zeta}' = 0$ is synonimous with the assertion that the skew-symmetric part of the tensor

γ'_{ji} is equal to zero $(\gamma_{<ij>} = 0)$. The tensor γ'_{ji} is thus symmetric. Therefore the strains σ'_{ji} compose the symmetric tensor. However the assumption $\zeta' = 0$ leads to the relation $\varphi' = \frac{1}{2}\operatorname{curl}\underline{u}'$. Since $\underline{\varphi}' \neq 0$ therefore also $\varkappa'_{ji} \neq 0$. Hence the following moment stress exist in the body

$$\mu'_{ji} = 2\gamma\,\varkappa_{(ji)} + 2\varepsilon\,\varkappa'_{<ji>} + \beta\,\varkappa'_{kk}\delta_{ij} \ . \tag{3.8}$$

As a rule the condition $m_i = \mu'_{ji} n_j$ does not hold. Since the functions $\underline{u}'$ do not satisfy all the boundary conditions the solutions $\underline{u}''$, $\underline{\zeta}''$ satisfying the system of equations (3.6) and the boundary conditions

$$\sigma''_{ji} n_j = 0 \qquad , \qquad (\mu_{ji} + \mu''_{ji})\, n_j = m_i \ , \tag{3.9}$$

should be added to the solutions $\underline{u}'$, $\underline{\zeta}'$.

H. Schaefer assumed the solution of eqs. (3.6) in the form

$$\underline{\zeta}'' = \operatorname{grad}\Phi + \operatorname{curl}\operatorname{curl}\underline{\Omega} \tag{3.10}$$

where the functions Φ , $\underline{\Omega}$ satisfy the differential equations

$$(h^2\nabla^2 - 1)\Phi = 0 \ , \quad (\nu^2\nabla^2 - 1)\Omega = 0 \ ,$$
$$h^2 = \frac{2\gamma + \beta}{4\alpha} \ , \quad \nu^2 = \frac{\gamma + \varepsilon}{4\alpha} \ . \tag{3.11}$$

However the proof of the completeness of the solutions $\underline{\zeta}''$ is lacking. Here, as in the elastokinetics, the displacements and rotations can be represented by two vector functions $\underline{G}$, $\underline{F}$ constituting a generalization of Galerkin's functions. If we substistute the representation given by N. Sandru [20]

$$
\begin{aligned}
\underline{u} &= (\lambda+2\mu)\nabla^2[(\gamma+\varepsilon)\nabla^2-4\alpha]\underline{F} - [(\gamma+\varepsilon)(\lambda+\mu-\alpha)\nabla^2 + \\
&\quad -4\alpha(\lambda+\mu)]\,\mathrm{grad\,div}\,\underline{F} - 2\alpha[(\beta+2\gamma)\nabla^2-4\alpha]\,\mathrm{curl}\,\underline{G}\ , \\
\underline{\varphi} &= (\mu+\alpha)\nabla^2[(\beta+2\gamma)\nabla^2-4\alpha]\underline{G} - [(\mu+\alpha)(\beta+\gamma-\varepsilon)\nabla^2 + \\
&\quad -4\alpha]\,\mathrm{grad\,div}\,\underline{G} - 2\alpha(\lambda+2\mu)\nabla^2\,\mathrm{curl}\,\underline{F}\ ,
\end{aligned}
\tag{3.12}
$$

into the system of equations (3.1) we obtain the following simple equations for the functions $\underline{F}$ and $\underline{G}$

$$
\begin{aligned}
&(\lambda+2\mu)\nabla^2\nabla^2(\ell^2\nabla^2-1)\underline{F} + \underline{X} = 0\ , \\
&16\,\alpha^2\mu\nabla^2(h^2\nabla^2-1)(\ell^2\nabla^2-1)\underline{G} + \underline{Y} = 0\ .
\end{aligned}
\tag{3.13}
$$

Let us observe that the assumption $\underline{X}=0$ entails also that $\underline{F}=0$. Similarly for $\underline{Y}=0$ we have $\underline{G}=0$. Eqs. (3.13) allow us to determine, in a very simple way, the Green functions in an infinite micropolar space. Below we list only the final results of the singular solutions. Let a concentrated unit force, directed along the x_i axis, act at a point $\underline{\xi}$. The displacements and rotations generated by the force take the following form [20]

$$
\begin{aligned}
u_j^{(i)} &= -\frac{\lambda+\mu}{8\pi\mu(\lambda+2\mu)}\partial_i\partial_j\left[R\,\frac{(\gamma+\varepsilon)(\lambda+2\mu)}{2\mu(\lambda+\mu)}\left(\frac{1-e^{-\frac{R}{\ell}}}{R}\right)\right] - \frac{1}{4\mu\pi}\left(\frac{\alpha}{\mu+\alpha}\,\frac{e^{-\frac{R}{\ell}}}{R} - \frac{1}{R}\right)\delta_{ij} \\
\varphi_j^{(i)} &= \frac{1}{8\pi\mu}\,\epsilon_{ijk}\,\frac{\partial}{\partial x_k}\left(\frac{1-e^{-\frac{R}{\ell}}}{R}\right)\ .
\end{aligned}
\tag{3.14}
$$

Here R is the distance between the points $\underline{x}$ and $\underline{\xi}$. Passing to the classical theory of elasticity we have

$$u_j^{(i)} = -\frac{\lambda+\mu}{8\pi\mu(\lambda+2\mu)}\partial_i\partial_j(R) + \frac{1}{4\pi\mu R}\delta_{ij}\,, \quad \varphi_j^{(i)} = 0\ .$$

In the case of a concentrated unit moment applied at a point ξ and acting in the direction of the x_i axis we have the following displacements $\hat{u}_j^{(i)}$ and rotations $\hat{\varphi}_j^{(i)}$ [20] :

$$\begin{aligned} \hat{u}_j^{(i)} &= \frac{1}{8\pi\mu}\epsilon_{ijk}\frac{\partial}{\partial x_k}\left(\frac{1-e^{-\frac{R}{\ell}}}{R}\right)\,, \\ \hat{\varphi}_j^{(i)} &= \frac{1}{16\pi\mu}\partial_i\partial_j\left[\frac{1-e^{-\frac{R}{\tau}}}{R} + \frac{\mu}{\alpha}\left(\frac{e^{-\frac{R}{h}} - e^{-\frac{R}{\ell}}}{R}\right)\right] + \frac{e^{-\frac{R}{\ell}}}{4\pi(\gamma+\varepsilon)R}\delta_{ij}\ . \end{aligned} \tag{3.15}$$

Beside the Galerkin function also the Papkovitch-Neuber type functions have been introduced in the micropolar elastostatics. H. Neuber [8] has generalized his functions on micropolar elastostatics and applied them to a series of problems referring to the stress concentration problems around holes and notches [38] - [40]. A certain variation of this type function has been given by N. Sandru [20] and S. C. Cowin [41] .

Parallely to the equations in terms of the displacements and rotations in elastostatics one can use the stress equations analogous to those of Beltrami-Mitchell. In this respect the discussion of H. Schaefer is interesting, he introduced a very general type of stress functions known in the classical elastostatics [42]. Also the paper of S. Kassel [51] on the stress functions deserves attention.

In a more detailed way we shall discuss the two-dimensional problems, namely the problems of the plane state of strain, and the axially symmetric problems. Consider the plane state of strain for which the deformation does not depend on the variable x_3. As we know, in this case two mutually independent systems of equations are obtained. In the first problem there appear the vectors $\underline{u} = (u_1, u_2, 0)$ $\underline{\varphi} = (0, 0, \varphi_3)$, while in the second one the vectors $\underline{u} = (0, 0, u_3)$, $\underline{\varphi} = (\varphi_1, \varphi_2, 0)$.

The following compatibility equations

$$
\begin{aligned}
&\partial_1^2\sigma_{22} + \partial_2^2\sigma_{11} - \frac{\lambda}{2(\lambda+\mu)}\nabla^2(\sigma_{11}+\sigma_{22}) = \partial_1\partial_2(\sigma_{12}+\sigma_{21}) \,, \\
&(\partial_2^2 - \partial_1^2)(\sigma_{12}+\sigma_{21}) + \frac{\mu}{\alpha}\nabla^2(\sigma_{12}-\sigma_{21}) + 2\partial_1\partial(\sigma_{11}-\sigma_{22}) + \\
&+ 4\mu/(\gamma+\varepsilon)\cdot(\partial_1\mu_{13} + \partial_2\mu_{23}) = 0 \,, \quad \partial_1\mu_{23} = \partial_2\mu_{13} \,,
\end{aligned}
\tag{3.16}
$$

and the equilibrium equations

$$
\begin{aligned}
&\partial_1\sigma_{11} + \partial_2\sigma_{21} = 0 \,, \qquad \partial_1\sigma_{12} + \partial_2\sigma_{22} = 0 \,, \\
&\sigma_{12} - \sigma_{21} + \partial_1\mu_{13} + \partial_2\mu_{23} = 0 \,.,
\end{aligned}
\tag{3.17}
$$

constitute the point of departure for the first problem. We have the system of six equations for the determination of six unknown functions σ_{11}, σ_{22}, σ_{12}, σ_{21}, μ_{13}, μ_{23}. The equilibrium equations are satisfied identically by two functions F and Ψ connected with the stresses by the following relations

$$\sigma_{11} = \partial_2^2 F - \partial_1 \partial_2 \Psi , \quad \sigma_{22} = \partial_1^2 F + \partial_1 \partial_2 \Psi ,$$
$$\sigma_{12} = -\partial_1 \partial_2 F - \partial_2^2 \Psi , \quad \sigma_{21} = -\partial_1 \partial_2 F + \partial_1^2 \Psi , \tag{3.18}$$
$$\mu_{\alpha\beta} = \partial_\alpha \Psi , \qquad \alpha, \beta = 1, 2$$

The function F is Airy's function known from the classical elastokinetics. The function Ψ has been introduced by R. D. Mindlin [43] and H. Schaefer [7] for the plane state of strain in the Cosserat continuum and pseudo-continuum. Substituting eqs. (3.18) into the compatibility equations we obtain the following equations in terms of the functions Ψ and F

$$\nabla_1^2 \nabla_1^2 F = 0 \quad , \quad \nabla_1^2 (l^2 \nabla_1^2 - 1) \Psi = 0 . \tag{3.19}$$

The functions F and Ψ are mutually dependent and satisfy the Cauchy-Riemann conditions

$$\partial_1 (l^2 \nabla_1^2 - 1) \Psi = A \partial_2 \nabla_1^2 F , \qquad A = \frac{(\lambda + 2\mu)(\gamma + \varepsilon)}{4\mu(\lambda + \mu)} . \tag{3.20}$$
$$\partial_2 (l^2 \nabla_1^2 - 1) \Psi = -A \partial_1 \nabla_1^2 F .$$

Since the functions $\nabla_1^2 F$ and $(l^2 \nabla_1^2 - 1)\Psi$ are harmonic, it is not difficult to observe that the method of complex variable is particularly useful for solving eqs. (3.19). This method has been successfully applied by G. N. Savin [44] - [46] and his coworkers in the problems of stress concentration around the holes. D. E. Carlson [47] has investigated the completeness of the solutions by means of the functions F and Ψ . Let us

Observe that the form of the plane problem is identical for both the pseudo-continuum and the continuum of Cosserats'. This is why there exists a number of special problems solved concerning the stress concentration, the state of stress in an elastic semi-space, and the singular solutions. First of all we mention here the papers by R. D. Mindlin [48], H. Schaefer [7], R. Muki and E. Sternberg [49], P. N. Kaloni and T. Ariman [50].

The first plane problem can be solved also by another method suitable in the case of the displacements and rotations prescribed on the boundary [52]. Using the differential equations in terms of displacements and rotations

$$
\begin{aligned}
&(\mu+\alpha)\nabla_1^2 u_1 + (\lambda+\mu-\alpha)\partial_1 e + 2\alpha\,\partial_2\varphi_3 = 0\ ,\\
(3.21)\quad &(\mu+\alpha)\nabla_1^2 u_2 + (\lambda+\mu-\alpha)\partial_2 e - 2\alpha\,\partial_1\varphi_3 = 0\ ,\\
&[(\gamma+\varepsilon)\nabla_1^2 - 4\alpha]\varphi_3 + 2\alpha(\partial_1 u_2 - \partial_2 u_1) = 0\ ,\quad e = \partial_1 u_1 + \partial_2 u_2\ ,
\end{aligned}
$$

and introducing the potentials Φ , Ψ related to the displacements

$$
(3.22)\qquad u_1 = \partial_1\Phi + \partial_2\Psi\ ,\qquad u_2 = \partial_2\Phi - \partial_1\Psi
$$

we obtain the following simple differential equations

$$
(3.23)\qquad \nabla_1^2\nabla_1^2\Phi = 0\ ,\qquad \nabla_1^2(l^2\nabla_1^2 - 1)\varphi_3 = 0\ .
$$

The functions Φ and φ_3 are not mutually independent, they should satisfy the conditions in the form .

$$\begin{aligned} \partial_1 \nabla_1^2 \Phi &= -\frac{2\mu}{\lambda+2\mu}\partial_2(l^2\nabla_1^2-1)\varphi_3 , \\ \partial_2 \nabla_1^2 \Phi &= \frac{2\mu}{\lambda+2\mu}\partial_1(l^2\nabla_1^2-1)\varphi_3 . \end{aligned} \tag{3.24}$$

The conditions (3.24) are the Cauchy-Riemann conditions for the functions $\nabla_1^2\Phi$ and $(l^2\nabla_1^2-1)\varphi_3$.

The potential Ψ is related to the function φ_3 :

$$\nabla_1^2\Psi = \frac{1}{2\alpha}\left[(\gamma+\varepsilon)\nabla^2-4\alpha\right]\varphi_3 . \tag{3.25}$$

The solution of the first plane problem is obtained by the following procedure. Solving eqs. (3.23) for example for an elastic semi-space we get four constants of integration. To determine the constants we have three boundary conditions and the Cauchy-Riemann conditions.

In the case of the second plane state of strain we have the following system of equations

$$\begin{aligned} &(\mu+\alpha)\nabla_1^2 u_3 + 2\alpha(\partial_1\varphi_2-\partial_2\varphi_1) = 0 , \\ &\left[(\gamma+\varepsilon)\nabla_1^2-4\alpha\right]\varphi_1 + (\beta+\gamma-\varepsilon)\partial_1 x + 2\alpha\partial_2 u_3 = 0 , \\ &\left[(\gamma+\varepsilon)\nabla_1^2-4\alpha\right]\varphi_2 + (\beta+\gamma-\varepsilon)\partial_2 x + 2\alpha\partial_1 u_3 = 0 . \end{aligned} \tag{3.26}$$

The simplest way to solve the above system of equations is to make use of the potentials Ω , Ξ [53] , where

$$(3.27) \qquad \varphi_1 = \partial_1 \Omega + \partial_2 \Xi \quad , \quad \varphi_2 = \partial_2 \Omega - \partial_1 \Xi \ .$$

Substituting the relations (3.21) intc eqs. (3.20) we cbtain the system of equations

$$(3.28) \qquad \nabla_1^2 (h^2 \nabla_1^2 - 1)\Omega = 0 \ , \ \nabla_1^2 (l^2 \nabla_1^2 - 1)\Xi = 0 \ .$$

The functions Ω and Ξ satisfy the following Cauchy-Riemann conditions

$$(3.29) \qquad \begin{aligned} -\partial_1 (h^2 \nabla_1^2 - 1)\Omega &= \frac{\mu}{\mu + \alpha} \partial_2 (l^2 \nabla_1^2 - 1)\Xi \ , \\ \partial_2 (h^2 \nabla_1^2 - 1)\Omega &= \frac{\mu}{\mu + \alpha} \partial_1 (l^2 \nabla_1^2 - 1)\Xi \ . \end{aligned}$$

The quantity u_3 is related with the potential Ξ by the equation

$$(3.30) \qquad \nabla_1^2 u_3 = \frac{2\alpha}{\mu + \alpha} \nabla_1^2 \Xi \ .$$

Another method of solution of the "second" problem of the plane state of strain belongs to M. Suchar [54]. Five compatibility equations and three equilibrium equations serve as the point of departure of his discussion. The system of equilibrium equations is satisfied by four functions Φ , Ψ , χ , Ω related to the stresses by the following formulae

$$(3.31) \qquad \begin{aligned} &\sigma_{13} = \partial_2 \Phi \ , \ \sigma_{23} = -\partial_1 \Phi \ , \ \sigma_{31} = -\partial_2 \Phi + \partial_1 \Psi \ , \ \sigma_{32} = \partial_1 \Phi + \partial_2 \Psi \ , \\ &\mu_{11} = 2\Phi + \partial_2 \chi \ , \ \mu_{21} = \Psi - \partial_1 \chi \ , \ \mu_{12} = -\Psi + \partial_2 \Omega \ , \ \mu_{22} = 2\Phi - \partial_1 \Omega \ . \end{aligned}$$

Substituting these relations into the equations of compatibility one obtains the following differential equations in terms of the functions Φ , Ψ , χ , and Ω

$$\nabla_1^2(h^2\nabla_1^2-1)\Phi=0\ ,\ \nabla_1^2(l^2\nabla_1^2-1)\Psi=0\ ,\ L(\chi)=0\ ,\ L(\Omega)=0, \quad (2.32)$$

where

$$L(\ )=(h^2\nabla^2-1)(l^2\nabla^2-1)\nabla^2\nabla^2(\ )\ .$$

One should add that the stress functions Φ , Ψ , χ , Ω are not mutually independent but combined by four additional differential relations.

In turn we consider the axially symmetric problems. We know from the preceding point that the system of six differential equations in terms of displacement and rotations can be split, in this case, into two mutually independent systems of equations. In the first one the deformation is determined by the vectors $\underline{u}=(u_r,0,u_z)$, $\underline{\varphi}=(0,\varphi_\theta,0)$ while in the second is given by $\underline{u}=(0,u_\theta,0)$, $\underline{\varphi}=(\varphi_r,0,\varphi_z)$.

In the classical theory of elasticity, solving the first problem we are very frequently making use of the Love function $\chi(r,z)$ satisfying the biharmonic equation. In the micropolar theory of elasticity we introduce two functions of Love's type. In the first axially symmetric problem the function $\chi(r,z)$ is related to the displacements and rotations in the following manner [55]

$$u_r = -\frac{\partial^2}{\partial r \partial z}(\Gamma\chi) \quad , \quad \varphi_\theta = 2\alpha(\lambda+2\mu)\frac{\partial}{\partial r}\nabla^2\chi \quad ,$$

(3.33)

$$u_z = \Theta\,\chi - \frac{\partial}{\partial z^2}(\Gamma\chi) \quad .$$

Here

$$\Gamma = (\gamma+\varepsilon)(\lambda+\mu-\alpha)\nabla^2 - 4\alpha(\lambda+\mu)\,,\ \Theta = (\lambda+2\mu)\nabla^2[(\gamma+\varepsilon)\nabla^2 - 4\alpha]\,,$$

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}\ ,\ r = (x_1^2 + x_2^2)^{1/2}\ .$$

Susbtituting the above relations into the equations of the elastostatics we obtain the following equations in terms of the function χ

(3.34) $4\alpha\mu(\lambda+2\mu)\nabla^2\nabla^2(l^2\nabla^2 - 1)\chi(r,z) + X_z(r,z) = 0\ .$

Similarly, in the second axially symmetric problem we assume

$$\varphi_r = -\frac{\partial^2}{\partial r \partial z}(\Omega\Psi)\ ,\quad u_\theta = 8\alpha^2\frac{\partial}{\partial r}(h^2\nabla^2 - 1)\Psi\ ,$$

(3.35)

$$\varphi_z = \Phi\Psi - \frac{\partial}{\partial z^2}(\Omega\Psi)\ ,$$

where

$$\Omega = (\mu+\alpha)(\beta+\gamma-\varepsilon)\nabla^2 - 4\alpha\ ,\quad \Phi = (\mu+\alpha)\nabla^2[(\beta+2\gamma)\nabla^2 - 4\alpha]\ ,$$

and we obtain, from the system of the elastostatic equations, the following equation in terms of the second generalized function of Love

$$16\alpha^2\mu(h^2\nabla^2-1)(l^2\nabla^2-1)\Psi(r,z)+Y_z(r,z)=0 \qquad (3.36)$$

J. Stefaniak [57] derived the analogous functions from the vector functions of N. Sandru.

Another method of solving the equations of the axially symmetric problems consists in the introduction of elastic potentials [56]. We shall explain this method shortly on the example of the first axially symmetric problem. Let us express the displacements $\underline{u}=(u_r, 0, u_z)$ and the rotations $\underline{\varphi}=(0, \varphi_\theta, 0)$ by means of the potentials Φ, Ψ, ϑ

$$u_r = \frac{\partial\Phi}{\partial r}+\frac{\partial^2\Psi}{\partial r\,\partial z} \quad , \quad u_z = \frac{\partial\Phi}{\partial z}-\frac{\partial^2\Psi}{\partial r^2}-\frac{1}{r}\frac{\partial\Psi}{\partial r} \; ,$$
$$\varphi_0 = \frac{\partial\vartheta}{\partial r} \; . \qquad (3.37)$$

Substituting eqs. (3.37) into the equations of elastostatics we obtain the system of two equations

$$\nabla^2\nabla^2\Phi = 0 \quad , \quad \nabla^2(l^2\nabla^2-1) = 0 \quad . \qquad (3.38)$$

The functions Φ and ϑ are connected by the relations

$$\nabla^2\Phi-\frac{2\mu}{\lambda+2\mu}\frac{\partial}{\partial z}(l^2\nabla^2-1)\vartheta = 0 \; ,$$
$$\frac{\partial}{\partial z}\nabla^2\Phi+\frac{2\mu}{\lambda+\mu}\left(\nabla^2-\frac{\partial^2}{\partial z^2}\right)(l^2\nabla^2-1)\vartheta = 0 \; . \qquad (3.39)$$

The function Ψ will be determined by the equation

$$\nabla^2\Psi = -\frac{1}{2\alpha^2}\left[(\gamma+\varepsilon)\nabla^2-4\alpha\right]\vartheta \; . \qquad (3.40)$$

Exactly the same procedure can be applied in the second axially-symmetric problem. This purpose can be attained also by different methods, as the application of Neuber's function, or, finally by the direct integration of the system of differential equations by means of the Hankel integral transform or Hankel-Fourier transform. A number of special problems, concerning the state of stress in an elastic semi-space has been solved (the generalized Boussinesq problem).

We conclude the review of the micropolar elastostatics devoting a few words to the general variational theorems, the theorem on reciprocity, and so on. The theorems of this type turned out to be easy to extend on the micropolar theory of elasticity by the addition of the corresponding terms connected with the work of moments and the moment stresses.

The principle of the virtual work of the virtual displacements δu_i and the virtual rotations $\delta \varphi_i$ takes the form

$$(3.41) \quad \int_V (X_i \delta u_i + Y_i \delta \varphi_i)\, dV + \int_A (p_i \delta u_i + m_i \delta \varphi_i)\, dA = \delta W_\varepsilon \ ,$$

where

$$W_\varepsilon = \int \left(\mu\, \gamma_{(ij)} \gamma_{(ij)} + \alpha\, \gamma_{\langle ij \rangle} \gamma_{\langle ij \rangle} + \frac{\lambda}{2} \gamma_{kk} \gamma_{nn} + \ldots \right) dV \ .$$

The minimum of potential energy theorem can be derived from the principle of virtual work

$$\delta \Gamma = 0 \quad , \tag{3.42}$$

where

$$\Gamma = W_\varepsilon - \int_V (X_i u_i + Y_i \varphi_i) dV - \int_{A_\sigma} (p_i u_i + m_i \varphi_i) dA \ .$$

Here A_σ denotes the part of the surface bounding the body where the loadings are prescribed.

In the classical elastostatics an important role is played by the theorem of minimum complementary energy. Here in the micropolar elastostatics we have the following variational principle

$$\delta \Pi = 0 \quad , \tag{3.43}$$

where

$$\Pi = W_\sigma - \int_{A_u} (p_i u_i + m_i \varphi_i) dA \quad ,$$

and

$$W_\sigma = \int_V \left(\mu' \sigma_{(ij)} \sigma_{(ij)} + \varepsilon' \sigma_{<ij>} \sigma_{<ij>} + \frac{\lambda'}{2} \sigma_{kk} \sigma_{nn} + \ . \ . \ . \right) dV \ .$$

A_u denotes the part of the bounding surface for which the displacements and rotations are prescribed.

N. Sandru [20] derived the reciprocity theorem

$$\begin{aligned} \int_V (X_i u'_i + Y_i \varphi'_i) dV + \int_A (p_i u'_i + m_i \varphi'_i) dA = \\ = \int_V (X'_i u_i + Y'_i \varphi_i) dV + \int_A (p'_i u_i + m'_i \varphi_i) dA \ , \end{aligned} \tag{3.44}$$

and the particular case of the theorem, namely the generalized formulae of Somigliano. Much attention has been devoted to the uniqueness of solution and the existence theorem (M. Hlavacek [58], D. Iesan [59]).

The research work on the theory of dislocations in the Cosserat continuum needs a separate treatment. W. Günther was the first [6] who noticed the importance of the micropolar elasticity for the theory of dislocations. Certain concepts concerning the dislocations were discussed in the paper of W. D. Claus and A. C. Eringen [60] and in the paper of S. Minagawa [61] .

The theory of anisotropic Cosserat continuum was investigated in papers by S. Kessel [62] and D. Iesan [63] .

The theory of the Cosserat continuum is well developed. At the present day it constitutes a complete superstructure of the classical theory of elasticity. However, the complete experimental verification of the theory is still lacking. The material constants μ, λ, α, β, γ, ε have not been determined for particular materials. We only know the order of these constants and the mutual ratios of the six constants. Thus we have here the extreme case when the theory outdistances the experiments.

In the further development of the Cosserat mechanics the main role should be played by the experimental research. The role of the theoreticians is here already exhausted.

REFERENCES

[1] Voigt W.: "Theoretische Studien über die Elastizitätverhältnisse der Kristalle", Abh. Ges. Wiss. Göttingen 34, (1887).

[2] Cosserat E. and Cosserat F.: "Théorie des corps déformables", A. Herrman, Paris, (1909).

[3] Truesdell C. and Toupin R.A.: "The classical field theories", Encyclopaedia of Physics 3, N°1, Springer Verlag, Berlin, 1960.

[4] Grioli G.: "Elasticité asymmetrique", Ann. di Mat. Pura et Appl. Ser. IV, 50 (1960).

[5] Mindlin R.D. and Tiersten H.F.: "Effects of couple stresses in linear elasticity", Arch. Mech. Analysis, 11 (1962), 385.

[6] Günther W.: "Zur Statik und Kinematik des Cosseratschen Kontinuums", Abh. Braunschweig, Wiss. Ges. 10 (1958), 85.

[7] Schaefer H.: "Versuch einer Elastizitätstheorie des zweidimensionalen Cosserat-Kontinuums, Misz. Angew. Mathematik Festschrift Tollmien, Berlin, (1962), Akademie Verlag.

[8] Neuber H.: "On the general solution of linear-elastic problems in isotropic and anisotropic Cosserat--continua", Int. Congress IUTAM, München, 1964.

[9] Kuvshinskii E.V. and Aero A.L.: "Continuum theory of asymmetric elasticity" (in Russian), Fizika Tvordogo Tela, 5 (1963).

[10] Palmov N.A.: "Fundamental equations of the theory of asymmetric elasticity" (in Russian), Prikl. Mat. Mekh. 28 (1964), 401.

[11] Eringen A.C. and Suhubi E.S.: "Nonlinear theory of simple microelastic solids", Int. J. of Engng. Sci. I, 2, 2 (1964), 189; II, 2, 4 (1964), 389.

[12] Stojanovic R.: "Mechanics of Polar Continua", CISM, Udine (1970).

[13] Nowacki W.: "Theory of micropolar elasticity", J. Springer, Wien (1970).

[14] Nowacki W.: "Propagation of rotation waves in asymmetric elasticity", Bull. Acad. Polon. Sci. Sér. Sci. Techn. 16, 10 (1968), 493.

[15] Ignaczak J.: "Radiation conditions of Sommerfeld type for elastic materials with microstructure", Bull. Acad. Polon. Sci. Sér. Sci. Techn. 17, 6 (1970), 251.

[16] Nowacki W. and Nowacki W.K.: "The generation of waves in an infinite micropolar elastic solid", Proc. Vibr. Probl. 10, 2 (1969), 170.

[17] Nowacki W.: "On the completeness of potentials in micropolar elasticity", Arch. Mech. Stos., 21, 2, (1969), 107.

[18] Galerkin B.: "Contributions à la solution générale du probleme de la théorie de l'élasticité dans le cas de trois dimensions", C. R. Acad. Sci. Paris, 190 (1970), 1047.

[19] Iacovache M.: "O extindere a metodei lui Galerkin pentru sistemul ecuatiilor elasticitatii", Bull. Stiint. Acad. Rep. Pop. Romane. Ser. A 1 (1949), 593.

[20] Sandru N.: "On some problems of the linear theory of asymmetric elasticity", Int. J. Eng. Sci. 4, 1, (1966), 81.

[21] Stefaniak J.: "Generalization of Galerkin's functions for asymmetric thermoelasticity", Bull. de l'Acad. Polon. Sci. Sér. Sci. Techn. 16, 8, (1968), 391.

[22] Olesiak Z.: "Stress differential equations of the micropolar elasticity", Bull. Ac. Polon. Sci. Sér. Sci. Techn. 18, 5, (1970), 172.

[23] Eringen A. C.: "Theory of micropolar elasticity", Fracture, 2 (1968), Academic Press, New York.

[24] Smith A. C.: "Waves in micropolar elastic solids", Int. J. Engng. Sci. 10, 5 (1967), 741.

[25] Parfitt V. R. and Eringen A. C.: "Reflection of plane waves from the flat boundary of a micropolar elastic halfspace", "Report N° 8-3, General Technology Corporation, (1966).

[26] Stefaniak J.: "Reflection of a plane longitudinal wave from a free plane in a Cosserat medium", Arch. Mech. Stos. 21, 6 (1969), 745.

[27] Kaliski S., Kapelewski J. and Rymarz C.: "Surface waves on an optical branch in a continuum with rotational degrees of freedom", Proc. Vibr. Probl. 9, 2, (1968), 108.

[28] Nowacki W. and Nowacki W. K.: "Propagation of monochromatic waves in an infinite micropolar elastic plate", Bull. Acad. Polon. Sci., Sér. Sci. Techn. 17, 1 (1969), 29.

[29] Nowacki W. and Nowacki W. K.: "The plane Lamb problem in a semi-infinite micropolar elastic body" Arch. Mech. Stos. 21, 3, (1969), 241.

[30] Eason G.: "Wave propagation in a material with microstructure", Proc. Vibr. Probl. 12, 4, (1971), 363.

[31] Eason G.: "The displacement produced in a semi-infinite linear Cosserat continuum by an impulsive force", Proc. Vibr. Probl., 11, 2, (1970), 199.

[32] Achenbach J. D.: "Free vibrations of a layer of micropolar continuum" Int. J. Engng. Sci. 10, 7 (1969), 1025.

[33] Nowacki W. and Nowacki W. K.: "Propagation of elastic waves in a micropolar cylinder", Bull. Acad. Polon. Sci., Sér. Sci. Techn. I-17, 1 (1969), 49.

[34] Nowacki W. and Nowacki W. K.: "The axially symmetric Lambs problem in a semi-infinite micropolar elastic solid", Proc. Vibr. Probl. 10, 2 (1969), 97.

[35] Iesan D.: "On the linear theory of micropolar elasticity", Int. J. Eng. Sci. 7 (1969), 1213.

[36] Graffi D.: "Sui teoremi di reciprocità nei fenomeni non stazionari", Atti Accad. Sci. Bologna, 10, 2, (1963).

[37] Schaefer H.: "Das Cosserat-Kontinuum", ZAMM, 47, 8 (1967), 485.

[38] Neuber H.: "Über Probleme der Spannungskonzentration im Cosserat-Körper", Acta Mechanika, 2, 1 (1966), 48.

[39] Neuber H.: "Die schubbeanspruchte Kerbe im Cosserat-Körper", ZAMM, 47, 5, (1967), 313.

[40] Neuber H.: "On the effects of stress concentration in Cosserat continua", IUTAM-Symposium (1967),

Freudenstadt-Stuttgart, Mechanics of generalized continua (1968), 109.

[41] Cowin S. C.: "Stress functions for elasticity", Int. J. Solids Structures, 6, (1970), 389.

[42] Schaefer H.: "Die Spannungsfunktionen eines Kontinuum mit Momentenspannungen", Bull. de l'Acad. Sci. Pol., Sér. Sci. Techn. I, 15, 1 (1967), 63; II, 15, 1, (1967), 485.

[43] Mindlin R. D.: "Influence of couple-stresses on stress concentrations", Exper. Mech. 3, 1, (1963), 1.

[44] Savin G. N.: "Foundation of couple stresses in elasticity" (in Ukrainian), Kiev, (1965).

[45] Savin G. N.: "Stress concentration in the neighbourhood of holes"(in Russian), Kiev (1966).

[46] Savin G. N. and Nemish J. N.: "Stress concentration in couple-stress elasticity" (in Russian), Prikl. Mech. 4, 12 (1968), 1.

[47] Carlson D. E.: "Stress functions for plane problems with couple-stresses", ZAMP, 17, 6, (1966), 789.

[48] Mindlin R. D.: "Representation of displacements and stresses in plane strain with couple-stresses", IUTAM Symposium (1963), Tbilisi (1954), 256.

[49] Muki R. and Sternberg E.: "The influence of couple-stress concentrations in elastic solids", ZAMP,, 16, 5, (1965), 611.

[50] Kaloni P. N. and Ariman T.: "Stress concentration effects in micropolar elasticity", ZAMP, 18, 1, (1967), 136.

[51] Kessel S.: "Die Spannungsfunktionen des Cosserat-Kontinuum", ZAMM, 47, 5 (1967), 329.

[52] Nowacki W.: "Plane problems of micropolar elasticity", Bull. Acad. Polon. Sci., Sér. Sci. Techn. 19, 6, (1971), 237.

[53] Nowacki W.: "The "second" plane problem of micropolar elasticity", Bull. Acad. Polon. Sci., Sér. Sci. Techn. 18, 11 (1970), 525.

[54] Suchar M.: "Stress functions in the "second" plane problem of micropolar elasticity", Bull. Acad. Polon. Sci. 20, (1972) (in print).

[55] Nowacki W.: "Generalized Love's functions in micropolar elasticity", Bull. Acad. Polon. Sci. Sér. Sci. Techn., 17, 4, (1969), 247.

[56] Nowacki W.: "Axially symmetric problem in micropolar elasticity", Bull. Acad. Polon. Sci., Sér. Sci. Techn. 19, 7/8, (1971), 317.

[57] Stefaniak J.: "Solving functions for axi-symmetric problems in the Cosserat medium", Bull. de l'Acad. Polon. Sci., Sér. Sci. Techn. 18, 9, (1970), 387.

[58] Hlavacek M.: "The first boundary-value problem of elasticity theory with couple stresses I. Cosserat Continuum", Bull. Acad. Polon. Sci., Sér. Sci. Techn. 18, 2, (1970), 75.

[59] Iesan D.: "Existence theorems in micropolar elastostatics", Int. J. Eng. Sc. 9, (1971), 59.

[60] Claus W. D. and Eringen A. C.: "Three dislocation concepts and micromorphic mechanics. Developments in Mechanics" Vol. 6. Proceedings of the 12th Midwestern Mechanics Conference.

[61] Minagawa S.: "On the force exerted upon continuously distributed dislocations in a Cosserat continuum", Phys. Stat. Sol., 39 (1970), 217.

[62] Kessel S.: "Lineare Elastizitätstheorie der anisotropen Cosserat-Kontinuums", Abh. der Braunschw. Wiss. Gesellschaft 16, (1964), 1.

[63] Iesan D.: "Thermal stresses in the generalized plane strain of anisotropic elastic solids", Bull. de l'Ac. Pol. Sc., Sér. Sci. Techn. 18, 6, (1970), 227.