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# A Distortion Problem of Micropolar Elasticity

by

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**Summary.** The theorem of reciprocity of works, valid for the case when external loadings are acting and permanent deformations (distortions) $\gamma^{\circ}$, $\kappa^{\circ}$ take place, is derived in this note. Making use of this theorem, formulae for displacements and rotations were given by means of the Green function. Two particular cases are investigated: one referring to increment of the volume of a simply connected body subject to action of distortion, the other, pertaining to displacements and rotations occurring in infinite elastic space, induced by simple concentrated defects. In the final part of the paper the above-mentioned theorem is developed to the dynamic distortion problem.

## 1. Introduction

The state of stress occurring inside a micropolar medium as a consequence of permanent distortion $\gamma_{ji}^{0}, \kappa_{ji}^{0}$ was investigated by the present author in paper [1]. Distortions of such a type may, a.o., take place in metals on exceeding the yield point as plastic strains. A special case of distortion constitute temperature strains $\gamma_{ji}^{0}=a_t\,\delta_{ij}\,\theta$, $\kappa_{ji}^{0}=0$, where $a_t$ stands for the coefficient of linear thermal dilatation and $\theta$ is increase of temperature. Distortions $\gamma_{ji}^{0}, \kappa_{ji}^{0}$ result in a field of deformations $\gamma_{ji}, \kappa_{ji}$ and stresses $\sigma_{ji}$, $\mu_{ji}$. The interrelations occurring among these quantities take the form

$$\sigma_{ji}=(\mu+a)\,(\gamma_{ji}-\gamma_{ji}^{0})+(\mu-a)\,(\gamma_{ij}-\gamma_{ij}^{0})+\lambda\delta_{ij}(\gamma_{kk}-\gamma_{kk}^{0}), \tag{1.1}$$

$$\mu_{ji}=(\gamma+\varepsilon)\,(\kappa_{ji}-\kappa_{ji}^{0})+(\gamma-\varepsilon)\,(\kappa_{ij}-\kappa_{ij}^{0})+\beta\delta_{ij}(\kappa_{kk}-\kappa_{kk}^{0}). \tag{1.2}$$

The notations introduced here, $\mu$, $\lambda$, $a$, $\beta$, $\gamma$, $\varepsilon$ are material constants. The strains $\gamma_{ji}, \kappa_{ji}$ are appertained to components of displacement vectors $\mathbf{u}$ and components of rotation vector $\boldsymbol{\varphi}$ expressed by the following equations

$$\gamma_{ji}=u_{i,j}-\varepsilon_{kji}\,\varphi_k, \tag{1.3}$$

$$\kappa_{ji}=\varphi_{i,j}. \tag{1.4}$$

Introducing (1.1), (1.2) and (1.3), (1.4) into equations of equilibrium

$$\sigma_{ji,j}+X_i=0, \tag{1.5}$$

$$\varepsilon_{ijk}\,\sigma_{jk}+\mu_{ji,j}+Y_i=0, \tag{1.6}$$

where $X_i$, $Y_i$ are the components of the vector of body forces and body couples, we arrive at

$$(1.7)\qquad (\mu+\alpha)\,\nabla^2 u_i+(\lambda+\mu-\alpha)\,u_{j,ji}+2\alpha\varepsilon_{ijk}\,\varphi_{k,j}+X_i=\sigma^0_{ji,j},$$

$$(1.8)\qquad [(\gamma+\varepsilon)\,\nabla^2-4\alpha]\,\varphi_i+(\gamma+\beta-\varepsilon)\,\varphi_{j,ji}+2\alpha\varepsilon_{ijk}\,u_{k,j}+Y_i=\varepsilon_{ijk}\,\sigma^0_{jk}+\mu^0_{ji,j}.$$

The other notations are introduced now in the following formula

$$(1.9)\qquad \sigma^0_{ji}=(\mu+\alpha)\,\gamma^0_{ji}+(\mu-\alpha)\,\gamma^0_{ij}+\lambda\delta_{ij}\,\gamma^0_{kk},$$

$$(1.10)\qquad \mu^0_{ji}=(\gamma+\varepsilon)\,\kappa^0_{ji}+(\gamma-\varepsilon)\,\kappa^0_{ij}+\beta\delta_{ij}\,\kappa^0_{kk}.$$

To Eqs. (1.7), (1.8) it is necessary to add the boundary conditions. If at the boundary $A$ of the body in question the displacements $U_i$ and rotations $\Phi_i$ are prescribed, then the boundary conditions take the form

$$(1.11)\qquad u_i(\mathbf{x})=U_i(\mathbf{x}),\qquad \varphi_i(\mathbf{x})=\Phi_i(\mathbf{x}),\qquad \mathbf{x}\in A.$$

Further, if loadings $p_i$ and moments $m_i$ are prescribed on the boundary considered, then the boundary conditions have the form:

$$(1.12)\qquad \sigma_{ji}(\mathbf{x})\,n_j(\mathbf{x})=p_i(\mathbf{x}),\qquad \mu_{ji}(\mathbf{x})\,n_j(\mathbf{x})=m_i(\mathbf{x}).$$

## 2. The theorem of the reciprocity of works

This theorem one may find in Ref. [1], where the author made use of what is called analogy of body forces and body couples. In the present note this theorem has been derived in another way.

Let an elastic body be acted upon by the two independent sets of forces. To these we may number the external forces and superimposed on the boundary of the body displacements and rotations. Either of these causes (forces) and results obtained will be marked with "primes".

Accordingly, for the system of equations being marked with primes we have the constitutive equations

$$(2.1)\qquad \sigma'_{ji}=(\mu+\alpha)\,(\gamma'_{ji}-\gamma'^0_{ji})+(\mu-\alpha)\,(\gamma'_{ij}-\gamma'^0_{ij})+\lambda\delta_{ji}\,(\gamma'_{kk}-\gamma'^0_{kk}),$$

$$(2.2)\qquad \mu'_{ij}=(\gamma+\varepsilon)\,(\kappa'_{ji}-\kappa'^0_{ji})+(\gamma-\varepsilon)\,(\kappa'_{ij}-\kappa'^0_{ij})+\beta\delta_{ji}\,(\kappa'_{kk}-\kappa'^0_{kk}).$$

Multiply now Eq. (1.2) by $\gamma'_{ji}$, whereas Eq. (2.1) by $\gamma_{ji}$ and subtract them side by side. As the result we get

$$(2.3)\qquad \sigma_{ji}\,\gamma'_{ji}+\gamma^0_{ji}\sigma'_{ji}=\sigma'_{ji}\,\gamma_{ji}+\gamma'^0_{ji}\,\sigma_{ji}.$$

Applying the same procedure to Eqs. (1.2) and (2.2), we obtain the relations

$$(2.4)\qquad \mu_{ji}\,\kappa'_{ji}+\kappa^0_{ji}\,\mu'_{ji}=\mu'_{ji}\,\kappa_{ji}+\mu_{ji}\,\kappa'^0_{ji}.$$

The following equation results from the fact that we add side by side relations (2.3) and (2.4) and integrate over the surface of the body considered

$$(2.5)\qquad \int_V(\sigma_{ji}\,\gamma'_{ji}+\mu_{ji}\,\kappa'_{ji}+\gamma^0_{ji}\,\sigma'_{ji}+\kappa^0_{ji}\,\mu'_{ji})\,dV=\int_V(\sigma'_{ji}\,\gamma_{ji}+\mu'_{ji}\kappa_{ji}+\gamma'^0_{ji}\,\sigma_{ji}+\kappa'^0_{ji}\,\mu_{ji})\,dV.$$

Take then advantage of relations (1.3), equations of equilibrium (1.4) and the Ostrogradzki—Gauss theorem for transformation of the integral given below

$$
\begin{aligned}
\int_V \sigma_{ji}\, \gamma'_{ji}\, dV &= \int_V \sigma_{ji} (u'_{i,j} - \varepsilon_{kji}\, \varphi'_k)\, dV, \\
&= \int_V [(\sigma_{ji}\, u'_i)_{,j} - \sigma_{ji,j}\, u'_i - \varepsilon_{kji}\, \varphi'_k\, \sigma_{ji}]\, dV, \qquad (2.6)\\
&= \int_A p_i\, u'_i\, dA + \int_V X_i\, u'_i\, dV - \int_V \sigma_{ji}\, \varepsilon_{kji}\, \varphi'_k, \qquad p_i = \sigma_{ji}\, n_j .
\end{aligned}
$$

Analogically we obtain the equation

$$
\begin{aligned}
\int_V \mu_{ji}\, \kappa'_{ji}\, dV &= \int_V \mu_{ji}\, \varphi'_{i,j}\, dV, \\
&= \int_V [(\mu_{ji}\, \varphi'_i)_{,j} - \mu_{ji,j}\, \varphi'_i]\, dV, \qquad (2.7)\\
&= \int_A m_i\, \varphi'_i\, dA + \int_V Y_i\, \varphi'_i\, dV + \int_V \varepsilon_{ijk}\, \sigma_{jk}\, \varphi'_i\, dV, \qquad m_i = \mu_{ji}\, n_j .
\end{aligned}
$$

Applying a similar procedure for the other integrals of Eq. (2.5), we obtain the following form of the theorem of reciprocity of works

$$
\begin{aligned}
(2.8)\qquad &\int_V (X_i\, u'_i + Y_i\, \varphi'_i)\, dV + \int_A (p_i\, u'_i + m_i\, \varphi'_i)\, dA + \int_V (\gamma^0_{ji}\, \sigma'_{ji} + \kappa^0_{ji}\, \mu'_{ji})\, dV = \\
&= \int_V (X'_i\, u_i + Y'_i\, \varphi_i)\, dV + \int_A (p'_i\, u_i + m'_i\, \varphi_i)\, dA + \int_V (\gamma'^0_{ji}\, \sigma_{ji} + \kappa'^0_{ji}\, \mu_{ji})\, dV .
\end{aligned}
$$

Further, consider some particular cases of this theorem. First let us be concerned with the strain of the body induced by distortions $\gamma^0_{ji}, \kappa^0_{ji}$; then assume that the body is fixed entirely on a certain part of its surface, whereas the remaining part of its surface is free from the loading. Therefore, it should be assumed that

$$\gamma'^0_{ji} = 0 \;\; \kappa'^0_{ji} = 0, \qquad X_i = 0, \; Y_i = 0$$

and

$$u_i = 0, \; \varphi_i = 0 \text{ on } A_u \quad \text{as well as} \quad p_i = 0, \, m_i = 0 \quad \text{on } A_\sigma, \quad A = A_u + A_\sigma .$$

Consider again the same elastic body under the same boundary conditions on $A_u$ and $A_\sigma$ and assume that the only cause of determations is the unit concentrated force, directed towards $x_k$-axis. Thus we have

$$X'_i = \delta(\mathbf{x} - \mathbf{x}')\, \delta_{ik}, \qquad Y'_i = 0, \, \mathbf{x} \in V; \;\; \text{and } u'_i = 0, \qquad \varphi'_i = 0 \text{ on } A_u,$$

$$p'_i = 0, \qquad m'_i = 0 \qquad \text{on} \qquad A_\sigma .$$

Putting the above quantities into the equation and keeping in mind that

$$\int_V \delta_{ik}\, \delta(\mathbf{x} - \mathbf{x}')\, u_i(\mathbf{x})\, dV(\mathbf{x}) = u_k(\mathbf{x}'),$$

and then introducing the notations

$$\sigma'_{ji} = \sigma^{(k)}_{ji}(\mathbf{x}, \mathbf{x}'), \qquad \mu'_{ji} = \mu^{(k)}_{ji}(\mathbf{x}, \mathbf{x}'),$$

we arrive at the following formula

$$u_k(\mathbf{x}') = \int_V [\gamma^0_{ji}(\mathbf{x})\, \sigma^{(k)}_{ji}(\mathbf{x}, \mathbf{x}') + \kappa^0_{ji}(\mathbf{x})\, \mu^{(k)}_{ji}(\mathbf{x}, \mathbf{x}')]\, dV(\mathbf{x}). \tag{2.9}$$

Here $\sigma^{(k)}_{ji}$ denotes force stresses, while $\mu^{(k)}_{ji}$ — couple stresses being induced by the action of the unit concentrated force applied to point $\mathbf{x}'$ and directed towards $x_k$-axis.

Functions $\sigma^{(k)}_{ji}$, $\mu^{(k)}_{ji}$ are expressed by means of the formulae

$$\sigma^{(k)}_{ji} = (\mu+\alpha)\, \gamma^{(k)}_{ji} + (\mu-\alpha)\, \gamma^{(k)}_{ij} + \lambda \delta_{ij}\, \gamma^{(k)}_{nn}, \tag{2.10}$$

$$\mu^{(k)}_{ji} = (\gamma+\varepsilon)\, \kappa^{(k)}_{ji} + (\gamma-\varepsilon)\, \kappa^{(k)}_{ij} + \beta \delta_{ij}\, \kappa^{(k)}_{nn}. \tag{2.11}$$

Accordingly, we have

$$\gamma^{(k)}_{ji} = G_{ik,j} - \varepsilon_{pji}\, \Phi_{pk}, \qquad \kappa_{ji} = \Phi_{ik,j}. \tag{2.12}$$

Notation $G_{ik}$ stands for the Green displacement tensor, $i$ is the $i$-th displacement component and $\Phi_{ik}$ — the Green rotation tensor. The action of force $X'_i = \delta(\mathbf{x}-\mathbf{x}')\, \delta_{ik}$ exerts an influence upon these quantities.

In turn, let the other set of forces be restricted to the concentrated body couple $Y'_i = \delta_{ik}\, \delta(\mathbf{x}-\mathbf{x}')$ assuming that

$$u'_i = 0,\ \varphi'_i = 0 \ \text{on}\ A_u \qquad \text{and} \qquad m'_i = 0,\ p'_i = 0 \ \text{on}\ A_\sigma.$$

The stresses induced by the action of the latter couple are here denoted by $\hat{\sigma}^{(k)}_{ji}$ and $\hat{\mu}^{(k)}_{ji}$, the displacement and rotational tensors being represented by $\hat{G}_{ik}$ and $\hat{\Phi}_{ik}$, respectively. Bearing in mind these assumptions, we can obtain from (2.8) the following formula for the rotation

$$\varphi_k(\mathbf{x}') = \int_V [\gamma^0_{ji}(\mathbf{x})\, \hat{\sigma}^{(k)}_{ji}(\mathbf{x}, \mathbf{x}') + \kappa^0_{ji}\, \hat{\mu}^{(k)}_{ji}(\mathbf{x}, \mathbf{x}')]\, dV(\mathbf{x}), \tag{2.13}$$

Hence we have

$$\hat{\sigma}^{(k)}_{ji} = (\mu+\alpha)\, \hat{\gamma}^{(k)}_{ji} + (\mu-\alpha)\, \hat{\gamma}^{(k)}_{ij} + \lambda \delta_{ij}\, \gamma^{(k)}_{nn}, \tag{2.14}$$

$$\hat{\mu}^{(k)}_{ji} = (\gamma+\varepsilon)\, \hat{\kappa}^{(k)}_{ji} + (\gamma-\varepsilon)\, \hat{\kappa}^{(k)}_{ij} + \beta \delta_{ij}\, \hat{\kappa}^{(k)}_{nn}, \tag{2.15}$$

where

$$\hat{\gamma}^{(k)}_{ji} = \hat{G}_{ik,j} - \varepsilon_{pji}\, \hat{\Phi}_{pk}, \qquad \hat{\kappa}^{(k)}_{ji} = \hat{\Phi}_{ik,j}. \tag{2.16}$$

The Green functions being known, accordingly, we are able to determine stresses $\sigma^{(k)}_{ji}$, $\mu^{(k)}_{ji}$, ... in virtue of formulae (2.10)—(2.12) as well as (2.14)—(2.15), and further from formulae (2.9) and (2.16) — both the displacement and the rotations are found. The sought state of stress in the elastic body considered may be obtained from formulae (1.1), (1.2), taking account of definitions (1.3) and (1.4).

Let us now analyze a simple case. Let the distortions $\gamma^0_{ji}$ act in a bounded and simply connected body, $X_i$, so we have the following relations:

$$X_i=0, \quad Y_i=0, \quad \mathbf{x}\in V \quad \text{and} \quad p_i=0, \quad m_i=0, \quad \mathbf{x}\in A.$$

Then, let in the "primed" system take place the all-sided stretching. Thus we have

$$\sigma'_{ji}=1\delta_{ij}, \quad p'_i=\sigma'_{ji}\,n_j=1n_i, \quad X'_i=0, \quad Y'_i=0, \quad \gamma'^0_{ji}=0, \quad \kappa'^0_{ji}=0, \quad \mu'_{ji}=0.$$

Eq. (2.8) yields the following relation

$$\int_A u_i\,n_i\,dA=\int_V \gamma^0_{ji}\,\delta_{ij}\,dV. \tag{2.17}$$

As can be seen the increment of the body volume may be described by the aid of the formula

$$\Delta V=\int_V u_{i,i}\,dV=\int_A u_i\,n_i\,dA.$$

From (2.17) we get

$$\Delta V=\int_V \gamma^0_{kk}\,dV. \tag{2.18}$$

It is noteworthy that this result is independent of the material constant. From relation (1.1) results that

$$\sigma_{kk}=(3\lambda+2\mu)(\gamma_{kk}-\gamma^0_{kk}). \tag{2.19}$$

If with the use of this relation we drop $\gamma^0_{kk}$ from (2.18), and simultaneously we take into consideration the dependence $\int_V \varepsilon_{kk}\,dV=V$, then we obtain

$$\int_V \sigma_{kk}\,dV=0. \tag{2.20}$$

In a particular case of thermal distorsions $\gamma^0_{ji}=a_t\,\delta_{ij}\,\theta, \quad \kappa^0_{ji}=0$, we can derive, in virtue of (2.18), the well-known relation [2]

$$\Delta V=3a_t\int_V \theta\,dV. \tag{2.21}$$

### 3. Distortions in the infinite space

Let in such an elastic space act the distortions $\gamma^0_{ji}$, $\kappa^0_{ji}$. They are situated in the bounded region $V'$ in a way, owing to which we are able to assume that for $r\to\infty$ displacements $u_k$ and rotations $\varphi_k$ tend to zero.

In this particular case the surface integrals vanish in Eq. (2.8). We have still the equation

$$\int_V (X_i u'_i+Y_i\varphi'_i)\,dV+\int_V (\gamma^0_{ji}\sigma'_{ji}+\kappa^0_{ji}\mu'_{ji})\,dV= \tag{3.1}$$
$$=\int_V (X'_i u_i+Y'_i\varphi_i)\,dV+\int_V (\gamma'^0_{ji}\sigma_{ji}+\kappa'^0_{ji}\mu_{ji})\,dV.$$

Let the "primed" system be confined to the concentrated force $X'_i = \delta_{ik}\,\delta(\mathbf{x}-\mathbf{x}')$ turned in the direction of $x_k$-axis. Thus making use of the latter equation, we arrive at

$$u_k(\mathbf{x}') = \int_V (\gamma^0_{ji}\,\sigma^{(k)}_{ji} + \kappa^0_{ji}\,\mu^{(k)}_{ji})\,dV. \tag{3.2}$$

Herein the stresses $\sigma^{(k)}_{ji}$, $\mu^{(k)}_{ji}$ take the form of (2.10) (2.12), respectively, where the Green functions $G_{ik}$, $\Phi_{ik}$ refer to the infinite space. They are expressed by the formulae [3]:

$$G_{ik} = \frac{1}{8\pi\mu}\left[\delta_{ik}R_{,pp} - \frac{\lambda+\mu}{\lambda+2\mu}R_{,ik}\right] + B\left[l^2\left(\frac{e^{-R/l}-1}{R}\right)_{,ik} - \delta_{ik}\frac{e^{-R/l}}{R}\right], \tag{3.3}$$

$$\Phi_{ik} = -\frac{1}{8\pi\mu}\,\varepsilon_{kij}\left(\frac{e^{-R/l}-1}{R}\right)_{,j}. \tag{3.4}$$

Here

$$R = [(x_i - x'_i)(x_i - x'_i)]^{1/2}, \qquad B = \frac{a}{4\mu\pi(\mu+a)}, \qquad l^2 = \frac{(\mu+a)(\gamma+\varepsilon)}{4\mu a}.$$

Notice that the displacement tensor $G_{ik}$ is composed of two members. The form of the first member is analogical to that of the displacement tensor in the classical theory of elasticity.

Now let the "primed" system be confined to a unit concentrated body couple $Y'_i = \delta_{ik}\,\delta(\mathbf{x}-\mathbf{x}')$ being directed towards $x_k$-axis.

From (3.1) we can derive the formula

$$\varphi_k(\mathbf{x}') = \int_V [\gamma^0_{ji}(\mathbf{x})\,\hat{\sigma}^{(k)}_{ji}(\mathbf{x},\mathbf{x}') + \kappa^0_{ji}(\mathbf{x})\,\hat{\mu}^{(k)}_{ji}(\mathbf{x},\mathbf{x}')]\,dV(\mathbf{x}). \tag{3.5}$$

Expressions $\hat{\sigma}^{(k)}_{ji}$, $\mu^{(k)}_{ji}$ are described by the formulae (2.13)—(2.16), where

$$\hat{G}_{ik} = -\frac{1}{8\pi\mu}\,\varepsilon_{kij}\left(\frac{e^{-R/l}-1}{R}\right)_{,j}, \tag{3.6}$$

$$\hat{\Phi}_{ik} = \frac{1}{16\pi\mu}\left(\frac{1-e^{-R/l}}{R}\right)_{,ik} + \frac{1}{16\pi a}\left(\frac{e^{-\frac{R}{h}}\,e^{-R/l}}{R}\right)_{,ik} + \frac{\mu+a}{16\pi a\mu l^2}\,\delta_{ik}\frac{e^{-R/l}}{R}, \tag{3.7}$$

where

$$h^2 = \frac{2\gamma+\beta}{4a}, \qquad l^2 = \frac{(\mu+a)(\gamma+\varepsilon)}{4\mu a}.$$

Let us take advantage of formulae (3.2) and (3.5) for the case when we consider the concentrated defect. Further, let the following distortion be given

$$\gamma^0_{ji} = \frac{1}{3}\,\delta(\mathbf{x})\,\delta_{ij}, \qquad \text{hence} \qquad \gamma^0_{kk} = \delta(\mathbf{x}). \tag{3.8}$$

From formulae (3.2) and (3.5) we obtain the following results

$$u_k(\mathbf{x}) = \frac{1}{3}\sigma_{nn}^{(k)}(\mathbf{x}, 0), \qquad \varphi_k(\mathbf{x}) = \frac{1}{3}\hat{\sigma}_{nn}^{(k)}(\mathbf{x}, 0), \tag{3.9}$$

whereas

$$\sigma_{nn}^{(k)} = KG_{nk,n}(\mathbf{x}, 0), \qquad \hat{\sigma}_{nn}^{(k)}(\mathbf{x}), 0 = K\hat{G}_{nk,n}(\mathbf{x}, 0), \qquad K = \lambda + \frac{2}{3}\mu. \tag{3.10}$$

In virtue of (3.3) and (3.6) we find that

$$\sigma_{nn}^{(k)} = \frac{3\lambda+2\mu}{24\pi(\lambda+2\mu)} R_{,pnn}, \qquad \hat{\sigma}_{nn}^{(k)} = 0, \qquad R = (x_i x_i)^{1/2}. \tag{3.11}$$

Hence

$$u_k(\mathbf{x}) = -\frac{1}{12\pi}\frac{1+\nu}{1-\nu}\frac{x_k}{R^3}, \qquad \varphi_k(\mathbf{x}) = 0, \qquad \nu = \frac{\lambda}{2(\lambda+\mu)}. \tag{3.12}$$

Proceeding, consider the defect $\kappa_{ji}^0 = \frac{1}{3}\delta(\mathbf{x})\delta_{ij}$. Basing on (3.2) and (3.5), we obtain $\kappa_{ji}^0 = \frac{1}{3}\delta(\mathbf{x})\delta_{ij}$.

$$u_k(\mathbf{x}) = \frac{2\gamma+3\beta}{3}\hat{\kappa}_{nn}^{(k)} = \frac{1}{3}(2\gamma+3\beta)\,\Phi_{nk,n}(\mathbf{x}, 0), \tag{3.13}$$

$$\varphi_k(\mathbf{x}) = \frac{2\gamma+3\beta}{3}\kappa_{nn}^{(k)} = \frac{1}{3}(2\gamma+3\beta)\,\hat{\Phi}_{nk,n}(\mathbf{x}, 0). \tag{3.14}$$

Bearing in mind that

$$\Phi_{nk,n}(\mathbf{x}, 0) = 0, \qquad \hat{\Phi}_{nk,n}(\mathbf{x}, 0) = -\frac{1}{4\pi(2\gamma+\beta)}\left(\frac{e^{-\frac{R}{h}}}{R}\right)_{,k},$$

we get, accordingly,

$$u_k = 0, \qquad \varphi_k = \frac{3\beta+2\gamma}{12\pi(2\gamma+\beta)}\left(\frac{e^{-\frac{R}{h}}}{R}\right)_{,k}. \tag{3.15}$$

### 4. The dynamic problem of distortion

Consider the case in which the distortions $\gamma_{ji}^0$, $\kappa_{ji}^0$ are not only functions of position $\mathbf{x}$ but also of time $t$. In this case the equations of equilibrium should be replaced by equations of motion

$$\sigma_{ji,j} + X_i = \rho\ddot{u}_i, \tag{4.1}$$

$$\varepsilon_{ijk}\sigma_{jk} + \mu_{ji,j} + Y_i = J\ddot{\varphi}_i. \tag{4.2}$$

As a consequence we obtain

$$(\mu+a)\nabla^2 u_i + (\lambda+\mu-a)u_{j,ji} + 2a\varepsilon_{ijk}\varphi_{k,j} + X_i = \sigma_{ji,j}^0 - \rho\ddot{u}_i, \tag{4.3}$$

$$((\gamma+\varepsilon)\nabla^2 - 4a))\varphi_i + (\beta+\gamma-\varepsilon)\varphi_{j,ji} + 2a\varepsilon_{ijk}u_{k,j} + Y_i = \varepsilon_{ijk}\sigma_{jk}^0 + \mu_{ji,j}^0 - J\ddot{\varphi}_i. \tag{4.4}$$

Perform on the constitutive relations (1.1)—(1.2) the Laplace transformation. Made the same transformation on the equations of motion (4.1) and (4.2). The latter equations after transformation, take the form

$$\bar{\sigma}_{ji,j}+\bar{X}_i=\rho p^2\,\bar{u}_i, \tag{4.5}$$

$$\varepsilon_{ijk}\,\bar{\sigma}_{jk}+\bar{\mu}_{ji,j}+\bar{Y}_i=Jp^2\,\bar{\varphi}_i. \tag{4.6}$$

We have assumed here that the initial boundary conditions are homogeneous. The following notations are introduced

$$\bar{\sigma}_{ji}(\mathbf{x},p)=\int_0^\infty \sigma_{ji}(\mathbf{x},t)\,e^{-pt}\,dt,\ \text{etc.} \tag{4.7}$$

Applying the same procedure as in point 2, operating only on the transformed quantities, we obtain

$$\begin{aligned}\int_V (\bar{\sigma}_{ji}\,\bar{\gamma}'_{ji}+\bar{\mu}_{ji}\,\bar{\kappa}'_{ji}+\bar{\gamma}^0_{ji}\,\bar{\sigma}'_{ji}+\bar{\kappa}^0_{ji}\,\bar{\mu}'_{ji})\,dV=\\ =\int_V (\bar{\sigma}'_{ji}\,\bar{\gamma}_{ji}+\bar{\mu}'_{ji}\,\bar{\kappa}_{ji}+\bar{\gamma}'^0_{ji}\,\bar{\sigma}_{ji}+\bar{\kappa}'^0_{ji}\,\bar{\mu}_{ji})\,dV.\end{aligned} \tag{4.8}$$

Performing analogical transformations as were carried out in point 2, and making use of Eqs. (4.5). (4.6), we get the equation

$$\begin{aligned}\int_V (\bar{X}_i\bar{u}'_i+\bar{Y}_i\bar{\varphi}'_i)\,dV+\int_A (\bar{p}_i\,\bar{u}'_i+\bar{m}_i\,\varphi'_i)\,dA+\int_V (\bar{\gamma}^0_{ji}\,\bar{\sigma}'_{ji}+\bar{\kappa}^0_{ji}\,\bar{\mu}'_{ji})\,dV=\\ =\int_V (\bar{X}'_i\bar{u}_i+\bar{Y}'_i\bar{\varphi}_i)\,dV+\int_A (\bar{p}'_i\,\bar{u}_i+\bar{m}'_i\,\bar{\varphi}_i)\,dA+\int_V (\bar{\gamma}'^0_{ji}\,\bar{\sigma}_{ji}+\bar{\kappa}'^0_{ji}\,\bar{\mu}_{ji})\,dV.\end{aligned} \tag{4.9}$$

Finally, we should perform the inverse Laplace transformation. In consequence we obtain

$$\begin{aligned}\int_V (X_i * u'_i+Y_i * \varphi'_i)\,dV+\int_A (p_i * u'_i+m_i * \varphi'_i)\,dA+\\ +\int_V (\gamma^0_{ji} * \sigma'_{ji}+\kappa^0_{ji} * \mu'_{ji})\,dV=\int_V (X'_i * u_i+Y'_i * \varphi_i)\,dV+\\ +\int_A (p'_i * u_i+m'_i * \varphi_i)\,dA+\int_V (\gamma'^0_{ji} * \sigma_{ji}+\kappa'^0_{ji} * \mu_{ji})\,dV\end{aligned} \tag{4.10}$$

Here

$$X_i * u'_i=\int_0^t X_i(\mathbf{x},\tau)\,u'_i(\mathbf{x},t-\tau)\,d\tau,\ \text{etc.}$$

After the same method as in point 2, on taking into account instantaneous concentrated forces and body couples we find the following formula

$$u_k(\mathbf{x}',t)=\int_V (\gamma^0_{ji} * \sigma^{(k)}_{ji}+\kappa^0_{ji} * \mu^{(k)}_{ji})\,dV(\mathbf{x}), \tag{4.11}$$

$$\varphi_k(\mathbf{x}',t)=\int_V (\gamma^0_{ji} * \hat{\sigma}^{(k)}_{ji}+\kappa^0_{ji} * \hat{\mu}^{(k)}_{ji})\,dV(\mathbf{x}). \tag{4.12}$$

The Green functions, occurring in the stresses $\sigma_{ji}^{(k)}$, $\hat{\sigma}_{ji}^{(k)}$, ..., are known merely for the unbounded elastic body, provided that we confine our considerations to the varying harmonically in time concentrated forces and body couples [3].

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В. Новацки, **Проблема дисторсий в микрополярной упругой среде**

**Содержание.** В настоящей работе выведена теорема о взаимности работ, справедливая для случая действия внешних нагрузок, а также начальных деформаций (дисторсий) $\gamma_{ji}^{0}$, $\kappa_{ji}^{0}$. Используя упомянутую теорему, приводятся формулы для перемещений и поворотов (вращений) при применении функции Грина. Рассмотрены два особых случая. Первый касается роста объема односвязного тела, подверженного действию дисторсии. Второй случай касается перемещений и поворотов (вращений) в бесконечном, упругом пространстве, вызванный простыми, сосредоточенными дефектами. В последней части работы расширена теорема о взаимности работ на динамические, дисторсионные проблемы.