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APPLIED MECHANICS
(ELASTICITY)

Thermal Stresses in a Micropolar Body Induced by the Action of a Discontinuous Temperature Field

by

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1. Introduction

The problem of thermal stresses induced by the action of a discontinuous temperature field was amply discussed in a number of papers [1]—[5]. All of them, however, dealt only with the Hooke's body.

In this paper we are concerned with thermal stresses due to the action of a discontinuous temperature field in an elastic micropolar body. The approach chosen for the solution of this problem will be explained with the help of a simple example of the bidimensional problem.

Under the effect of temperature $\theta(\mathbf{x})$ an elastic micropolar body suffers deformation characterized by two asymmetric tensors, namely the tensor of strain γ_{ji} and the curvature-twist tensor κ_{ji} [6].

$$\gamma_{ji}=u_{i,j}-\epsilon_{kji}\,\varphi_k, \qquad \kappa_{ji}=\varphi_{i,j} \tag{1.1}$$

ε_{ijk} being the known Levi—Civtá alternator.

The state of stress is characterized by two asymmetric tensors, namely the force-stress tensor σ_{ji} and the couple-stress tensor μ_{ji}.

The state of stress, the state of strain and the temperature are connected by the following relations [7]

$$\begin{aligned}\sigma_{ji}&=(\mu+a)\,\gamma_{ji}+(\mu-a)\,\gamma_{ij}+(\lambda\gamma_{kk}-\nu\theta)\,\delta_{ij},\\ \mu_{ji}&=(\gamma+\varepsilon)\,\kappa_{ji}+(\gamma-\varepsilon)\,\kappa_{ij}+\beta\kappa_{kk}\,\delta_{ij}, \qquad \nu=(3\lambda+2\mu)\,a_t.\end{aligned} \tag{1.2}$$

Here the symbols $\mu, \lambda, a, \beta, \gamma, \varepsilon$ denote the material constants of the micropolar body, while a_t stands for the coefficient of linear thermal dilatation. Introducing (1.2) into the equations of equilibrium

$$\sigma_{ji,j}=0, \qquad \epsilon_{ijk}\,\sigma_{jk}+\mu_{ji,j}=0, \tag{1.3}$$

and making use of the relations (1.1) we arrive at a system of differential equations in displacement and rotations [7].

$$(1.4)\qquad \begin{aligned} &(\mu+a)\,\nabla^2\,\mathbf{u}+(\lambda+\mu-a)\,\mathrm{grad}\,\mathrm{div}\,\mathbf{u}+2a\,\mathrm{rot}\,\boldsymbol{\varphi}=\nu\,\mathrm{grad}\,\theta\,,\\ &(\gamma+\varepsilon)\,\nabla^2\,\boldsymbol{\varphi}-4a\,\boldsymbol{\varphi}+(\gamma+\beta-\varepsilon)\,\mathrm{grad}\,\mathrm{div}\,\boldsymbol{\varphi}+2a\,\mathrm{rot}\,\mathbf{u}=0\,.\end{aligned}$$

The above equations have to be supplemented with boundary conditions. Assuming the boundary of the body to be free of loadings we get them in the following form

$$(1.5)\qquad \sigma_{ji}\,n_j=0\,,\qquad \mu_{ji}\,n_j=0\,,$$

where the symbol $\mathbf{n}$ denotes the unit vector of the normal to the surface A.

2. The solution of differential equations of thermoelasticity

The solution of the system of Eqs. (1.4) will be given in the form of a sum of two partial solutions

$$(2.1)\qquad \mathbf{u}=\mathbf{u}'+\mathbf{u}''\,,\qquad \boldsymbol{\varphi}=\boldsymbol{\varphi}'+\boldsymbol{\varphi}''\,.$$

The solution

$$(2.2)\qquad \mathbf{u}'=\mathrm{grad}\,\boldsymbol{\Phi}\,,\qquad \boldsymbol{\varphi}'=0$$

is the particular integral of the non-homogeneous system of Eqs. (1.4), while $\mathbf{u}''$, $\boldsymbol{\varphi}''$ are general integrals of the homogeneous equations

$$(2.3)\qquad \begin{aligned} &(\mu+a)\,\nabla^2\,\mathbf{u}''+(\lambda+\mu-a)\,\mathrm{grad}\,\mathrm{div}\,\mathbf{u}''+2a\,\mathrm{rot}\,\boldsymbol{\varphi}''=0\,,\\ &(\gamma+\varepsilon)\,\nabla^2\,\boldsymbol{\varphi}''-4a\,\boldsymbol{\varphi}''+(\beta+\gamma-\varepsilon)\,\mathrm{grad}\,\mathrm{div}\,\boldsymbol{\varphi}''+2a\,\mathrm{rot}\,\mathbf{u}''=0\,.\end{aligned}$$

Substituting (2.2) into the system of equations (1.4), we obtain the Poisson's equation, which makes it possible to determine the potential of the thermoelasticity Φ

$$(2.4)\qquad \nabla^2\,\Phi=m\theta\,,\qquad m=\frac{\nu}{\lambda+2\mu}\,.$$

Let us remark that this equation is identical — as to its form — with that describing the potential Φ in classical thermoelasticity (in Hooke's body).

The stresses connected with the function

$$(2.5)\qquad \sigma''_{ji}=2\mu\,(\Phi_{ij}-\delta_{ij}\,\Phi_{,kk})\,,\qquad \mu'_{ji}=0\,,\qquad i,j,k=1,2,3$$

have also identical forms for Hooke's and micropolar Cosserat bodies. The asymmetric stresses σ''_{ji}, μ''_{ji} are connected with the solutions $\mathbf{u}''$, $\boldsymbol{\varphi}''$ of the system of Eqs. (2.3).

Consider an infinite elastic Hooke's body. Assume the temperature discontinuity $\theta^{(i)}-\theta^{(e)}$ on its Γ surface. The symbols $\theta^{(i)}$ and $\theta^{(e)}$ denote the values of the function θ on the surface Γ, when approaching this surface from its internal or external side, respectively.

Let us assume a rectangular coordinate system y_1, y_2, y_3 situated at an arbitrary point S on the surface Γ, the y_3-axis being normal to this surface.

It results from Goodier's considerations [1] that — owing to the properties of the volume potential Φ — the displacements $\mathbf{u}'$ are continuous functions also in the case of a discontinuous temperature field. Similarly, the shear stresses $\sigma_{\alpha\beta}$ ($\alpha, \beta=1, 2, 3$; $\alpha\neq\beta$) and the normal stresses σ_{33} are continuous within the whole region. Instead, the stresses σ_{11} and σ_{22} when passing through the surface Γ display a jump of the value

$$\sigma_{11}^{(i)}-\sigma_{11}^{(e)}=\sigma_{22}^{(i)}-\sigma_{22}^{(e)}=-2\mu m\left(\theta^{(i)}-\theta^{(e)}\right). \tag{2.6}$$

Goodier's considerations hold true also for an infinite elastic micropolar medium in view of the identical forms of Eqs. (2.4) for both media.*)

The above considerations hold true for a bounded body. Hooke's and micropolar bodies as well as a bounded body may be considered as a region separated from an infinite space. The surface of a separated body A will not be free — in the general case — of loadings ($p_i'=\sigma_{ji}' n_j$) due to the potential Φ. These loadings have to be removed by supplementing the state of stresses σ_{ji}', μ_{ji}'' by the state of stresses σ_{ji}'' μ_{ji}'' connected with the solutions $\boldsymbol{\mu}'', \boldsymbol{\varphi}''$ of the system of Eqs. (2.3). The integration constants appearing in the solutions of the system of Eqs. (2.3) will be obtained from the boundary conditions

$$(\sigma_{ji}'+\sigma_{ji}'')\, n_j=0, \qquad \mu_{ji}''\, n_j=0. \tag{2.7}$$

In classical thermoelasticity still another method of determining the Φ potential is known. It consists in making use of Green's function $\hat{\Phi}$ verifying the differential equation

$$\nabla^2\, \hat{\Phi}\,(\mathbf{x}, \boldsymbol{\xi})=m\delta\,(\mathbf{x}-\boldsymbol{\xi}) \tag{2.8}$$

with the boundary condition $\hat{\Phi}=0$. On the right-hand side of Eq. (2.8) there is the nucleus of the temperature $\theta=\delta\,(\mathbf{x}-\boldsymbol{\xi})$ at the point $\boldsymbol{\xi}$. Integrating the equation

$$\Phi\,(\mathbf{x})=\int_V \theta\,(\boldsymbol{\xi})\, \hat{\Phi}\,(\boldsymbol{\xi}, \mathbf{x})\, dV\,(\boldsymbol{\xi}) \tag{2.9}$$

we obtain the potential $\Phi\,(\mathbf{x})$ for the prescribed distribution of the discontinuous temperature.

After the stresses $\hat{\sigma}_{ji}'\,(\mathbf{x}, \boldsymbol{\xi})$ connected with the potential $\hat{\Phi}$ have been determined we get

$$\sigma_{ji}'\,(\mathbf{x})=\int_V \theta\,(\boldsymbol{\xi})\, \hat{\sigma}_{ji}'\,(\boldsymbol{\xi}, \mathbf{x})\, dV\,(\boldsymbol{\xi}). \tag{2.10}$$

Let us point to the fact that the function $\hat{\Phi}$ will appear in the solution of Eq. (2.8) as a singular function. The singularity will appear also in stresses $\hat{\sigma}_{ji}'$. It deserves

*) The only difference consists in different values of Lamé's constants μ, λ for either of these media.

attention that the singularity may appear only in σ'_{ij} stresses, while the σ''_{ji} and μ''_{ji} are regular.

It should be stressed that certain solutions pertaining to thermal stresses due to a discontinuous distribution of temperature are identical for both the Hooke's and micropolar media. It was shown in [8] that in a hollow sphere — provided the symmetry of the temperature field with respect to a given point is preserved — as well as in a hollow cylinder — provided there is the axial symmetry of the temperature field — the system of Eqs. (1.4) simplifies to a single equation, identical as to its form with that valid for the Hooke's body. The same considerations hold true for the discontinuous temperature field.

3. The discontinuous temperature field in an elastic half-space

Let us consider *exempli modo* a simple bidimensional problem. Let a temperature nucleus $\theta=\delta(x_1-\xi_1)\,\delta(x_2)$ act in an elastic half-space $x_1\geqslant 0$ along the $x_1=\xi_1, x_2=0$ line. We have to determine in the half-space the stresses $\hat{\sigma}_{ji}, \hat{\mu}_{ji}$ assuming the boundary $x_1=0$ to be free of loadings:

$$\hat{\sigma}_{11}=0, \quad \hat{\sigma}_{12}=0, \quad \hat{\mu}_{13}=0 \quad \text{for} \quad x_1=0. \tag{3.1}$$

We are going first to solve the Poisson's Eq. (2.8)

$$\nabla^2\hat{\Phi}=m\delta(x_1-\xi_1)\,\delta(x_2) \tag{3.2}$$

with the boundary condition $\hat{\Phi}(0,x_2)=0$ and regularity condition $\hat{\Phi}=0$ for $|x_1^2+x_2^2|\to\infty$.

The solution of Eq. (3.2) is known [9]. It reads as follows

$$\hat{\Phi}=-\frac{2m}{\pi^2}\int_0^\infty\int_0^\infty \frac{\sin a_1\xi_1\sin a_1 x_1}{a_1^2+a_2^2}\cos a_2x_2\,da_1\,da_2 \tag{3.3}$$

or else

$$\hat{\Phi}=-\frac{m}{4\pi}\ln\frac{x_2^2+(x_1+\xi_1)^2}{x_2^2+(x_1-\xi_1)^2}. \tag{3.4}$$

The function $\hat{\Phi}$ shows the singularity of logarithmic type at the point $(\xi_1, 0)$; for $|x_1^2+x_2^2|\to\infty$ the function $\hat{\Phi}$ tends to zero.

The components of the state of stress $\hat{\sigma}'_{ji}$ may be described by the following formulae

$$\begin{aligned}\hat{\sigma}'_{11}&=-\hat{\sigma}'_{22}=-\frac{m\mu}{\pi}\left[\frac{(x_1+\xi_1)^2-x_2^2}{r_2^4}-\frac{(x_1-\xi_1)^2-x_2^2}{r_1^4}\right],\\ \hat{\sigma}'_{12}&=\frac{2\mu m}{\pi}x_2\left(\frac{x_1+\xi_1}{r_2^4}-\frac{x_1-\xi_1}{r_1^4}\right),\qquad r_{1,2}=[(x_1\mp\xi_1)^2+x_2^2]^{1/2}.\end{aligned} \tag{3.5}$$

For $x_1=0$ the normal stresses $\hat{\sigma}'_{11}$ and $\hat{\sigma}'_{22}$ vanish, while the stress $\hat{\sigma}'_{12}$ remains different from zero. When approaching the point $(\xi_1, 0)$ the stresses increase boundlessly.

At the boundary $x_1=0$ we have

$$\hat{\sigma}'_{12}(0, x_2)=\frac{4\mu m}{\pi}\,\frac{\xi_1 x_2}{(x_1^2+\xi_1^2)^2}=\frac{2\mu m}{\pi}\int_0^\infty e^{-a_2\xi_1}a_2\sin a_2 x_2\,da_2. \tag{3.6}$$

In order to suppress the stresses $\hat{\sigma}'_{12}(0, x_2)$ we supplement the state of stress $\hat{\sigma}'_{ji}$ by the state of stress $\hat{\sigma}''_{ji}$, $\hat{\mu}''_{ji}$. This additional state of stress will be expressed in the micropolar medium by the function of stresses F and Ψ in the following way [10]

$$\begin{aligned}
&\hat{\sigma}''_{11}=\partial_2^2 F-\partial_1\partial_2\Psi, && \hat{\sigma}''_{22}=\partial_1^2 F+\partial_1\partial_2\Psi,\\
&\hat{\sigma}''_{12}=-\partial_1\partial_2 F-\partial_2^2\Psi, && \hat{\sigma}''_{21}=-\partial_1\partial_2 F+\partial_1^2\Psi,\\
&\hat{\mu}''_{13}=\partial_1\Psi, && \hat{\mu}''_{23}=\partial_2\Psi.
\end{aligned} \tag{3.7}$$

The functions F and Ψ have to satisfy the differential equations

$$\nabla_1^2\nabla_1^2 F=0, \qquad \nabla_1^2(1-l^2\nabla_1^2)\Psi=0. \tag{3.8}$$

Here

$$\nabla_1^2=\partial_1^2+\partial_2^2, \qquad l^2=\frac{(\mu+a)(\gamma+\varepsilon)}{4\mu a}.$$

The functions F and Ψ are connected with each other by the relations

$$\begin{aligned}
&-\partial_1(1-l^2\nabla_1^2)\Psi=A\partial_2\nabla_1^2 F,\\
&\partial_2(1-l^2\nabla_1^2)\Psi=A\partial_1\nabla_1^2 F, \qquad A=\frac{(\lambda+2\mu)(\gamma+\varepsilon)}{4\mu(\lambda+\mu)}.
\end{aligned} \tag{3.9}$$

The state of stress $\hat{\sigma}''_{ji}$, $\hat{\mu}''_{ji}$ should be chosen so as to ensure at the boundary $x_1=0$ the following conditions:

$$\hat{\sigma}''_{11}=0, \qquad \hat{\sigma}''_{12}+\hat{\sigma}'_{12}=0, \qquad \hat{\mu}''_{13}=0. \tag{3.10}$$

The functions F and Ψ will be chosen in the form

$$\begin{aligned}
F&=\int_0^\infty (M+Nx_1a_2)e^{-a_2x_1}\cos a_2x_2\,da_2,\\
\Psi&=\int_0^\infty (Ce^{-a_2x_1}+De^{-\varrho x_1})\sin a_2x_2\,da_2, \qquad \rho=\left(a_2^2+\frac{1}{l^2}\right)^{1/2}.
\end{aligned} \tag{3.11}$$

For $|x_1^2+x_2^2|\to\infty$ these functions tend to zero. The M, N, C, D constants will be determined from the boundary conditions (3.10) and from the relation (3.9).

There is

$$\begin{aligned}
&M=0, \qquad Ca_2+\rho D=0,\\
&a_2^2(N+C+D)=-\frac{2\mu m}{\pi}a_2 e^{-a_2\xi_1'},\\
&C=2a_2^2AN.
\end{aligned} \tag{3.12}$$

Whence

$$C=-\frac{4\mu mA}{\pi\Delta_0}a_2\,e^{-\alpha_2\xi_1},\qquad N=-\frac{2\mu m}{\pi a_2\,\Delta_0}e^{-\alpha_2\xi_1},\qquad D=-\frac{a_2}{\rho}C,$$

$$\Delta_0=1+2Aa_2^2\left(1-\frac{a_2}{\rho}\right).$$

Taking profit of the formulae (3.7), we determine successively:

$$
(3.13)\qquad
\begin{aligned}
\hat{\sigma}''_{11}&=\frac{2\mu m}{\pi}\int_0^\infty\frac{a_2\,e^{-\alpha_2\xi_1}}{\Delta_0}\left[a_2\,x_1\,e^{-\alpha_2x_1}-2Aa_2^2\,(e^{-\alpha_2x_1}-e^{-\rho x_1})\right]\cos a_2\,x_2\,da_2\,,\\
\hat{\sigma}''_{22}&=\frac{2\mu m}{\pi}\int_0^\infty\frac{a_2\,e^{-\alpha_2\xi_1}}{\Delta_0}\left[(2-a_2\,x_1)\,e^{-\alpha_2x_1}+\right.\\
&\qquad\left.+2Aa_2^2\,(e^{-\alpha_2x_1}-e^{-\rho x_1})\right]\cos a_2\,x_2\,da_2\,,\\
\hat{\sigma}''_{12}&=\frac{2\mu m}{\pi}\int_0^\infty\frac{a_2\,e^{-\alpha_2\xi_1}}{\Delta_0}\left[(1-a_2\,x_1)\,e^{-\alpha_2x_1}-\right.\\
&\qquad\left.-2Aa_2^2\left(e^{-\alpha_2x_1}-\frac{a_2}{\rho}e^{-\rho x_1}\right)\right]\sin a_2\,x_2\,da_2\,,\\
\hat{\sigma}''_{21}&=\frac{2\mu m}{\pi}\int_0^\infty\frac{a_2\,e^{-\alpha_2\xi_1}}{\Delta_0}-\left[(1a_2\,x_1)\,e^{-\alpha_2x_1}+\right.\\
&\qquad\left.+2Aa_2\,(a_2\,e^{-\alpha_2x_1}-\rho e^{-\rho x_1})\right]\sin a_2\,x_2\,da_2\,,\\
\hat{\mu}''_{13}&=\frac{4\mu mA}{\pi}\int_0^\infty\frac{a_2^2\,e^{-\alpha_2x_1}}{\Delta_0}\,(e^{-\alpha_2x_1}-e^{-\rho x_1})\sin a_2\,x_2\,da_2\,,\\
\hat{\mu}''_{23}&=\frac{4\mu mA}{\pi}\int_0^\infty\frac{a_2^2\,e^{-\alpha_2x_1}}{\Delta_0}\left(e^{-\alpha_2x_1}-\frac{a_2}{\rho}e^{-\rho x_1}\right)\cos a_2\,x_2\,da_2\,.
\end{aligned}
$$

Let us remark that the singularity appears solely in σ'_{ji} stresses, whereas in the half--space considered the $\hat{\sigma}''_{ji}$, $\hat{\mu}''_{ji}$ stresses are regular functions.

In the particular case of Hooke's body the $\hat{\sigma}''_{ji}$, $\hat{\mu}''_{ji}$ stresses assume a particularly simple form. We pass to the Hooke's body putting in the formulae (3.13):$a=0$, $\rho=\xi$, $\Delta_0=1$. The stresses $\hat{\sigma}''_{ji}$, $\hat{\mu}''_{ji}$ for the Hooke's body may be written in a closed form:

$$
(3.14)\qquad
\begin{aligned}
\hat{\sigma}''_{11}&=\frac{4\mu m}{\pi r_2^6}\,x_1\,(x_1+\xi_1)\,[(x_1+\xi_1)^2-3x_2^2]\,,\\
\hat{\sigma}''_{22}&=\frac{4\mu m}{\pi r_2^4}\left[(x_1+\xi_1)\,(3x_1+\xi_1)-x_2^2-x_1\,(x_1+\xi_1)\,\frac{3\,(x_1+\xi_1)^2-x_2^2}{r_2^2}\right],\\
\hat{\sigma}''_{12}&=\hat{\sigma}''_{21}=-\frac{4\mu m}{\pi r_2^4}\,x_2\left[(x_1+\xi_1)+x_1\,\frac{x_2^2-3\,(x_1+\xi_1)^2}{r_2^2}\right],\\
\hat{\mu}''_{13}&=0,\qquad \hat{\mu}''_{23}=0\,.
\end{aligned}
$$

The final stresses $\hat{\sigma}_{ji}, \hat{\mu}_{ji}$ due to the action of the temperature nucleus situated at the point (ξ_1, ξ_2) will be obtained by adding the formulae (3.6) and (3.13) and substituting $x_2 - \xi_2$ for x_2. If within the region Ω the temperature distribution $\theta(x_1, x_2)$ is given, while outside this region there is $\theta = 0$, then the stresses σ_{ji} may be determined from the formula

$$(3.15) \qquad \sigma_{ji}(x_1, x_2) = \iint_{\Omega} \theta(\xi_1 \xi_2) \left[\hat{\sigma}'_{ji}(\xi_1, \xi_2; x_1, x_2) + \hat{\sigma}''_{ji}(\xi_1 \xi_2; x_1, x_2)\right] d\xi_1\, d\xi_2 .$$

A similar formula for the μ_{ji} stresses reads as follows

$$(3.16) \qquad \mu_{ji}(x_1, x_2) = \iint_{\Omega} \theta(\xi_1, \xi_2)\, \hat{\mu}''_{ji}(\xi_1, \xi_2; x_1, x_2)\, d\xi_1\, d\xi_2 .$$

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В. НОВАЦКИЙ, ТЕРМИЧЕСКИЕ НАПРЯЖЕНИЯ В МИКРОПОЛЯРНОМ ТЕЛЕ, ВЫЗВАННЫЕ ДЕЙСТВИЕМ РАЗРЫВНОГО ПОЛЯ ТЕМПЕРАТУРЫ

В настоящей заметке автор обсуждает проблему термических напряжений в микрополярном теле, вызванных действием разрывного поля температуры.

Решение вопроса состоит из двух частей: 1) симметрические напряжения σ'_{ji}, связанные с потенциалом перемещения и 2) не-симметрические напряжения σ''_{ji}, μ''_{ji}, связанные с решением системы уравнений (в перемещениях и оборотах) микрополярной упругости.