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APPLIED MECHANICS

Problem of Linear Coupled Magneto-thermoelasticity I. Energetic Theorem and Uniqueness Theorem of Solutions

by

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1. Introduction

In the present paper we shall derive the energetic theorem; it will be used subsequently to obtain the theorem on uniqueness of solutions of equations of magneto-thermoelasticity. Our considerations refer to an homogeneous and isotropic medium, its electric conductivity being assumed finite.

The system of basic equations of magneto-thermoelasticity consists of:

a) equations of electrodynamics of slowly moving media [1], [2]

$$\text{rot}\,\boldsymbol{h} = \frac{4\pi}{c}\boldsymbol{j}, \tag{1.1}$$

$$\text{rot}\,\boldsymbol{E} = -\frac{\mu_0}{c}\frac{\partial \boldsymbol{h}}{\partial t}, \tag{1.2}$$

$$\boldsymbol{j} = \lambda_0\left[\boldsymbol{E} + \frac{\mu_0}{c}\left(\frac{\partial \boldsymbol{u}}{\partial t}\times\boldsymbol{H}\right)\right], \tag{1.3}$$

$$\text{div}\,\boldsymbol{h} = 0, \tag{1.4}$$

b) equations of motion

$$\sigma_{ij,j} + X_i + T_{ij,j} = \varrho\ddot{u}_i, \qquad i,j = 1, 2, 3, \tag{1.5}$$

and

c) equation of heat conductivity [3] in a coupled form

$$\nabla^2\theta - \frac{1}{\varkappa}\frac{\partial\theta}{\partial t} - \eta\,\text{div}\,\dot{\boldsymbol{u}} = -\frac{Q}{\varkappa}. \tag{1.6}$$

In the equations above the following notations are used: $\boldsymbol{h}$ and $\boldsymbol{E}$ stand for the vectors of magnetic and electric field intensities, respectively, $\boldsymbol{j}$ denotes the vector of current density, $\boldsymbol{H}$ — the vector of primary, constant field, $\boldsymbol{u}$ — the displacement vector,

μ_0 — the magnetic permeability, c — velocity of light and, finally, λ_0 — the electric conductivity.

The symbol σ_{ij} denotes, as usually in our papers, the stress tensor and T_{ij} — the Maxwell's tensor of the tension of electromagnetic field. $\boldsymbol{X}$ — is the vector of body forces and ϱ the density.

$\theta = T - T_0$ is the difference between the absolute temperature T and that T_0 of the natural thermic state of the body; Q means the function describing the intensity of heat sources, $Q = W/\varrho c_\varepsilon$, where W denotes the quantity of heat generated per volume unit of the body in a time unit; c_ε denotes the specific heat of the body its deformation being assumed constant and $\varkappa = k/\varrho c_\varepsilon$ is a coefficient, k denoting the heat conductivity.

The equations given above should be supplemented with relations between the stresses, deformations and temperature. They are called Duhamel—Neumann relations

$$\sigma_{ij} = 2\mu\varepsilon_{ij} + (\lambda e - \gamma\theta)\,\delta_{ij}, \qquad e = \varepsilon_{kk} \tag{1.7}$$

Moreover, the following relations are to be taken into consideration

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}), \qquad i,j = 1,2,3. \tag{1.8}$$

The tensor T_{ij} appearing in Eq. (1.5) may be expressed by the components of the $\boldsymbol{h}$ and $\boldsymbol{H}$ vectors in the following form:

$$T_{ij} = \frac{\mu_0}{4\pi}\left[h_i H_j + h_j H_i - \delta_{ij}(h_k H_k)\right], \qquad i,j,k = 1,2,3. \tag{1.9}$$

In the sequal we shall make use of the following equation

$$\Delta^2 \boldsymbol{h} - \beta\dot{\boldsymbol{h}} = -\beta \operatorname{rot}(\dot{\boldsymbol{u}} \times \boldsymbol{H}), \qquad \beta = \frac{4\pi\mu_0\lambda_0}{c^2}; \tag{1.10}$$

it is obtaied by elimination of the quantities j and $\boldsymbol{E}$ from Eqs. (1.1)—(1.3) and by taking advantage of Eq. (1.4). Introducing the relations (1.7)—(1.9) into Eqs. (1.5), we arrive at the following displacement equations

$$\mu\nabla^2\boldsymbol{u} + (\lambda+\mu)\operatorname{grad}\operatorname{div}\boldsymbol{u} + \frac{\mu_0}{c}(j \times \boldsymbol{H}) + \boldsymbol{X} = \varrho\ddot{\boldsymbol{u}}. \tag{1.11}$$

μ and λ are here Lamé's constants measured in isothermic conditions.

2. Basic energetic theorem

Let us multiply the equation of motion, Eq. (1.5), by the velocity of displacement $v_i = \dot{u}_i$ and integrate it with respect to the volume of the body.

The integral

$$\int_B (\sigma_{ij,j} + T_{ij,j} + X_i)\, v_i\, dV = \varrho \int_B \dot{v}_i v_i\, dV, \tag{2.1}$$

expresses here the principle of momentum conservation. Applying the theorem on divergence, we transform the integral (2.1) to the form

$$\int_B X_i v_i dV + \int_A p_i v_i dA = \varrho \int_B \dot{v}_i v_i dV + \int_B T_{ij} \dot{\varepsilon}_{ij} dV + \int_B \sigma_{ij} \dot{\varepsilon}_{ij} dV, \tag{2.2}$$

where $p_i = (\sigma_{ij} + T_{ij}) n_j$.

Expressing the stress σ_{ij} by the Duhamel—Neumann equation, we obtain

$$\int_B X_i v_i dV + \int_B p_i v_i dA = \varrho \int_B \dot{v}_i v_i dV + \int_B T_{ij} \dot{\varepsilon}_{ij} dV + \int_B (2\mu \varepsilon_{ij} \dot{\varepsilon}_{ij} + \lambda e \dot{e})\, dV - \gamma \int_B \theta \dot{e}\, dV. \tag{2.3}$$

Eq. (2.3) may be written as well in the form

$$\frac{dK}{dt} + \frac{dW}{dt} = \int_B X_i v_i\, dV + \int_A p_i v_i\, dA + \gamma \int_B \theta \dot{e}\, dV - \int_B T_{ij} \dot{\varepsilon}_{ij}\, dV. \tag{2.4}$$

The symbol K means here kinetic energy and W — the deformation work. These quantities will be expressed as follows:

$$K = \frac{1}{2} \varrho \int_B v_i v_i\, dV, \qquad W = \int_B \left(\mu \varepsilon_{ij} \varepsilon_{ij} + \frac{\lambda e^2}{2} \right) dV.$$

On the right-hand side of Eq. (2.4) there appear in explicit form the following magnitudes: causes inducing motion, body forces X_i and surface loads, p_i. As regards the causes of thermic and electromagnetic type, they appear in an implicit form. To have them in an explicit form, too, we take advantage of Eqs. (1.6) and (1.10).

To this end we multiply Eq. (1.6) by θ and integrate it over the body region

$$\varkappa \int_B \theta \nabla^2 \theta dV = \int_B (\dot{\theta} + \eta \varkappa \dot{e} - Q)\, \theta dV. \tag{2.5}$$

Performing on Eq. (2.5) Green's trnasformation, we obtain

$$\int_B \dot{e}\, \theta dV = \frac{1}{\eta \varkappa} \int_B Q\, \theta dV + \frac{1}{\eta} \int_A \theta \theta_{,n}\, dA - \frac{1}{\eta \varkappa} \int_B \theta \dot{\theta}\, dV - \frac{1}{\eta} \int_B \theta_{,i} \theta_{,i}\, dV \tag{2.6}$$

Substituting Eq. (2.6) into Eq. (2.4) and introducing the function of thermic energy P and dissipation function χ_0

$$P = \frac{\gamma}{2\eta\varkappa} \int_B \theta^2\, dV, \qquad \chi_0 = k T_0 \int_B \left(\frac{\theta_{,i}}{T_0} \right)^2 dV,$$

we get

$$\frac{dK}{dt} + \frac{dW}{dt} + \frac{dP}{dt} + \chi_0 = \int_B X_i v_i\, dV + \int_A p_i v_i\, dA + \frac{c_\varepsilon}{T_0} \int_B Q\theta dV + \frac{k}{T_0} \int_A \theta \theta_{,n}\, dA - \int_B T_{ij} \dot{\varepsilon}_{ij}\, dV. \tag{2.7}$$

Let us remark that for $T_{ij} \to 0$ Eq. (2.7) reduces to the energetic equation derived by J. H. Weiner in [4] for the thermoelastic problem.

We have now to transform the last integral in the right-hand part of Eq. (2.7). To simplify our subsequent considerations we assume $\boldsymbol{H} = (0, 0, H)$, i.e. we assume the vector of the primary magnetic field to act along the x_3-axis. It does not encroach on the generality of our considerations. The $\boldsymbol{H}$-field being assumed as indicated above we take into account the relation (1.9) and in this way we arrive at

$$(2.8) \qquad \int_B T_{ij}\dot{\varepsilon}_{ij}\,dV = \frac{\mu_0 H}{4\pi}\int_B [h_j(\dot{u}_{j,3} + \dot{u}_{3,j}) - \dot{e}h_3]\,dV = \\ = \frac{\mu_0 H}{4\pi}\int_B (h_j\dot{u}_{j,3} - h_3\dot{e})\,dV + \frac{\mu_0 H}{4\pi}\int_A \dot{u}_3 h_j n_j\,dA.$$

Use was made here of Eq. (1.4), i.e. $h_{j,j} = 0$.
Assuming $\boldsymbol{H} = (0, 0, H)$ Eq. (1.10), reads as below

$$(2.9) \qquad \nabla^2 \boldsymbol{h} - \beta\dot{\boldsymbol{h}} = -\beta H\boldsymbol{\varphi}, \quad \boldsymbol{\varphi} = (\dot{u}_{1,3}, \dot{u}_{2,3}, -\dot{u}_{1,1} - \dot{u}_{2,2}).$$

We multiply now the first of Eqs. (2.9) by h_1 and integrate it over the body region. We obtain

$$(2.10) \qquad \int_B (\nabla^2 h_1 - \beta\dot{h}_1)h_1\,dV = -\beta H\int_B \dot{u}_{1,3}h_1\,dV.$$

and, after performing Green's transformation, we have

$$(2.11) \qquad \int_A h_1\frac{\partial h_1}{\partial n}\,dA - \int_B h_{1,i}h_{1,i}\,dV - \beta\int_B h_1\dot{h}_1\,dV = -\beta H\int_B h_1\dot{u}_{1,3}\,dV.$$

Performing similar operations on two subsequent Eqs. (2.9) and additionning Eqs. (2.11) with two subsequent analogous equations, we have

$$(2.12) \qquad \int_A h_j\frac{\partial h_j}{\partial n}\,dA - \int_B h_{j,i}h_{j,i}\,dV - \beta\int_B \dot{h}_j h_j\,dV = -\beta H\int_B (h_j\dot{u}_{j,3} - h_3\dot{e})\,dV.$$

Taking into account Eqs. (2.8) and (2.12), we get

$$(2.13) \qquad \int_B T_{ij}\dot{\varepsilon}_{ij}\,dV = \frac{\mu_0}{4\pi\beta}\int_B h_{j,i}h_{j,i}\,dV + \frac{\mu_0}{4\pi}\int_B h_j\dot{h}_j\,dV - \frac{\mu_0}{4\pi}\int_A h_j h_{j,n}\,dA + \\ + \frac{\mu_0 H}{4\pi}\int_A \dot{u}_3 h_j n_j\,dA.$$

Now inserting Eq. (2.13) into Eq. (2.7) and introducing the functions

$$D = \frac{\mu_0}{8\pi}\int_B h_j h_j\,dV, \qquad \chi_h = \frac{\mu_0}{4\pi\beta}\int_B h_{j,i}h_{j,i}\,dV,$$

we obtain the final form of the energetic theorem which reads as follows

$$\frac{d}{dt}(K+W+P+D)+\chi_\theta+\chi_h = \int_B X_i v_i \, dV + \int_A p_i v_i \, dA + \frac{c_\varepsilon}{T_0}\int_B Q\theta \, dV + \frac{k}{T_0}\int_A \theta\theta_{,n} \, dA + \frac{\mu_0}{4\pi\beta}\int_A h_j h_{j,n} \, dA - \frac{\mu_0 H}{4\pi}\int_A \dot{u}_3 h_j n_j \, dA. \tag{2.14}$$

Here, in the right-hand part of the equation all caused inducing the movement of the body appear in explicit form. Several particular cases are included in Eq. (2.14) in its form as above.

Thus, assuming the absence of primary magnetic field, we have $h_i = 0$ and, consequently, Eq. (2.14) takes the following form:

$$\frac{d}{dt}(K+W+P)+\chi_\theta = \int_B X_i v_i \, dV + \int_A \hat{p}_i v_i \, dA + \frac{c_\varepsilon}{T_0}\int_B Q\theta \, dV + \frac{k}{T_0}\int_A \theta\theta_{,n} \, dA, \qquad \hat{p}_i = \sigma_{ij} n_j \tag{2.15}$$

which holds for problems of thermoelasticity. If the heat sources are lacking and the surface of the body is thermally insulated, Eq. (2.15) reduces to the energetic equation of elastokinetics

$$\frac{d}{dt}(K+W) = \int_B X_i v_i \, dV + \int_A \hat{p}_i v_i \, dA. \tag{2.16}$$

Let us observe that the quantities μ and λ appearing in the expression for the deformation energy, W, assume adiabatic values.

3. Uniqueness theorem

The demonstation of the uniqueness of solutions of equations of magneto-thermoelasticity may be derived basing on the energetic theorem, Eq. (2.14). Assume there are two solutions, one of them being characterized by the quantities u_i', θ', h_i' and the second — by u_i'', θ'', h_i''. Denoting the difference between these two solutions by

$$u_i^* = u_i' - u_i'', \qquad \theta^* = \theta' - \theta'', \qquad h_i^* = h_i' - h_i'', \tag{3.1}$$

it is easily seen that the functions u_i^*, θ^*, h_i^* satisfy homogeneous differential equations, homogeneous boundary conditions and homogeneous initial conditions. Thus, the solutions u_i^*, θ^*, h_i^* refer to a body wherein body forces and heat sources inside the body are lacking. At the same time there are no loads, Maxwell's pressures and heatings.

Thus, it remains to be proved that inside the body the values for the stresses σ_{ij}^*, temperature θ^* and the function h_i^* are zero-values.

The energetic equation (2.4) for the functions μ_i^*, θ^*, h_i^* will assume the following form

$$(3.2) \qquad \frac{d}{dt}(K^*+W^*+P^*+D^*) = -(\chi_0^*+\chi)_h^* \leqslant 0.$$

or

$$(3.3) \qquad \frac{d}{dt}\int_B \left\{\frac{1}{2}\varrho v_i^* v_i^* + \left(\mu\varepsilon_{ij}^*\varepsilon_{ij}^* + \frac{\lambda}{2}e^{*2}\right) + \frac{\gamma}{2\eta\varkappa}\theta^{*2} + \frac{\mu_0}{8\pi}h_j^* h_j^*\right\} dV \leqslant 0.$$

Consequently, we infer from the inequality (3.3) that the integral cannot rise for $t > 0$. Since the initial conditions for the integrand functions are homogeneous, the integral itself at the initial moment should be equal to zero. Its values cannot be negative as the integrand expression is a sum of squares with positive coefficients. Thus, the value of the integral appearing in (3·3) is necessarily zero for $t \geqslant 0$. It leads to the following equalities

$$v_i^* = 0, \quad \varepsilon_{ij}^* = 0, \quad \theta^* = 0 \quad h_i^* = 0,$$

or

$$(3.4) \qquad v_i' = v_i'', \quad \varepsilon_{ij}' = \varepsilon_{ij}'', \quad \theta' = \theta'', \quad h_i' = h_i'' \quad \text{for} \quad t \geqslant 0 \quad \text{in the}$$

region B.

Making use of the Duhamel—Neumann relations, we see that $\sigma_{ij}' = \sigma_{ij}''$, too.

Thus we may conclude on the uniqueness of the solution of the problem of magneto-thermoelasticity as regards the deformations, stresses, temperature and the values for h_i. For the displacements we obtain

$$(3.5) \qquad u_i' = u_i'' + \text{linear term}.$$

The linear term describes here the rotation and translation of the body considered as perfectly rigid. In the case of displacements prescribed on A, the linear term vanishes.

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В. НОВАЦКИЙ, ПРОБЛЕМА ЛИНЕЙНОЙ СОПРЯЖЕННОЙ ТЕРМОУПРУГОСТИ. I. ЭНЕРГЕТИЧЕСКАЯ ТЕОРЕМА И ТЕОРЕМА ОБ ОДНОЗНАЧНОСТИ РЕШЕНИЙ.

В сообщении выведена энергетическая теорема для упругого тела, находящегося в первичном и постоянном магнитном поле, при предположении, что это тело обладает конечной электропроводностью. В упомянутом теле под влиянием механических, термических и электромагнитных причин образуются поля деформации, температурное и электромагнитное поля, сопряженные между собой. Энергетическая теорема (2.14) дает возможность показать однозначность решений магнитотермоупругости.