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APPLIED MECHANICS

Mixed Boundary Value Problems of Thermoelasticity

by

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We shall consider in the present paper a region (body) B bounded by a surface S. On two parts of this surface, S_1 and S_2, various boundary conditions are prescribed. It will be assumed here that on S_1 the prescribed values for the displacements u_i and temperature θ are zero-values. Our further assumptions are the following: let be on the surface $S_2 = S - S_1$ the prescribed loadings p_i and temperature gradient $\theta_{,n}$; inside the body considered let act body forces X_1 and a heat source Q; the initial moment of any action provoking motion and heating of the body will be assumed at $t = 0_+$. Thus, the initial conditions for displacements and temperature are assumed to be homogeneous. *)

Let us agree upon the following notations: The coordinates of a point inside the region B will be denoted by $x \equiv (x_1, x_2, x_3)$ and those on the surface S — by $y \equiv (y_1, y_2, y_3)$. The displacements u_i and temperature θ should satisfy the basic equation of thermoelasticity (equation of motion)

$$D_{ij}(u_j(x,t)) + X_i(x,t) = \gamma\theta_{,i}(x,t) \tag{1}$$

and the equation of heat conduction

$$D\theta(x,t) - \eta\dot{u}_{k,k}(x,t) = -\frac{1}{\varkappa}Q(x,t), \quad x \in B, \quad t > 0. \tag{2}$$

The following notations have been introduced in Eqs. (1) and (2):

$$\begin{gathered} D_{ij} = \mu\delta_{ij}\,\Box_2^2 + (\lambda+\mu)\,\partial_i\,\partial_j, \quad D = \nabla^2 - \frac{1}{\varkappa}\,\partial_t, \\ \Box_2^2 = \nabla^2 - \frac{1}{c_2^2}\,\partial_t^2, \quad c_2^2 = \frac{\mu}{\varrho}. \end{gathered} \tag{3}$$

The magnitudes μ and λ in Eqs. (1) and (2) as well as in Eq. (3) are Lamé constants measured in isothermic conditions, $\eta = T_0\gamma/\lambda_0$, where T_0 denotes the absolute

*) The assumption of the homogeneity of the initial and boundary conditions is made solely for the clarity's sake. Nothing stands, however, in the way of the assumption of heterogeneous boundary conditions on the S_1 surface relating to the displacements as well as to the temperature.

temperature of the body in natural state and λ_0 — the thermal conductivity coefficient, $\gamma = (3\lambda+2\mu)\,\alpha_t$, where α_t stands for the coefficient of linear thermal dilatation.

We have further $\varkappa = \lambda_0/\varrho\, c_\varepsilon$, where ϱ denotes the density and c_ε — the specific heat of the body, the deformation being constant. Finally, δ_{ij} means the Kronecker's symbol and the point above the symbol denotes the first partial derivative with respect to time.

The solution of Eqs. (1) and (2) is sought for under following boundary conditions:

$$
(4)\qquad
\begin{aligned}
&\begin{cases} u_i(y,t)=0 \\ \theta(y,t)=k(y,t)=0 \end{cases} && \text{on} \quad \begin{matrix} S_1 \\ S_1 \end{matrix} && \text{for} \quad y\in S_1, \quad t>0,\\
&\begin{cases} p_i=\sigma_{ij}(y,t)\,n_j \\ \theta_{,n}=k_{,n}(y,t) \end{cases} && \text{on} \quad \begin{matrix} S_2 \\ S_2 \end{matrix} && \text{for} \quad y\in S_2, \quad t>0.
\end{aligned}
$$

and initial conditions

$$
(5)\qquad u_i(x,0)=0, \quad \dot{u}_i(x,0)=0, \quad \theta(x,0)=0, \quad x\in B, \quad t=0.
$$

Having thus formulated the boundary value problem, we shall now attempt to reduce it to the solution of simpler problems on two adjacent surfaces, S_1 and S_2, where boundary conditions of the same type occur. To this end we introduce what is called the "fundamental system", i.e., a thermoelastic body of the same shape but fixed in the point x_0 and free of loadings on the surfaces S_1 and S_2.

Let us assume that in such a hypothetical "initial system" a concentrated instantaneous force is acting at the point x' along the x_k-axis. This force induces in the body the displacement $U_i^{(k)}(x,x',t)$ and the temperature $C^{(k)}(x,x',t)$.

These functions should verify the basic differential equations of thermoelasticity

$$
(6)\qquad D_{ij}\,[U_j^{(k)}(x,x',t)]+\delta(x-x')\,\delta(t)\,\delta_{ik}=\gamma C_{,i}^{(k)}(x,x',t),
$$

$$
(7)\qquad DC^{(k)}(x,x',t)-\eta\dot{U}_{j,j}^{(k)}(x,x',t)=0, \quad x,x'\in B, \quad t>0
$$

with the following boundary conditions on the surface $S=S_1+S_2$

$$
(8)\qquad \sigma_{ij}^{(k)}(y,x',t)\,n_j=0, \quad C_{,n}^{(k)}(y,x',t)=0 \quad \text{on} \quad y\in S, \quad t>0,
$$

and initial conditions

$$
(9)\qquad U_i^{(k)}(x,x',0)=0, \quad \dot{U}_i^{(k)}(x,x',0)=0, \quad C^{(k)}(x,x',0)=0, \quad x,x'\in B, \quad t=0.
$$

The symbol $\sigma_{ij}^{(k)}$ appearing in Eq. (8) denotes the stress induced by the action of the concentrated force. The first condition of (8) may be written in the following form:

$$
(10)\qquad u\,(U_{i,j}^{(k)}+U_{i,j}^{(k)})\,n_j+\lambda n_i\,U_{j,j}^{(k)}-\gamma n_i\,C^{(k)}=0.
$$

The Green functions $U_i^{(k)}$ and $C^{(k)}$ obtained from the solution of Eqs. (6) and (7) will be considered in the sequel as known functions.

Let us now assume that in our "fundamental system" a concentrated instantaneous heat source $Q' = \delta(x - x')\,\delta(t)$ is placed in the point x'. Due to the action of this heat source, there will appear in the thermoelastic body the displacements $W_i(x, x', t)$ and temperature $H(x, x', t)$. These functions should satisfy the following system of basic differential equations of thermoelasticity:

$$D_{ij}\big(W_j(x, x', t)\big) = \gamma H_{,i}(x, x', t), \tag{11}$$

$$DH(x, x', t) - \eta \dot{W}_{j,j}(x, x', t) = -\frac{1}{\varkappa}\,\delta(x - x')\,\delta(t), \quad x, x' \in B, \quad t > 0 \tag{12}$$

with the following boundary conditions:

$$\sigma_{ij}^{(Q)}(y, x', t)\, n_j = 0, \quad H_{,n}(y, x', t) = 0, \quad y \in S, \quad t > 0 \tag{13}$$

and with the following initial conditions:

$$W_i(x, x', 0) = 0, \quad \dot{W}_i(x, x', 0) = 0, \quad H(x, x', 0) = 0, \quad x, x' \in B, \quad t = 0. \tag{14}$$

The symbol $\sigma_{ij}^{(Q)}$ (see Eq. (13)) denotes the thermal stresses caused by the action of the concentrated and instantaneous heat source. As may be seen from Eq. (13), the surface S is free of loadings and heating.

We assume also that the functions W_i and H are determined in our "fundamental system", thus in the sequel we may consider them as known functions.

In what follows we shall take advantage of the theorem on reciprocity written down in a transformed expression, namely after the integral transformation has been performed, [1]:

$$\begin{aligned}\eta\varkappa p \int_B (\bar{X}_i \bar{u}_i' - \bar{X}_i' \bar{u}_i)\, dV + \eta\varkappa p \int_S (\bar{p}_i \bar{u}_i' - \bar{p}_i' \bar{u}_i)\, dS = \\ = \gamma \int_B (\bar{Q}\bar{\theta}' - \bar{Q}'\bar{\theta})\, dV + \varkappa\gamma \int_S (\bar{k}\,\bar{k}'_{,n} - \bar{k}'\,\bar{k}_{,n})\, dS,\end{aligned} \tag{15}$$

where

$$\bar{u}_i(x, p) = \mathcal{L}[u_i(x, t)] = \int_0^\infty u_i(x, t)\, e^{-pt}\, dt, \quad \text{and so on.}$$

In the above formula, Eq. (15), we assume as the system with "primes" our "fundamental system", wherein the concentrated force $X_i' = \delta(x - x')\,\delta(t)\,\delta_{ik}$ is acting. It induces the displacement $U_i^{(k)}$ and temperature $C^{(k)}$. Thus we obtain

$$\begin{aligned}\eta\varkappa p \int_B \big(\bar{X}_i(x, p)\, \bar{U}_i^{(k)}(x, x', p) - \delta(x - x')\,\delta_{ik}\, \bar{u}_i(x, p)\big]\, dV(x) + \\ + \eta\varkappa p \int_{S_1+S_2} \bar{p}_i(y, p)\, \bar{U}_i^{(k)}(y, x', p)\, dS(y) = \gamma \int_B \bar{Q}(x, p)\, \bar{C}^{(k)}(x, x', p)\, dV(x) + \\ + \varkappa\gamma \int_{S_1+S_2} \bar{k}_{,n}(y, p)\, \bar{C}^{(k)}(y, x', p)\, dS(y),\end{aligned} \tag{16}$$

the boundary conditions, as given in Eq. (8), being taken into account. Let us denote by $R_i(y, t) = p_i(y, t)$ the unknown bearing reaction on the surface S_1 and by $\theta(y, t) = k_{,n}(x, t)$ — the unknown distribution of the temperature gradient on this surface. From Eq. (16) we obtain the following relation

$$(17)\quad \bar{u}_k(x', p) = \bar{u}_k^0(x', p) + \int_{S_1} \bar{R}_i(y, p)\, \bar{U}_i^{(k)}(y, x', p)\, dS(y) + \frac{\gamma}{\eta p}\int_{S_1} \bar{\Theta}(y, p)\, \bar{C}_{,n}^{(k)}(y, x', p)\, dS(y).$$

The term

$$(18)\quad \bar{u}_k^0(x', p) = \int_B \bar{X}_i(x, p)\, \bar{U}_i^{(k)}(x, x', p)\, dV(x) + \int_{S_1} \bar{p}_i(y, p)\, \bar{U}_i^{(k)}(y, x', p)\, dS(y) + \frac{\gamma}{\eta p}\int_{S_2} \bar{k}_{,n}(y, p)\, \bar{C}^{(k)}(y, x', p)\, dS(y);$$

is to be considered as a known magnitude since all subintegral functions are known. Eq. (17) is a functional equation wherein the unknown functions R_i and $\bar{\theta}$ appear as subscripts to the surface integrals. Let us return now to the theorem on reciprocity, Eq. (15), and let us introduce therein $u_i' = W_i$, $\theta' = H$, $Q' = \delta(x - x')\,\delta(t)$, $X' = 0$. After some simple transformations — the boundary conditions (13) being taken into account — we obtain the following functional relation

$$(19)\quad \bar{\theta}(x', p) = \bar{\theta}_0(x', p) - \frac{\eta\varkappa p}{\gamma}\int_{S_1} \bar{R}_i(y, p)\, \bar{W}_i(y, x', p)\, dS(y) + \varkappa\int_{S_1} \bar{\Theta}(y, p)\, \bar{H}(y, x', p)\, dS(y),$$

where

$$(20)\quad \bar{\theta}_0(x', p) = \int_B \bar{Q}(x, p)\, H(x, x', p)\, dV(x) - \frac{\eta\varkappa p}{\gamma}\int_B \bar{X}_i(x, p)\, \bar{W}_i(x, x', p)\, dV(x) - \frac{\eta\varkappa p}{\gamma}\int_{S_2} \bar{p}_i(y, p)\, \bar{W}_i(y, x', p)\, dS(y) + \varkappa\int_{S_2} \bar{k}_{,n}(y, p)\, \bar{H}(y, x', p)\, dS(y).$$

The $\bar{\theta}_0$ function may be considered as known, as all subintegral magnitudes are known.

In the functional relations (17) and (18) two unknown functions appear, namely the bearing reactions, $R_i(y, t)$, and the temperature gradient, Θ, both on the S_1 surface. In order to determine these functions we take advantage of the formula (4) for the boundary conditions on the S_1 surface. We have

$$(21)\quad R_i(y', t) = 0, \quad k(y', t) = 0, \quad y' \in S_1, \quad t > 0.$$

By a translation in Eqs. (17) and (19) of the point $x' \in B$ to the point $y' \in S_1$ we obtain a system of two integral Fredholm equations of the first kind:

$$(22)\quad 0 = \bar{u}_k^0(y', p) + \int_{S_1} \bar{R}_i(y, p)\, \bar{U}_i^{(k)}(y, y', p)\, dS(y) + \frac{\gamma}{\eta p}\int_{S_1} \bar{\Theta}(y, p)\, \bar{C}^{(k)}(y, y', p)\, dS(y),$$

$$(23)\qquad 0 = \bar{\theta}_0(y', p) - \frac{\eta\varkappa p}{\gamma}\int_{S_1} \bar{R}_i(y, p)\,\bar{W}_i(y, y', p)\,dS(y) + \varkappa\int_{S_1} \bar{\Theta}(y, p)\,\bar{H}(y, y', p)\,dS(y).$$

On solving this system of equations, we get the functions $\bar{R}_i$ and θ on the surface S_1. Now, introducing them into the functional relations (17) and (19), we obtain the functions $u_k(x, p)$ and $\bar{\theta}(x, p)$. Finally, after performing on the relations (17) and (19) the inverse Laplace transformation we arrive at

$$(23)\qquad \dot{u}_k(x', t) = \dot{U}_k^0(x', t) + \int_0^t d\tau \int_{S_1} R_i(y, t-\tau)\frac{\partial U_i^{(k)}(y, x', \tau)}{\partial \tau}\,dS(y) + \frac{\gamma}{\eta}\int_0^t d\tau \int_{S_1} \Theta(y, t-\tau)\frac{\partial C^{(k)}(y, x', \tau)}{\partial \tau}\,dS(y),$$

$$(24)\qquad \theta(x', t) = \theta_0(x', t) - \frac{\eta\varkappa}{\gamma}\int_0^t d\tau \int_{S_1} R_i(y, t-\tau)\frac{\partial W_i(y, x', \tau)}{\partial \tau}\,dS(y) + \varkappa\int_0^t d\tau \int_{S_1} \bar{\Theta}(y, \tau)\,H(y, x', t-\tau)\,dS(y).$$

Let us observe that the relations (17) and (19) include a number of particular cases. Two boundary cases will be considered here. If the problem is to be considered as uncoupled one, the body being subjected solely to the body forces X_i and to the loading p_i acting on the S_2 surface, then we have to put $\theta = 0$, $C^{(k)} = 0$ in the region B. Thus, Eq. (17) will be simplified to the form, [2]:

$$(25)\qquad \bar{u}_k(x', p) = \bar{u}_k^0(x', p) + \int_{S_1} \bar{R}_i(y, p)\,\bar{V}_i^{(k)}(y, x', p)\,dS(y).$$

where

$$\bar{u}_k^0(x', p) = \int_B \bar{X}_i(x, p)\,\bar{V}_i^{(k)}(x, x', p)\,dV(x) + \int_{S_2} \bar{p}_i(y, p)\,\bar{V}_i^{(k)}(y, x', p)\,dS(y),$$

the $V_i^{(k)}$ functions fulfilling the equation of classical elastokinetics

$$(26)\qquad D_{ij}[V_j^{(k)}(x, x', t)] + \delta(x - x')\,\delta(t)\,\delta_{ik} = 0,$$

with initial conditions

$$(27)\qquad n_j\,\sigma_{ij}^{(k)}(y, x', t) = 0 \quad \text{on} \quad S_1 \text{ and } S_2, \quad y \in S, \quad t > 0,$$

and with homogeneous boundary conditions. Since no heat source is acting and in view of $C^{(k)} = 0$, Eq. (19) becomes immaterial. The second limit case concerns the determination of the temperature field in a bounded body with mixed boundary conditions. Considering this problem as uncoupled, we put in Eq. (19) $\eta = 0$. We obtain then the following functional relation [3]:

$$\bar{\theta}(x', p) = \bar{\theta}_0(x', p) + \varkappa \int_{S_1} \bar{\Theta}(y, p)\, \bar{S}(y, x', p)\, dS(y), \tag{28}$$

where

$$\bar{\theta}_0(x', p) = \int_B \bar{Q}(x', p)\, \bar{S}(x, x', p)\, dV(x) + \varkappa \int_{S_2} \bar{k}_{,n}(y, p)\, \bar{S}(y, x', p)\, dS(y).$$

The function $S(x, x', t)$ should satisfy the classical equation of heat conductivity

$$\left(\nabla^2 - \frac{1}{\varkappa}\partial_t\right) S(x, x', t) = -\frac{1}{\varkappa}\delta(x - x')\,\delta(t), \tag{29}$$

with the boundary condition

$$S_{,n}(y, x', t) = 0 \quad \text{on} \quad S_1, \quad y \in S_1, \quad t > 0, \tag{30}$$

and the initial condition

$$S(x, x', 0) = 0, \quad x, x' \in B, \quad t = 0. \tag{31}$$

Translating the point $x' \in B$ to the point $y' \in S_1$, we obtain the integral equation

$$0 = \bar{\theta}_0(y', p) + \varkappa \int_{S_1} \bar{\Theta}(y, p)\, \bar{S}(y, y', p)\, dS(y), \tag{32}$$

which makes possible the determination of the unknown temperature gradient $\bar{\Theta}(y, p)$ on the surface S_1. In this particular case we succeeded in elaborating a method of solving the classical problem of heat conductivity, mixed boundary conditions in the body being assumed, [3].

The method advanced in this paper may easily be extended to a thermoelastic body, where mixed boundary conditions occur on more numerous parts of the surface S, as, e.g., $S_1, S_2, \ldots, S_n$, where $S = S_1 + S_2 + \ldots + S_n$. Similarly, we can pass easily to the case of loadings varying harmonically in time, to the quasi-static and stationary problems. In the latter case, the functional relations (17) and (19) will be uncoupled with one another.

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В. НОВАЦКИЙ, О ЗАДАЧАХ ТЕРМОУПРУГОСТИ СО СМЕШАННЫМИ КРАЕВЫМИ УСЛОВИЯМИ

В настоящей работе рассматривается проблема смешанных краевых условий, различных на поверхностях S_1 и S_2 термоупругого тела. Определяя функции Грина в основной системе с одинаковыми краевыми условиями на S_1 и S_2 и используя теорему о взаимности получены функциональные выражения на перемещение u_i и температуру θ внутри тела. Используя краевые условия на S_1 получена система двух интегральных уравнений Фредгольма первого порядка для определения неизвестной функции опорных реакций R_i, а также неизвестной функции температуры θ на поверхности S_1.