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APPLIED MECHANICS

A Plane Dynamic Distortion Problem in Stresses

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Let us consider an elastic body subjected to initial stresses produced by a distortional field characterized by initial strains ε_{11}^0, ε_{22}^0, ε_{33}^0, ε_{12}^0, depending on the coordinates x_1, x_2 and time t.

In order to determine the stress components σ_{ij} and the strain components ε_{ij}, we make use of the dynamic stress functions. This way is particularly useful in the case when the external loadings are given or for a cylinder free from stresses on its curved surface. For the plane strain state the relations between the stresses σ_{ij}, strains ε_{ij} and distortions $\overset{0}{\varepsilon}_{ij}$ are given by the equations

$$\sigma_{ij} = 2\mu(\varepsilon_{ij} - \overset{0}{\varepsilon}_{ij}) + \lambda\delta_{ij}(e - \bar{e}^0),\ i, j = 1, 2, \tag{1}$$

$$\sigma_{33} = -2\mu\overset{0}{\varepsilon}_{33} + \lambda(e - \bar{e}^0),\ e = \varepsilon_{11} + \varepsilon_{22},\ \bar{e}^0 = \varepsilon_{11}^0 + \varepsilon_{22}^0 + \varepsilon_{33}^0. \tag{2}$$

We assume that the plane strain state $(\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{12}) \neq 0$ is produced by the state $(\varepsilon_{11}^0, \varepsilon_{22}^0, \varepsilon_{12}^0, \varepsilon_{33}^0) \neq 0$ and by a surface loading.

In Eqs. (1), (2) μ, λ are the Lamé constants and δ_{ij} is the Kronecker delta. The strains $\varepsilon_{ij}\,(i, j = 1,2)$ are connected with the displacements by the relations

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \quad i, j = 1, 2. \tag{3}$$

The strains should satisfy the equation of geometrical compatibility

$$\varepsilon_{11,22} + \varepsilon_{22,11} = 2\varepsilon_{12,12}. \tag{4}$$

Let us differentiate the first equation of motion

$$\sigma_{ij,j} = \varrho\ddot{u}_i \tag{5}$$

with respect to x_1, the second with respect to x_2 and then first add them, further subtract these equations. We obtain two equations

$$\sigma_{11,11} + \sigma_{22,22} + 2\sigma_{12,12} = \varrho\ddot{e}, \tag{6}$$

$$\mathcal{D}_1^2\sigma_{11} - \mathcal{D}_2^2\sigma_{22} = \varrho(\ddot{\varepsilon}_{11}^0 - \ddot{\varepsilon}_{22}^0), \tag{7}$$

where

$$\mathcal{D}_i^2 = \partial_i^2 - \frac{1}{2c_2^2}\partial_t^2, \quad i = 1, 2, \quad c_2^2 = \frac{\mu}{\varrho}.$$

ϱ is the density of a body, c_2 is the velocity of the elastic equivoluminal wave.

Let us differentiate the first equation of the system (5) with respect to x_2, the second with respect to x_1, then add one to the other. As the result we obtain the equation

$$s_{,12} + \Box_2^2 \sigma_{12} = 2\varrho \ddot{\varepsilon}_{12}^0, \tag{8}$$

where

$$s = \sigma_{11} + \sigma_{22}, \quad \Box_2^2 = \nabla^2 - \frac{1}{c_2^2}\partial_t^2.$$

If we substitute relations (1) in Eq. (6) and make use of Eq. (4), then after some simple calculations we obtain the equation

$$\begin{cases} \Box_1^2 s = -2\mu\beta\left(\varepsilon_{11,22}^0 + \varepsilon_{22,11}^0 - 2\varepsilon_{12,12}^0 - \dfrac{1}{2c_2^2}\ddot{e}^0 + \gamma\,\Box_2^2 \varepsilon_{33}^0\right), \\ \beta = \dfrac{2(\lambda+\mu)}{\lambda+2\mu}, \ \gamma = \dfrac{\lambda}{2(\lambda+\mu)}, \ e^0 = \varepsilon_{11}^0 + \varepsilon_{22}^0, \ \Box_1^2 = \nabla^2 - \dfrac{1}{c_1^2}\partial_t^2, \ c_2^2 = \dfrac{\lambda+2\mu}{\varrho}. \end{cases} \tag{9}$$

Eqs. (6), (7) and (9) yield relations between three stress components σ_{11}, σ_{22}, σ_{12} and the components of the distortional field ε_{11}^0, ε_{12}^0, ε_{22}^0, ε_{33}^0. We introduce three stress functions Φ_i ($i = 1, 2, 3$) connected with the stresses σ_{ij} by the following relations

$$\sigma_{11} = \mathcal{D}_1^2 \Phi_1 + \Box_1^2 \Phi_2, \quad \sigma_{22} = \mathcal{D}_2^2 \Phi_1 - \Box_1^2 \Phi_2, \quad \sigma_{12} = -\partial_1\partial_2 \Phi_1 + \Box_1^2 \Phi_3. \tag{10}$$

Substituting relations (10) into Eq. (9), (8) and (7), we obtain the system of three equations

$$\Box_1^2 \Box_2^2 \Phi_i = A_i \quad i = 1, 2, 3, \tag{11}$$

where

$$\begin{cases} A_1 = -2\mu\beta\left(\varepsilon_{11,22}^0 + \varepsilon_{22,11}^0 - 2\varepsilon_{12,12}^0 - \dfrac{1}{2c_2^2}\ddot{e}_0 + \gamma\,\Box_2^2 \varepsilon_{33}^0\right), \\ A_2 = \varrho(\ddot{\varepsilon}_{11}^0 - \ddot{\varepsilon}_{22}^0), \quad A_3 = 2\varrho\ddot{\varepsilon}_{12}^0. \end{cases} \tag{12}$$

From Eq. (11) we determine the particular solutions Φ_i, then by means of relations (10) the stresses σ'_{ij}. The functions Φ'_i do not satisfy all boundary conditions; therefore we add to stresses σ'_{ij} stresses σ''_{ij} expressed by the functions

$$\sigma''_{ij} = -\partial_i\partial_j \varphi + \delta_{ij}\left(\nabla^2 - \frac{1}{2c_2^2}\partial_t^2\right)\varphi, \tag{13}$$

where the function φ should satisfy the homogeneous equation

$$\Box_1^2 \Box_2^2 \varphi = 0. \tag{14}$$

According to the T. Boggio theorem [1], Eq. (14) can be replaced by the system of equations

$$\Box_1^2 \varphi_1 = 0, \quad \Box_2^2 \varphi_2 = 0, \tag{15}$$

where

$$\varphi = \varphi_1 + \varphi_2 .$$

The particular solution of Eq. (11) for the infinite elastic space may be given in the form of the Fourier repeated integral [2]

$$\begin{aligned} \Phi_i(x_1, x_2, t) = \\ = (2\pi)^{-3/2} \iiint_{-\infty}^{\infty} \frac{\bar{A}_i(a_1, a_2, \omega)}{D(a_1, a_2, \omega)} \exp[-i(a_1 x_1 + a_2 x_2 + \omega t)]\, da_1\, da_2\, d\omega , \end{aligned} \tag{16}$$

where

$$D(a_1, a_2, \omega) = \left(a_1^2 + a_2^2 - \frac{\omega^2}{c_1^2}\right)\left(a_1^2 + a_2^2 - \frac{\omega^2}{c_2^2}\right)$$

and

$$\bar{A}_1 = 2\mu\beta\,[(a_2^2 - \varkappa\omega^2)\,\bar{\varepsilon}_{11}^0 + (a_1^2 - \varkappa\omega^2)\,\bar{\varepsilon}_{22}^0 - 2\,a_1 a_2\,\bar{\varepsilon}_{12}^0 +$$

$$+ \gamma(a_1^2 + a_2^2 - 2\varkappa\omega^2)\,\bar{\varepsilon}_{33}^0], \quad \varkappa = \frac{1}{2c_2^2},$$

$$\bar{A}_2 = -\varrho\omega^2(\bar{\varepsilon}_{11}^0 - \bar{\varepsilon}_{22}^0), \quad \bar{A}_3 = -2\varrho\omega^2\,\bar{\varepsilon}_{12}^0,$$

$$\bar{\varepsilon}_{ij}^0(a_1, a_2, \omega) = (2\pi)^{-3/2} \iiint_{-\infty}^{\infty} \varepsilon_{ij}^0(x_1, x_2, t) \exp[i(a_1 x_1 + a_2 x_2 + \omega t)]\, dx_1\, dx_2\, dt .$$

Let us consider two particular cases of the distortional field ε_{ij}^0.

a) Let $\varepsilon_{11}^0 = \varepsilon_{22}^0 = \varepsilon_{33}^0 = \eta^0$, $\varepsilon_{12}^0 = 0$. In this case we have $A_2 = A_3 = 0$, and

$$A_1 = -2\mu\,\frac{3\lambda + 2\mu}{\lambda + 2\mu}\,\Box_2^2\,\eta^0 .$$

In order to determine the stresses, it is sufficient to take one function Φ satisfying the equation

$$\Box_1^2\,\Box_2^2\,\Phi + 2\mu\beta_0\,\Box_2^2\,\eta^0 = 0, \quad \beta_0 = \frac{3\lambda + 2\mu}{\lambda + 2\mu} . \tag{17}$$

The stresses are given by the relations

$$\sigma_{ij} = -\partial_i \partial_j \Phi + \delta_{ij}\left(\nabla^2 - \frac{1}{2c_2^2}\,\partial_t^2\right)\Phi, \quad i, j = 1, 2. \tag{18}$$

The solution of Eq. (17) is composed of the particular solution Φ' of the non-homogeneous Eq. (17) and of the general solution Φ'' of the homogeneous Eq. (17), i.e. for $\eta^0 = 0$.

The case here considered contains a thermal distortion, where

$$\varepsilon_{ij}^0 = a_t\,\delta_{ij}\,T, \quad i, j = 1, 2, 3.$$

In the infinite space the solution of Eq. (17) is confined only to the particular solution

(19) $$\Phi(x_1, x_2, t) =$$
$$= 2\mu\beta_0 (2\pi)^{-3/2} \iiint\limits_{-\infty}^{\infty} \frac{\bar{\eta}^0(a_1, a_2, \omega)}{a_1^2 + a_2^2 - \frac{\omega^2}{c_1^2}} \exp\left[-i(a_1 x_1 + a_2 x_2 + \omega t)\right] da_1\, da_2\, d\omega,$$
where
$$\bar{\eta}^0(a_1, a_2, \omega) = (2\pi)^{-3/2} \iiint\limits_{-\infty}^{\infty} \eta^0(x_1, x_2, t) \exp\left[i(a_1 x_1 + a_2 x_2 + \omega t)\right] dx_1\, dx_2\, dt.$$

If the distortional field varies harmonically in time and p is frequency

(20) $$\eta^0(x_1, x_2, t) = e^{ipt}\, \eta^*(x_1, x_2).$$

We have also

(21) $$\Phi(x_1, x_2, t) = e^{ipt}\, \Phi^*(x_1, x_2),$$

and Eq. (17) takes the form

(22) $$(\nabla^2 + k_1^2)(\nabla^2 + k_2^2)\,\Phi^* + 2\mu\beta_0 (\nabla^2 + k_2^2)\,\eta^* = 0,$$

where

(23) $$k_i^2 = \frac{p^2}{c_i^2}, \quad i = 1, 2.$$

For the infinite space the following function is the solution of Eq. (22)

(24) $$\Phi^*(x_1, x_2) = 2\mu\beta_0 (2\pi)^{-1} \iint\limits_{-\infty}^{\infty} \frac{\bar{\eta}^*(a_1, a_2)}{a_1^2 + a_2^2 - k_1^2} \exp\left[-i(a_1 x_1 + a_2 x_2)\right] da_1\, da_2,$$

where
$$\bar{\eta}^*(a_1, a_2) = \frac{1}{2\pi} \iint\limits_{-\infty}^{\infty} \eta^*(x_1, x_2) \exp\left[i(a_1 x_1 + a_2 x_2)\right] dx_1\, dx_2.$$

If the distortional field $\eta^0(x_1, x_2, t)$ moves in the x_1-axis direction with constant velocity c, then transforming the co-ordinates

(25) $$\xi_1 = x_1 - ct, \quad \xi_2 = x_2,$$

we reduce Eq. (17) to the form

(26) $$(\partial_1^2 + \gamma_1^2 \partial_2^2)(\partial_1^2 + \gamma_2^2 \partial_2^2)\,\Phi(\xi_1, \xi_2) + 2\mu\beta_0 \gamma_1^2 (\partial_1^2 + \gamma_2^2 \partial_2^2)\,\eta^0(\xi_1, \xi_2) = 0,$$

where
$$\gamma_i^2 = \left(1 - \frac{c^2}{c_i^2}\right)^{-1}, \quad \partial_i = \frac{\partial}{\partial \xi_i}, \quad i = 1, 2.$$

The solution of Eq. (26) for the infinite space is given by the integral

(27) $$\Phi(\xi_1, \xi_2) = \frac{\mu\beta_0 \gamma_1^2}{\pi} \iint\limits_{-\infty}^{\infty} \frac{\bar{\eta}^0(a_1, a_2)}{a_1^2 + a_2^2 \gamma_1^2} \exp\left[-i(a_1 \xi_1 + a_2 \xi_2)\right] da_1\, da_2,$$

where

$$\bar{\eta}^0(\alpha_1, \alpha_2) = (2\pi)^{-1} \int\limits_{-\infty}^{\infty}\int \eta^0(\xi_1, \xi_2) \exp[i(\alpha_1 \xi_1 + \alpha_2 \xi_2)]\, d\xi_1\, d\xi_2 .$$

Observe that the distortional field $\eta^0(x_1, x_2, t)$ produces in the infinite space only longitudinal waves.

b) Let $\varepsilon^0_{11} = \varepsilon^0_{22} = \varepsilon^0_{33} = 0$, $\varepsilon^0_{12} = \vartheta^0$. In this case we have

$$A_1 = 4\mu\beta\,\vartheta^0_{,12}, \quad A_2 = 0, \quad A_3 = 2\varrho\,\ddot{\vartheta}^0_{,12}. \tag{28}$$

In order to determine the state of stresses, it is sufficient to take two functions satisfying the equations

$$\Box^2_1 \Box^2_2 \Phi_1 = A_1, \qquad \Box^2_1 \Box^2_2 \Phi_3 = A_3. \tag{29}$$

The stresses are given by the relations

$$\sigma_{11} = \mathcal{D}^2_2 \Phi_1, \quad \sigma_{22} = \mathcal{D}^2_1 \Phi_1, \quad \sigma_{12} = -\partial_1 \partial_2 \Phi_1 + \Box^2_1 \Phi_3. \tag{30}$$

In the infinite space the solutions of Eq. (29) are given by Fourier's integrals (16), for $i = 1, 2, 3$, where

$$\bar{A}_1 = -4\mu\beta\alpha_1 \alpha_2 \bar{\vartheta}^0, \quad \bar{A}_2 = 0, \quad A_3 = 2\varrho\omega^2 \alpha_1 \alpha_2 \bar{\vartheta}^0. \tag{31}$$

If the distortional field varies slowly in time, then in Eqs. (6), (7) and (9), we can neglect the inertial terms. The system of equations

$$\begin{cases} \nabla^2 s = -2\mu\beta(\varepsilon^0_{11,22} + \varepsilon^0_{22,11} - 2\varepsilon^0_{12,12} + \gamma\nabla^2 \varepsilon^0_{33}), \\ \partial^2_1 \sigma_{11} - \partial^2_2 \sigma_{22} = 0, \qquad s_{,12} + \nabla^2 \sigma_{12} = 0, \end{cases} \tag{32}$$

is satisfied by function F; moreover, [3]

$$\nabla^2 \nabla^2 F = -2\mu\beta(\varepsilon^0_{11,22} + \varepsilon^0_{22,11} - 2\varepsilon^0_{12,12} + \gamma\nabla^2 \varepsilon^0_{33}), \tag{33}$$

and

$$\sigma_{ij} = -(\partial_i \partial_j - \delta_{ij} \nabla^2) F. \tag{34}$$

The solution of Eq. (33) can be represented in the form

$$F = -2\mu\beta \iint\limits_{(r)} \left(\varepsilon^0_{11} \frac{\partial^2 F^*}{\partial \xi_2^2} + \varepsilon^0_{22} \frac{\partial^2 F^*}{\partial \xi_1^2} - 2\varepsilon^0_{12} \frac{\partial^2 F^*}{\partial \xi_1 \partial \xi_2} + \gamma \varepsilon^0_{33} \nabla^2_{\xi_1 \xi_2} F^* \right) d\xi_1 d\xi_2, \tag{35}$$

where F^* is the Green function satisfying the equation

$$\nabla^2 \nabla^2 F^* = \delta(x_1 - \xi_1)\, \delta(x_2 - \xi_2). \tag{36}$$

If the boundary of the infinite cylinder (the axis parallel to x_3) is free from stresses on the curved surface ($F = F_{,n} = 0$, $F^* = F^*_{,n} = 0$), then solution (35) can be represented in the form [4]

$$F = -2\mu\beta \iint_{(r)} F^* \left(\frac{\partial^2 \varepsilon^0_{11}}{\partial \xi_2^2} + \frac{\partial^2 \varepsilon^0_{22}}{\partial \xi_1^2} - 2 \frac{\partial^2 \varepsilon^0_{12}}{\partial \xi_1 \partial \xi_2} + \gamma \nabla^2 \varepsilon^0_{33} \right) d\xi_1 \, d\xi_2 . \tag{37}$$

In the particular case of the steady temperature field in the absence of heat sources ($\varepsilon^0_{ij} = \delta_{ij}\, \alpha_t T$, $\nabla^2 T = 0$) we have $F = 0$ at each point of the cylinder. Thus we have only one component of the stresses σ_{33}.

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