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A Quasi-Stationary Thermo-Elastic Problem in Three Dimensions

by

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The object of this paper is the determination of the state of displacement and stress in an elastic semi-space, due to the action of a source of heat of intensity W, moving with a constant velocity v in the plane bounding the semi-space.

The material is assumed to be thermally and elastically isotropic; the values characterising the material are assumed to be constant and independent of the temperature and stress.

The temperature and stress field will vary in time due to the motion of the source of heat. In a fixed co-ordinate system ξ, η, ζ the heat equation is

$$\frac{\partial^2 T}{\partial \xi^2}+\frac{\partial^2 T}{\partial \eta^2}+\frac{\partial^2 T}{\partial \zeta^2}=\frac{1}{k}\,\frac{\partial T}{\partial t}, \tag{1.1}$$

where $k=\lambda/\varrho c$, λ—coefficient of heat conduction ϱ-density, and c-specific heat. Assuming the co-ordinate system x, y, z as a moving one (in the origin of which the heat source is located and the x and y-axes lie in the plane bounding the elastic semi-space) moving together with the heat source at a constant velocity v in the direction of the ξ-axis, and using the transformation

$$x=\xi-vt, \quad y=\eta, \quad z=\zeta,$$

we obtain the Eq. (1.1) in the form, [1],

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}=-\frac{v}{k}\,\frac{\partial T}{\partial x}. \tag{1.2}$$

The time t does not appear in this equation. For an observer moving along the ξ-axis together with the heat source, the temperature and stress field will be steady.

Consider first the auxiliary problem of determining the state of stress due to the action of a moving heat source in an infinite elastic space.

The solution of (1.2) is, [2],

$$T=\frac{W}{4\pi\lambda}R^{-1}e^{-\mu(x+R)},\quad \text{where}\quad \mu=\frac{v}{2k}\quad \text{and}\quad R=(x^2+y^2+z^2)^{1/2}. \tag{1.3}$$

This solution can also be represented in the form of Fourier integrals which will be useful in the subsequent considerations:

$$\begin{aligned} T&=\frac{W}{2\pi^2\lambda}e^{-\mu x}\int\limits_0^\infty K_0\left(r\sqrt{\beta^2+\mu^2}\right)\cos\beta y\,d\beta,\\ &=\frac{W}{2\pi^2\lambda}e^{-\mu x}\int\limits_0^\infty\int\limits_0^\infty\frac{e^{-r\delta}}{\delta}\cos\beta y\cos\gamma z\,d\beta\,d\gamma,\\ &=\frac{W}{\pi^3\lambda}\int\limits_0^\infty\int\limits_0^\infty\int\limits_0^\infty\frac{\alpha\sin\alpha x\cos\beta y\cos\gamma z\,d\alpha\,d\beta\,d\gamma}{\alpha^2+\eta^2}, \end{aligned} \tag{1.4}$$

where

$$r=(y^2+z^2)^{1/2},\qquad \delta=(\beta^2+\gamma^2)^{1/2},\qquad \eta=\mu+(\mu^2+\delta^2)^{1/2}.$$

In order to determine the stress components it will be convenient to use the so-called potential of thermo-elastic displacement Φ. This function is related to the components u, v, w of the state of displacement by the equations

$$\frac{\partial\Phi}{\partial x}=u,\qquad \frac{\partial\Phi}{\partial y}=v,\qquad \frac{\partial\Phi}{\partial z}=w. \tag{1.5}$$

Introducing the relations (1.5) in the three displacement equations of the theory of elasticity, we reduce them to the unique equation, [3],

$$\nabla^2\Phi=\frac{1+\nu}{1-\nu}\alpha_t T, \tag{1.6}$$

where α_t is the coefficient of thermal dilatation and ν—Poisson's ratio. Differentiating (1.6) with respect to x and using the Eq. (12), we obtain

$$\nabla^2\left(\frac{\partial\Phi}{\partial x}\right)=\frac{1+\nu}{1-\nu}\alpha_t\frac{\partial T}{\partial x}=-\frac{1+\nu}{1-\nu}\frac{\alpha_t}{2\mu}\nabla^2 T. \tag{1.7}$$

Hence,

$$\frac{\partial\Phi}{\partial x}=-\frac{1+\nu}{1-\nu}\frac{\alpha_t}{2\mu}T\quad\text{and}\quad \Phi=-\frac{1+\nu}{1-\nu}\frac{\alpha_t}{2\mu}\int T\,dx. \tag{1.8}$$

The particular integral of the Eq. (1.6) will therefore be

$$(1.9)\qquad \Phi = -\frac{1+\nu}{1-\nu}\frac{a_t W}{8\mu\pi\lambda}\int_0^x R^{-1}\exp\left[-\mu(x+R)\right]dx = -\frac{1+\nu}{1-\nu}\frac{a_t W}{8\mu\pi\lambda}Ei\left[-\mu(x+R)\right],$$

where

$$Ei(-s) = \int_s^{\infty}\frac{e^{-u}}{u}du.$$

The function Φ can also be expressed by an integral

$$(1.10)\qquad \Phi = -\frac{1+\nu}{1-\nu}\frac{a_t W}{\pi^3\lambda}\int_0^\infty\int_0^\infty\int_0^\infty \frac{\alpha\sin\alpha x\cos\beta y\cos\gamma z}{(\alpha^2+\eta^2)(\alpha^2+\gamma^2+\beta^2)}d\alpha\, d\beta\, d\gamma.$$

The stress components due to the action of the heat source will be obtained from, [3],

$$(1.11)\qquad \sigma_{ij} = -2G\left(\nabla^2\Phi\delta_{ij}-\frac{\partial^2\Phi}{\partial i\,\partial j}\right),\quad i,j=x,y,z,$$

where δ_{ij} is Kronecker's delta.

We obtain successively

$$\sigma_{xx} = -KG\frac{e^{-\mu(x+R)}}{R(x+R)}\left[1-\frac{r^2}{R^2}\left(\mu R+\frac{R}{x+R}\right)\right],$$

$$\sigma_{yy} = -KG\frac{e^{-\mu(x+R)}}{R(x+R)}\left[1-\frac{z^2}{R^2}\left(1+\frac{R}{R+x}+\mu R\right)-\left(\frac{x+R}{R}\right)\left(\frac{x}{R}+\mu(x+R)\right)\right],$$

$$(1.12)\qquad \sigma_{zz} = -KG\frac{e^{-\mu(x+R)}}{R(x+R)}\left[1-\frac{y^2}{R^2}\left(1+\frac{R}{R+x}+\mu R\right)-\left(\frac{x+R}{R}\right)\left(\frac{x}{R}+\mu(x+R)\right)\right],$$

$$\sigma_{xy} = KGy\frac{e^{-\mu(x+R)}}{R^3}(1+\mu R),\qquad \sigma_{xz} = KGz\frac{e^{-\mu(x+R)}}{R^3}(1+\mu R),$$

$$\sigma_{yz} = KGzy\frac{e^{-\mu(x+R)}}{R^3(R+x)}\left(1+\mu R+\frac{R}{x+R}\right),$$

where

$$K = \frac{1+\nu}{1-\nu}\frac{a_t W}{8\pi\mu\lambda}.$$

The displacement components will be obtained from Eqs. (1.5)

$$(1.13)\quad u = K\frac{e^{-\mu(x+R)}}{R},\quad v = -Ky\frac{e^{-\mu(x+R)}}{R(x+R)},\quad w = -Kz\frac{e^{-\mu(x+R)}}{R(x+R)}.$$

In the particular case of $v \to 0$ ($\mu \to 0$) we obtain from (1.11) and (1.12) the known relation for the stationary problem, [3].

Consider a heat dipole at the origin of a co-ordinate system connected with the elastic space. In the plane $z=0$ we obtain $T=0$ (except for the origin). We have also $T=0$ at infinity. Thus, we have realised the temperature field in an elastic space, due to the action of a heat source in the plane $z=0$ bounding the semi-space.

According to (1.3) and (1.4) we have

$$(1.14)\quad T = -\frac{W}{4\pi\lambda}\frac{\partial}{\partial z}(R^{-1}e^{-\mu(x+R)}) = \frac{W}{4\pi\lambda}\frac{z}{R^3}e^{-\mu(x+R)}(1+\mu R),$$

or

$$(1.15)\quad T = \frac{W}{\pi^3\lambda}\int_0^\infty\int_0^\infty\int_0^\infty \frac{\alpha\gamma\sin\alpha x\cos\beta y\sin\gamma z}{\alpha^2+\eta^2}\,d\alpha\,d\beta\,d\gamma.$$

In an analogous manner the function Φ is obtained in the form

$$(1.16)\quad \Phi = \frac{1+\nu}{1-\nu}\frac{a_t W}{8\mu\lambda\pi}\frac{\partial}{\partial z}[Ei(-\mu(x+R))] = \frac{1+\nu}{1-\nu}\frac{a_t}{8\mu}\frac{W}{\pi\lambda}\frac{ze^{-\mu(x+R)}}{R(x+R)},$$

or in the form of the Fourier integral

$$(1.17)\quad \Phi = -\frac{1+\nu}{1-\nu}\frac{a_t W}{\pi^3\lambda}\int_0^\infty\int_0^\infty\int_0^\infty \frac{\alpha\gamma\sin\alpha x\cos\beta y\sin\gamma z}{(\alpha^2+\eta^2)(\alpha^2+\beta^2+\gamma^2)}\,d\alpha\,d\beta\,d\gamma.$$

The knowledge of the function Φ enables us to determine the stress components $(\bar{\sigma}_{ij})$ from the Eqs. (1.11).

We have

$$\bar{\sigma}_{xx} = -2G\left(\frac{\partial^2\Phi}{\partial y^2}+\frac{\partial^2\Phi}{\partial z^2}\right) = \frac{KGze^{-\mu(x+R)}}{R^2(R+x)}\left\{\left(\mu+\frac{1}{R}+\frac{1}{R+x}\right)\left(1+\frac{3x^2}{R^2}\right) - \right.$$

$$\left. -\frac{z^2+y^2}{R(x+R)^2}\cdot[2+2\mu(x+R)+\mu^2(x+R)^2]\right\},$$

$$\bar{\sigma}_{yy} = -2G\left(\frac{\partial^2\Phi}{\partial x^2}+\frac{\partial^2\Phi}{\partial z^2}\right) = -\frac{KGze^{-\mu(x+R)}}{R^2}\left\{\left(\mu+\frac{1}{R}\right)\left(\mu+\frac{3}{R}-\right.\right.$$

$$\left.-\frac{3}{x+R}-\frac{3y^2}{R^2(x+R)}\right)-\frac{1}{R^2(x+R)^2}\left[3(x^2+y^2)-\right.$$

$$(1.18)\quad \left.\left.-z^2\left(\frac{2R}{R+x}+2\mu R+\mu^2R(x+R)\right)\right]\right\},$$

$$\bar{\sigma}_{zz}=-2G\left(\frac{\partial^2 \Phi}{\partial x^2}+\frac{\partial^2 \Phi}{\partial y^2}\right)=-\frac{KGze^{-\mu(x+R)}}{R^2}\left[\left(\mu+\frac{1}{R}\right)\left(\mu+\frac{3}{R}-\right.\right.$$

$$\left.-\frac{1}{R+x}+\frac{2y^2}{R(x+R)^2}-\frac{3z^2}{R^2(x+R)}\right)-\frac{z^2+x^2}{R^2(R+x)^2}+$$

$$\left.+\frac{2y^2}{R(R+x)^3}+\frac{\mu^2 y^2}{R(x+R)}\right],$$

$$\bar{\sigma}_{xy}=2G\frac{\partial^2 \Phi}{\partial x\,\partial y}=\frac{KGyze^{-\mu(x+R)}}{R^3}\left[\mu^2+3\left(\frac{1}{R}+\mu\right)\right],$$

$$\bar{\sigma}_{xz}=2G\frac{\partial^2 \Phi}{\partial x\,\partial z}=\frac{KGe^{-\mu(x+R)}}{R^2}\left[\frac{\mu^2 z^2}{R}-\left(\mu+\frac{1}{R}\right)\left(1-\frac{3z^2}{R^2}\right)\right],$$

$$\bar{\sigma}_{yz}=2G\frac{\partial^2 \Phi}{\partial y\,\partial z}=\frac{KGye^{-\mu(x+R)}}{R^2(x+R)}\left\{\left(\mu+\frac{1}{R}+\frac{1}{x+R}\right)\left(\frac{3z^2}{R^2}+\right.\right.$$

$$\left.\left.+\frac{2z^2}{R(x+R)}-1\right)-\frac{z^2}{R^2(x+R)}\left[2-\mu^2 R(x+R)\right]\right\}.$$

Let us observe that the state of stress $(\bar{\sigma}_{ij})$ does not satisfy all the boundary conditions in the plane $z=0$. The stress $\bar{\sigma}_{zz}$ vanishes in that plane, but the shear stress $\bar{\sigma}_{xz}$, $\bar{\sigma}_{yz}$ does not vanish. The state of stress $(\bar{\bar{\sigma}}_{ij})$ should be superposed on $(\bar{\sigma}_{ij})$. This state constitutes the solution of the following isothermal space problem: determine the stresses $(\bar{\bar{\sigma}}_{ij})$ in an elastic semi-space due to shear stresses $-\bar{\sigma}_{xz}$ and $-\bar{\sigma}_{yz}$ acting in the plane $z=0$. The stress components $\bar{\bar{\sigma}}_{ij}$, can be found by using B. G. Galerkin's displacement function φ. They are expressed in terms of φ by the following relations:

$$\bar{\bar{\sigma}}_{xx}=\frac{\partial}{\partial z}\left(\nu\nabla^2\varphi-\frac{\partial^2\varphi}{\partial x^2}\right),\quad \bar{\bar{\sigma}}_{yy}=\frac{\partial}{\partial z}\left(\nu\nabla^2\varphi-\frac{\partial^2\varphi}{\partial y^2}\right),$$

$$\bar{\bar{\sigma}}_{zz}=\frac{\partial}{\partial z}\left[(1-\nu)\nabla^2\varphi+\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}\right],\quad \sigma_{xy}=-\frac{\partial^3\varphi}{\partial x\,\partial y\,\partial z},\tag{1.19}$$

$$\bar{\bar{\sigma}}_{xz}=\frac{\partial}{\partial x}\left[\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}-\nu\nabla^2\varphi\right],\quad \bar{\bar{\sigma}}_{yz}=\frac{\partial}{\partial y}\left[\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}-\nu\nabla^2\varphi\right].$$

The function φ should satisfy the biharmonic equation

$$\nabla^2\nabla^2\varphi=0,\tag{1.20}$$

with the boundary conditions

$$\bar{\sigma}_{xz}+\bar{\bar{\sigma}}_{xz}\big|_{z=0}=0,\quad \bar{\sigma}_{yz}+\bar{\bar{\sigma}}_{yz}\big|_{z=0}=0,\quad \bar{\bar{\sigma}}_{zz}\big|_{z=0}=0\tag{1.21}$$

and $\varphi=0$ at infinity.

The function φ is assumed in the form

$$\varphi = \int_0^\infty \int_0^\infty Z(\alpha, \beta, z) \sin \alpha x \cos \beta y \, d\alpha \, d\beta, \tag{1.22}$$

where

$$Z = (A + B\vartheta z) e^{-\vartheta z}, \quad \vartheta = (\alpha^2 + \beta^2)^{1/2}.$$

The functions $\bar{\sigma}_{xz}$, $\bar{\sigma}_{yz}$ appearing in the boundary conditions will also be expressed by means of a Fourier integral

$$\begin{aligned} \bar{\sigma}_{xz}|_{z=0} &= 2G \left. \frac{\partial^2 \Phi}{\partial x \partial z} \right|_{z=0} = -2GK \int_0^\infty \int_0^\infty \alpha^2 P(\alpha, \beta) \cos \alpha x \cos \beta y \, d\alpha \, d\beta, \\ \bar{\sigma}_{yz}|_{z=0} &= 2G \left. \frac{\partial^2 \Phi}{\partial y \partial z} \right|_{z=0} = 2GK \int_0^\infty \int_0^\infty \alpha\beta P(\alpha, \beta) \sin \alpha x \sin \beta y \, d\alpha \, d\beta, \end{aligned} \tag{1.23}$$

where

$$P(\alpha, \beta) = \int_0^\infty \frac{\gamma^2 \, d\gamma}{(\alpha^2 + \gamma^2)(\alpha^2 + \beta^2 + \gamma^2)}.$$

The third boundary condition (1.21) leads to the relation $A = -B(1 - 2\nu)$.

The first two conditions of the group (1.21) lead to the same relation:

$$B(\alpha, \beta) = -2GK \frac{\alpha P(\alpha, \beta)}{\vartheta^2}. \tag{1.24}$$

Then,

$$\varphi = -2KG \int_0^\infty \int_0^\infty \frac{\alpha}{\vartheta^2} P(\alpha, \beta) (1 - 2\nu - \vartheta z) e^{-\vartheta z} \sin \alpha x \cos \beta y \, d\alpha \, d\beta. \tag{1.25}$$

Hence, from the Eqs. (1.19) we determine the stress components $(\bar{\bar{\sigma}}_{ij})$

$$\begin{aligned} \bar{\bar{\sigma}}_{xx} &= -2KG \int_0^\infty \int_0^\infty \frac{\alpha P(\alpha, \beta)}{\vartheta^2} [2(\alpha^2 + \nu\beta^2) - \\ &\qquad - \alpha^2 \vartheta z] e^{-\vartheta z} \sin \alpha x \cos \beta y \, d\alpha \, d\beta, \\ \bar{\bar{\sigma}}_{yy} &= 2KG \int_0^\infty \int_0^\infty \frac{\alpha P(\alpha, \beta)}{\vartheta^2} [2(\beta^2 + \nu\alpha^2) - \\ &\qquad - \beta^2 \vartheta z] e^{-\vartheta z} \sin \alpha x \cos \beta y \, d\alpha \, d\beta, \\ \bar{\bar{\sigma}}_{zz} &= -2KGz \int_0^\infty \int_0^\infty \alpha P(\alpha, \beta) \vartheta^2 e^{-\vartheta z} \sin \alpha x \cos \beta y \, d\alpha \, d\beta, \\ \bar{\bar{\sigma}}_{xz} &= -2KG \int_0^\infty \int_0^\infty P(\alpha, \beta) \alpha^2 (1 - \vartheta z) e^{-\vartheta z} \cos \alpha x \cos \beta y \, d\alpha \, d\beta, \\ \bar{\bar{\sigma}}_{yz} &= -2KG \int_0^\infty \int_0^\infty P(\alpha, \beta) \alpha\beta (1 - \vartheta z) e^{-\vartheta z} \sin \alpha x \cos \beta y \, d\alpha \, d\beta, \\ \bar{\bar{\sigma}}_{xy} &= 2KG \int_0^\infty \int_0^\infty \frac{\alpha^2 \beta}{\vartheta} P(\alpha, \beta) (2 - 2\nu - \vartheta z) e^{-\vartheta z} \cos \alpha x \sin \beta y \, d\alpha \, d\beta. \end{aligned} \tag{1.26}$$

Superposing the stress components $(\bar{\sigma}_{ij})$ and $(\bar{\bar{\sigma}}_{ij})$, we obtain our stresses (σ_{ij}). Note that for $v \to 0$ $(\mu \to 0)$ we pass to a steady-state problem. In this case, it was shown in the [4] that the stresses $\sigma_{xz}, \sigma_{yz}, \sigma_{zz}$ vanish, and the stresses $\sigma_{xx}, \sigma_{yy}, \sigma_{xy}$ can be represented in a closed form.

Consider the particular case of a linear heat source moving with a constant velocity v in an infinite elastic space. Let this source be distributed with a uniform intensity W along the z-axis.

Then the temperature field is determined by

$$(1.27)\qquad T = \frac{W}{4\pi\lambda} e^{-\mu x} \int\limits_{-\infty}^{\infty} \frac{e^{-\mu(x^2+y^2+z^2)^{1/2}}}{(x^2+y^2+z^2)^{1/2}} dz = \frac{W}{2\pi\lambda} e^{-\mu x} K_0(\mu\sqrt{x^2+y^2}),$$
$$r_0 = (x^2+y^2)^{1/2},$$

where $K_0(\mu r_0) = \pi i/2\, H_0^{(1)}(i\mu r_0)$ and $H_0^{(1)}(i\mu r_0)$ is Hankel's function of the first kind and zero order.

In a similar manner the function Φ takes the following form, according to the Eq. (1.8):

$$(1.28)\qquad \Phi = -N \int\limits_0^x e^{-\mu x} K_0[\mu(x^2+y^2)^{1/2}]\, dx, \qquad N = \frac{1+\nu}{1-\nu}\frac{a_t}{4\mu}\frac{W}{\pi\lambda},$$

which is in agreement with the result obtained by E. Melan [5].

Now let a linear heat source uniformly distributed along the z-axis move in the x-direction with constant velocity v in the plane $y=0$ bounding the elastic semi-space. The temperature field is determined by

$$(1.29)\qquad T = -\frac{W}{2\pi\lambda} e^{-\mu x} \frac{\partial}{\partial y}[K_0(\mu\sqrt{x^2+y^2})] = \frac{W\mu}{2\pi\lambda}\frac{y}{r_0} e^{-\mu x} K_1(\mu r_0),$$
$$r_0 = (x^2+y^2)^{1/2}$$

or

$$(1.30)\qquad T = \frac{W}{\pi^2\lambda}\int\limits_0^\infty\int\limits_0^\infty \frac{\alpha\beta\sin\alpha x\sin\beta y}{\alpha^2+\eta_0^2}\, d\alpha\, d\beta, \qquad \text{where} \qquad \eta_0 = \mu + \sqrt{\mu^2+\beta^2}.$$

We find in an analogous way that

$$(1.31)\qquad \Phi = -N\mu y \int\limits_0^x \frac{e^{-\mu x} K_1[\mu(x^2+y^2)^{1/2}]}{(x^2+y^2)^{1/2}}\, dx,$$

or

$$(1.32)\qquad \Phi = -\frac{1+\nu}{1-\nu}\frac{W a_t}{\pi^2\lambda}\int\limits_0^\infty\int\limits_0^\infty \frac{\alpha\beta\sin\alpha x\sin\beta y}{(\alpha^2+\beta^2)(\alpha^2+\eta_0^2)}\, d\alpha\, d\beta.$$

The knowledge of the function Φ enables us to determine the stress components $(\bar{\sigma}_{ij})$.

Thus,

$$
(1.33)\quad
\begin{aligned}
\bar{\sigma}_{xx} &= -2G\frac{\partial^2 \Phi}{\partial y^2} = 2GN\mu^2\frac{e^{-\mu x}}{r_0^2}\,xy\,K_2(\mu r_0),\\
\bar{\sigma}_{yy} &= -2G\frac{\partial^2 \Phi}{\partial x^2} = -2GN\mu^2\frac{e^{-\mu x}}{r_0}\left[K_1(\mu r_0) + \frac{x}{r_0}K_2(\mu r_0)\right],\\
\bar{\sigma}_{zz} &= -2G\left(\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2}\right) = -4GN\frac{\mu^2 y e^{-\mu x}}{r_0}K_1(\mu r_0),\\
\bar{\sigma}_{xy} &= 2G\frac{\partial^2 \Phi}{\partial x\,\partial y} = -2GN\frac{\mu}{r_0}e^{-\mu x}\left[(1-\mu y)K_1(\mu r_0) - \frac{\mu y^2}{r_0}K_2(\mu r_0)\right],\\
\bar{\sigma}_{xz} &= 0, \qquad \bar{\sigma}_{yz} = 0.
\end{aligned}
$$

For $y = 0$ we have $\bar{\sigma}_{yy} = 0$ and $\bar{\sigma}_{xy} \neq 0$.

Over the state $(\bar{\sigma}_{ij})$ the state $(\bar{\bar{\sigma}}_{ij})$ is superposed, being chosen in such a way that all boundary conditions are satisfied at the edge $y = 0$. In order to determine the state $(\bar{\bar{\sigma}}_{ij})$ we use the Airy function. The biharmonic equation,

$$
(1.34)\qquad \nabla^2\nabla^2 F = 0,
$$

should be satisfied with the following boundary conditions for $y = 0$:

$$
(1.35)\qquad \bar{\sigma}_{xy} - \frac{\partial^2 F}{\partial x\,\partial y} = 0, \qquad \frac{\partial^2 F}{\partial x^2} = 0.
$$

The function F will be assumed in the form

$$
(1.36)\qquad F = \int_0^\infty (A + B\alpha y)e^{-\alpha y}\sin\alpha x\,d\alpha.
$$

From the second condition of the group (1.35) it follows that $A = 0$. From the first condition, (1.35), and bearing in mind (1.32), we obtain

$$
(1.37)\qquad F = \frac{8\mu GNy}{\pi}\int_0^\infty P(\alpha)\,\alpha\,e^{-\alpha y}\sin\alpha x\,d\alpha,
$$

where

$$
P(\alpha) = \int_0^\infty \frac{\beta^2\,d\beta}{(\alpha^2+\beta^2)(\alpha^2+\eta_0^2)}.
$$

The knowledge of the function F enables us to determine the stress components $(\bar{\bar{\sigma}}_{ij})$.

We obtain

$$\bar{\sigma}_{xx} = \frac{\partial^2 F}{\partial y^2} = -\frac{8\,\mu G N}{\pi} \int_0^\infty P(a)\, a^2 e^{-ay} (2 - a\,y) \sin a x\, da,$$

$$\bar{\sigma}_{yy} = \frac{\partial^2 F}{\partial x^2} = -\frac{8\,\mu G N}{\pi}\, y \int_0^\infty P(a)\, a^3 e^{-ay} \sin a x\, da,$$

$$\bar{\sigma}_{zz} = -\nu \nabla^2 F = \frac{16\,\mu\nu N G}{\pi} \int_0^\infty P(a)\, a^2 e^{-ay} \sin a x\, da, \tag{1.38}$$

$$\bar{\sigma}_{xy} = -\frac{\partial^2 F}{\partial x\, \partial y} = -\frac{8\,\mu G N}{\pi} \int_0^\infty P(a)\, a^2 e^{-ay} (1 - a\,y) \cos a x\, da,$$

$$\bar{\sigma}_{xz} = 0, \qquad \bar{\sigma}_{yz} = 0.$$

The stresses are obtained finally from the equations $\sigma_{ij} = \bar{\sigma}_{ij} + \bar{\bar{\sigma}}_{ij}$. For $v \to 0\,(\mu \to 0)$, in other words, for a fixed, steady heat source the stresses σ_{xx}, σ_{yy}, σ_{xy} vanish, the only non-vanishing stress being σ_{zz}.

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