BULLETIN
DE
L'ACADÉMIE POLONAISE DES SCIENCES

CLASSE QUATRIÈME

VOLUME I
NUMÉROS 1—2

VARSOVIE 1953

BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Cl. IV — Vol. I, Nos 1—2, 1953

APPLIED MECHANICS

On Certain Cases of Torsion of Bars

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Communicated at the meeting of October 6, 1952

The object of this note*) is to obtain an exact solution of the torsional problem for prismatical bars with cross-sections in the form of rectangles, or in the form of rectangles with slots, or cross-sections in the form of annular sectors with slots, or finally circular cross-sections with slots.

The material is assumed to be anisotropic, of the orthogonal type.

It is known from the theory of (pure) torsion of prismatical bars [1] that shear stresses τ_{zx}, τ_{zy} can be expressed by a stress function $\Psi(x, y)$ satisfying the differential equation

$$\frac{1}{G_{23}} \frac{\partial^2 \Psi}{\partial x^2} + \frac{1}{G_{13}} \frac{\partial^2 \Psi}{\partial y^2} = -2\vartheta \tag{1}$$

with the boundary condition

$$\Psi = \text{const}$$

(in the case of a cross-section representing a simply connected region $c = 0$). Thus

$$\tau_{xz} = \frac{\partial \Psi}{\partial y}, \quad \tau_{yz} = -\frac{\partial \Psi}{\partial x}. \tag{2}$$

In equation (1) G_{13}, G_{23} denote the moduli of elasticity in shear for the directions x, y of the orthotropic structure, and ϑ, the angle of twist.

According to an analogy used by Prandtl, the problem is identical with that of finding the deflection surface for a membrane subjected to tensions S_x, S_y at the edges and an uniform lateral pressure $p = \text{const.}$, the outline being identical with that of the cross-sections of the twisted bar.

The relation between the function $\Psi(x, y)$ and the deflection of the membrane $w(x, y)$, obtained by solving the differential equation

$$S_x \frac{\partial^2 w}{\partial x^2} + S_y \frac{\partial^2 w}{\partial y^2} = p \tag{3}$$

*) This note will be published in extenso in „Archiwum Mechaniki Stosowanej", **5** (1953), 1.

with the boundary conditon of $w = \text{const}$ (or $w = 0$ for a simply connected region), is as follows

$$\Psi = c \cdot w, \quad c = \frac{2\, G_{13} \cdot \vartheta \cdot S_y}{p}. \tag{4}$$

The shear stresses are obtained from the formulae

$$\tau_{zx} = c \frac{\partial w}{\partial y}, \quad \tau_{zy} = - c \frac{\partial w}{\partial x} \tag{5}$$

For more convenient mechanical interpretation of certain mathematical operations the results will be given for a membrane model of torsion.

1. A rectangular membrane supported at the edges and also along a straight line AB within the area of the membrane.

Let us denote by $R(\xi)$ the unknown function representing the reaction at the support AB, and by $w_1(x, y; \xi, \eta)$ the function of Green expressing the deflection surface for a concentrated force $R = 1$ at the point (ξ, η), and finally let us denote by $w_0(x, y)$ the deflection surface of the membrane corresponding to the lateral pressure $p = \text{const.}$ for the basic system, i. e.

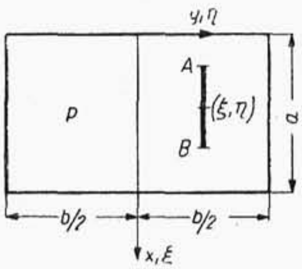

Fig. 1

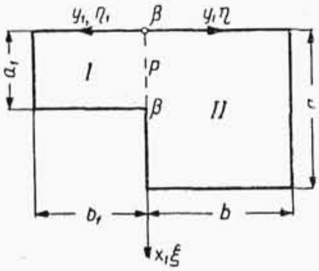

Fig. 2

for the membrane supported at the edges only. The deflection $w(x, y)$ of the membrane is expressed (with the boundary condition $w = 0$ for the edges and the straight line AB) by the integral equation

$$w(x, y) = w_0(x, y) + \int_{AB} R(\xi) \cdot w_1(x, y; \xi, \eta) d\xi. \tag{6}$$

From the condition of zero deflection at the support AB we obtain

$$w_0(x, 0) + \int_{AB} R(\xi) \cdot w_1(x, 0; \xi, \eta) d\xi = 0. \tag{7}$$

From the above equation, which is a Fredholm integral equation of the first type, we obtain the unknown function $R(\xi)$, and from equation (6) the deflection $w(x, y)$.

From equation (5) we obtain by differentiation the shear stresses in a rectangular bar with a very narrow slot AB. Using the symbols

$$\alpha_n = \frac{n\pi}{a}, \lambda^2 = \frac{S_x}{S_y} = \frac{G_{13}}{G_{23}}, \gamma = \lambda b \alpha_n$$

we have

$$w_1(x, y; \xi, \eta) = -\frac{P}{4\pi S_y \lambda} \ln \frac{\cosh\frac{\pi\lambda}{a}(y-\eta) - \cos\frac{\pi}{a}(x-\xi)}{\cosh\frac{\pi\lambda}{a}(y-\eta) - \cos\frac{\pi}{a}(x+\xi)} +$$

$$(8) \qquad + \frac{P}{\pi\lambda S_y} \sum_{n=1,2\ldots}^{\infty} \frac{\sin\alpha_n\xi}{n\sinh\gamma_n}[\cosh\lambda\alpha_n(y+\eta) - e^{\gamma_n}\cosh\lambda\alpha_n(y-\eta)]\sin\alpha_n x,$$

$$w_0(x,y) = \frac{4pa^2}{\pi^3\lambda S_y} \sum_{n=1,3\ldots}^{\infty} \frac{1}{n^3}\left(1 - \frac{\cosh\lambda\alpha_n y}{\cosh\frac{\gamma_n}{2}}\right)\sin\alpha_n x.$$

The above method can be extented to the majority of rectilinear supports within the area of the membrane and parallel to the x or y axis.

In the case of a membrane composed of two rectangles it will be convenient to assume as unknown the function of deflection $\overline{w}(\xi)$ for the section $\beta - \beta$ (Fig. 2).

For rectangle I we have

$$(9\text{ a}) \qquad w_{\mathrm{I}}(x, y) = w_{0,\mathrm{I}} + \int_0^{a_1} \overline{w}(\xi) K^{\mathrm{I}}(x, y; \xi, 0)\, d\xi,$$

and for rectangle II

$$(9\text{ b}) \qquad w_{\mathrm{II}}(x, y_1) = w_{0,\mathrm{II}} + \int_0^{a_1} \overline{w}(\xi) K^{\mathrm{II}}(x, y_1; \xi, 0)\, d\xi.$$

where $w_{0,\mathrm{I}}$ and $w_{0,\mathrm{II}}$ are the deflection surfaces of membranes I and II respectively, supported at the edges and subjected to a pressure $p = \text{const.}$, $K^{\mathrm{I}}(x\ y; \xi, 0)$ and $K^{\mathrm{II}}(x, y_1; \xi, 0)$ are functions of Green for the state of $\overline{w}(\xi) = 1$, and finally $w_{\mathrm{I}}(x, y)$, $w_{\mathrm{II}}(x, y_1)$ are the deflections of membranes I and II as a whole.

The unknown function of deflection $\overline{w}(\xi)$ is found from the condition of continuity of the membrane for the section $\beta - \beta$

$$(10) \qquad \left.\frac{\partial w_{\mathrm{I}}}{\partial y}\right|_{y=0} = \left.\frac{\partial w_{\mathrm{II}}}{\partial y_1}\right|_{y_1=0},$$

which leads to a Fredholm integral equation of the first order.

The above method can be used for a membrane composed of any number of rectangles.

2. For bars of the cylindrical type of orthotropy the shear stresses τ_{rz} and $\tau_{\varphi z}$ can be expressed by the formulae

$$(11) \qquad \tau_{rz} = \frac{1}{r_1}\frac{\partial \Psi}{\partial \varphi_1}, \quad \tau_{\varphi z} = -\frac{\partial \Psi}{\partial r_1},$$

where the function Ψ should satisfy the differential equation

$$a_{44}\frac{\partial^2 \Psi}{\partial r_1^2}+a_{55}\frac{\partial^2 \Psi}{\partial \varphi_1^2}+a_{44}\frac{1}{r_1}\frac{\partial \Psi}{\partial r_1}=-2\vartheta \tag{12}$$

where a_{44} and a_{55} are constants characteristic of the orthotropy of the material.

Equation (12) can be reduced, by means of the affine transformation $r=r_1,\ \varphi=\beta\varphi_1,\ \beta=\sqrt{\dfrac{a_{44}}{a_{55}}}$, to a form identical with the differential equation of deflection

$$\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}+\frac{1}{r^2}\frac{\partial^2 w}{\partial \varphi^2}=-\frac{p}{S},\ p=\frac{2S\vartheta}{a_{44}}. \tag{13}$$

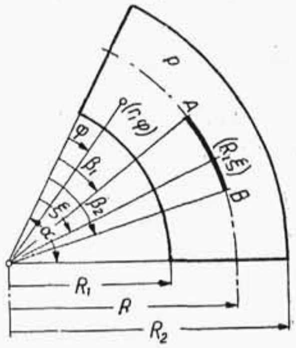

Fig. 3

It is now a simple matter to express the shear stresses in terms of first derivatives of the function $w(r,\varphi)$

$$\tau_{rz}=\frac{c}{r}\frac{\partial w}{\partial \varphi}\cdot\beta,\ \ \tau_{\varphi z}=-c\frac{\partial w}{\partial r},\ \ \Psi=c\cdot w,\ \ c=\frac{2S\vartheta}{a_{44}p} \tag{14}$$

For the annular sector (Fig. 3), supported at the edges and with an additional curvilinear support AB, the deflection $w(r,\varphi)$ is expressed by the formula

$$w(\varrho,\varphi)=w_0(\varrho,\varphi)+\int_{\beta_1}^{\beta_2}R(\xi)w_1(\varrho,\varphi;1,\xi)R\,d\xi \qquad \beta_1\leqq\xi\leqq\beta_2,\ \ \varrho=\frac{r}{R} \tag{15}$$

$w_0(\varrho,\varphi)$ being the deflection of the membrane supported at the edges and subjected to a lateral pressure $p=\text{const.}$, $R(\xi)$ being an unknown function expressing the reaction at the support AB, constituting a circular arc, and finally $w(\varrho,\varphi;1,\xi)$ representing the function of Green for the load $P=1$ acting at the point $(\varrho=1,\xi)$.

We have

$$w(\varrho,\varphi;1,\xi)=-\frac{1}{4\pi}\ln\frac{\cosh(k\ln\varrho)-\cos k(\varphi-\xi)}{\cosh(k\ln\varrho)-\cos k(\varphi+\xi)}-$$

$$-\frac{1}{\pi S}\sum_{n=1,2,\ldots}^{\infty}\frac{1}{n}\frac{(1-\varrho_1^{2nk})\varrho^{nk}-(1-\varrho_2^{2nk})\varrho_1^{2nk}\varrho^{-nk}}{\varrho_2^{2nk}-\varrho_1^{2nk}}\sin nk\xi\cdot\sin nk\varphi$$

$$\varrho_1=\frac{R_1}{R},\quad \varrho_2=\frac{R_2}{R},\quad k=\frac{\pi}{\alpha}.$$

$$w_0(r,\varphi)=\frac{4pr^2}{\pi S}\sum_{n=1,3,\ldots}^{\infty}\frac{\sin nk\varphi}{n(n^2k^2-4)}\cdot$$

$$\cdot\left[1-\frac{(R_2^{nk+2}-R_1^{nk+2})r^{nk-2}+R_1^{nk+2}\cdot R_2^{nk+2}(R_2^{nk-2}-R_1^{nk-2})r^{-nk+2}}{R_2^{2nk}-R_1^{2nk}}\right].$$

In the particular cases of $\varrho_1 = 0$, $\varrho_2 \neq 0$ and $\varrho_1 \neq 0$, $\varrho_2 = 0$ we obtain $w_1(\varrho, \varphi; 1, \xi)$ in a closed form.

The unknown reaction at the support $P(\xi)$ is obtained if the deflection at the curvilinear support AB is equal to zero:

$$w_0(1, \varphi) + \int_{\beta_1}^{\beta_2} P(\xi) w_1(1, \varphi; 1, \xi) R d\xi = 0, \qquad \beta_1 \leqq \xi \leqq \beta_2. \tag{16}$$

The problem of deflection of a circular membrane with an additional rectilinear support situated radially (Fig. 4) can be solved in a similar manner.

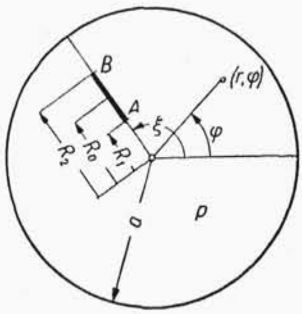

Fig. 4

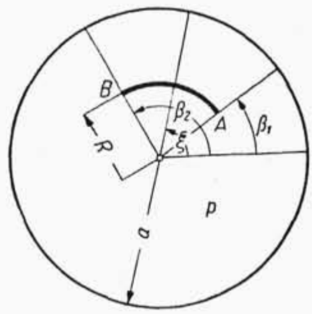

Fig. 5

The deflection caused by the lateral pressure $p = \text{const}$ and the unknown reaction $P(\eta)$ at the support AB is expressed by the following integral equation:

$$w(\varrho, \varphi) = w_0(\varrho) + \int_{\eta_1}^{\eta_2} P(\eta) \cdot w_1(\varrho, \varphi; \eta, \xi) a \, d\eta. \tag{17}$$

The unknown function expressing the reaction $P(\eta)$ at the support is found from the condition of zero deflection at the support AB

$$w_0(\varrho) + \int_{\eta_1}^{\eta_2} P(\eta) \cdot w_1(\varrho, \xi; \eta, \xi) a \, d\eta = 0. \tag{18}$$

Remembering that

$$w_0(\varrho) = \frac{p a^2}{4S}(1 - \varrho^2),$$

$$w_1(\varrho, \varphi; \eta, \xi) = \frac{1}{4\pi S} \ln \frac{\cosh(\ln \varrho\eta) - \cos(\varphi - \xi)}{\cosh(\ln \varrho/\eta) - \cos(\varphi + \xi)},$$

$$\varrho = \frac{r}{a}, \quad \eta = \frac{R}{a},$$

we obtain equation (18) in the form

$$\int_{\eta_1}^{\eta_2} R(\eta) \ln \frac{\varrho\eta - 1}{\varrho - \eta} d\eta = -\frac{p a \pi}{2}(1 - \varrho^2). \tag{19}$$

The deflection of a circular membrane supported at the periphery with an additional curvilinear support $R = \text{const}$ (Fig. 5), and subjected to a lateral pressure $p = \text{const}$ is expressed by the equation

$$w(\varrho, \varphi) = w_0(\varrho) + \int_{\beta_1}^{\beta_2} P(\xi) \cdot w_1(\varrho, \varphi; \varrho, \xi) R_0 \, d\xi, \tag{20}$$
$$\varrho_0 = \frac{R_0}{a}, \quad \varrho = \frac{r}{a}.$$

The unknown function $P(\xi)$ is found from the integral equation

$$w_0(\varrho) + \int_{\beta_1}^{\beta_2} (P(\xi) w_1(\varrho_0, \varphi; \varrho_0, \xi) R_0 \, d\xi = 0, \tag{21}$$
$$\int_{\beta_1}^{\beta_2} P(\xi) \ln \frac{\cosh(2 \ln \varrho_0) - \cos(\varphi - \xi)}{1 - \cos(\varphi - \xi)} \, d\xi = - \frac{p a^2 \pi}{R_0} (1 - \varrho_0^2).$$

The above method of solving complex cases of (pure) torsion of anisotropic bars or those of deflection of membranes can be applied to other physical phenomena whose mathematical expression leads to differential equations and boundary conditions of the kind discussed in this note.

REFERENCES

[1] Lekhnitzky C. G., *The Theory of Elasticity of an Anisotropic Body*, Moscow, 1950, 151.