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TRANSIENT THERMAL STRESSES IN VISCOELASTIC BODIES (I)

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The problem of transient thermal stresses in viscoelastic bodies has recently been the object of many investigations. Two method of solution may be distiguished in theoretical papers. The first consists in a direct solution of the displacement equations, [1], [2], or in a solution of such equations after the application of the Laplace transform, employing the principle of correspondence. The second method is based on introducing the thermoelastic potential, [3], [4], connected with the application of the Laplace transform.

In the present paper, a different way of solution will be chosen, constituting an extension of W. M. Maysel's method, [5], (invented for perfectly elastic bodies) for viscoelastic bodies. The essence of Maysel's method is a generalization of E. Betti's reciprocal theorem and a derivation of simple expressions for quasi-statical displacements. It will be proved that this method is equivalent to replacing the differential equations for displacements by appropriate integral equations.

We present a two-stage method of solution of dynamical problems due to non-stationary temperature fields in viscoelastic bodies. In the first stage, the solution of the quasi-statical problem is obtained; in the second stage, the influence of inertia forces is taken into account.

In the first part of the paper, an outline of the method is given and quasi-statical problems are considered in detail. The second part is devoted to dynamical problems.

1. Basic Equations and Relations

It is assumed that the deformations are small and physical constants (both mechanical and thermal) are independent of position and temperature. We shall consider a homogeneous, isotropic viscoelastic medium of linear characteristics, obeying the following stress-strain law, [6], [7]:

$$(1.1)\quad P_1(D)\,P_3(D)\,\sigma_{ij}^{(1)}(x_r,t) = P_2(D)\,P_3(D)\,\varepsilon_{ij}^{(1)}(x_r,t) +$$
$$+\,\delta_{ij}\left\{\frac{1}{3}\left[P_1(D)\,P_4(D) - P_2(D)\,P_3(D)\right]e^{(1)}(x_r,t) - P_1(D)\,P_4(D)\,\alpha_t T(x_r,t)\right\},$$

$$(1.2)\quad \sigma_{ij}^{(2)}(x_r, t) = 2\int_0^t a(t-\tau)\frac{\partial}{\partial\tau}\varepsilon_{ij}^{(2)}(x_r,\tau)\,d\tau +$$
$$+\delta_{ij}\int_0^t \left\{ b(t-\tau)\frac{\partial e^{(2)}(x_r,\tau)}{\partial\tau} - [3\,b(t-\tau)+2\,a(t-\tau)]\,\alpha_t \frac{\partial T(x_r,\tau)}{\partial\tau}\right\} d\tau,$$

The relations (1.1) follow from

$$(1.3)\qquad P_1(D)\,s_{ij}^{(1)} = P_2(D)\,e_{ij}^{(1)},$$

$$(1.4)\qquad P_3(D)\,s = P_4(D)\,(e - 3\,\alpha_t T).$$

Here $\sigma_{ij}^{(1)}$ denotes the stress tensor, $s_{ij}^{(1)}$ its deviatoric part, $e_{ij}^{(1)}$ that of the strain tensor, δ_{ij} is the Kronecker's symbol, α_t — the coefficient of thermal expansion, T — the temperature.

The operators $P_i(D)$ $(i = 1, 2, 3, 4)$ are defined by the formulae

$$(1.5)\qquad P_i(D) = \sum_{n=0}^{Ni} a_i^{(n)} D^{(n)} \quad a_i^{(Ni)} \neq 0,$$

where $D^n = \partial^n/\partial t^n$ denotes the n-th derivative with respect to the time t. The coefficients $a_i^{(n)}$ are independent of position and temperature; they are constant quantities. In the particular case of a perfectly elastic body, the operators $P_i(D)$, reduce to the first terms of the sum (1.5): where μ_0 and λ_0 denote the Lamé constants for a perfectly elastic body.

The relation (1.2) was derived by M. A. Biot, [8], and generalized to three-dimensional problems of viscoelasticity by D. S. Berry, [9]. They may be applied to bodies which are free of stresses at the initial moment. $a(t)$ and $b(t)$ are the relaxation functions which reduce to the Lamé constants for a perfectly elastic body.

Let us insert the relations (1.1) and (1.2) into the equations of motion:

$$(1.6)\qquad \sum_k \frac{\partial\sigma_{ki}}{\partial x_k} = \beta\frac{\partial^2 u_i}{\partial t^2},$$

[β denotes the density and $u_i\,(i = 1, 2, 3)$ the components of displacement vector], and let us express the stresses by the strains and the strains by the displacements according to the relations:

$$(1.7)\qquad \varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right).$$

As a consequence of these operations, the displacement equations are obtained:

$$(1.8)\quad P_2(D)\,P_3(D)\,\nabla^3 u_i^{(1)} + \frac{1}{3}\,[2\,P_4(D)\,P_1(D) + P_2(D)\,P_3(D)]\,\frac{\partial e^{(1)}}{\partial x_i} =$$
$$= 2\,P_1(D)\,P_3(D)\,\beta\frac{\partial^2 u_i^{(1)}}{\partial t^2} + 2\,P_4(D)\,P_1(D)\,\alpha_t\frac{\partial T}{\partial x_i},$$

$$(1.9)\quad \int_0^t \left\{ a(t-\tau)\frac{\partial}{\partial\tau}\nabla^2 u_i^{(2)} + [b(t-\tau)+a(t-\tau)]\frac{\partial}{\partial\tau}\frac{\partial e^{(2)}}{\partial x_i}\right\} d\tau = \\ = \beta\frac{\partial^2 u_i^{(2)}}{\partial t^2} + a_t\int_0^t [3\,b(t-\tau)+2\,a(t-\tau)]\frac{\partial}{\partial\tau}\frac{\partial T}{\partial x_i}\,d\tau.$$

Both displacement equations may be represented in the following operational form:

$$(1.10)\qquad L(u_i) = K\left(\frac{\partial^2 u_i}{\partial t^2}\right) + M\left(\frac{\partial T}{\partial x_i}\right).$$

The solution of (1.10) is the sum $u_i = u_i^* + u_i^{**}$, where u_i^* corresponds to the solution of the quasi-statical problem, i. e. it satisfies the equation

$$(1.11)\qquad L(u_i^*) = M\left(\frac{\partial T}{\partial x_i}\right),$$

and u_i^{**} satisfies the operational equation

$$(1.12)\qquad L(u_i^{**}) - K\left(\frac{\partial^2 u_i^{**}}{\partial t^2}\right) = K\left(\frac{\partial^2 u_i^*}{\partial t^2}\right).$$

Let us assume that the viscoelastic body was in its natural state at the initial moment, i. e. it was free of stresses. Applying the Laplace transform to (1.1) and (1.2) we obtain

$$(1.13)\quad \bar\sigma_{ij}(x_r, p) = [\bar\lambda(p)\,\bar e(x_r, p) - \bar\gamma(p)\,\bar T(x_r, p)]\,\delta_{ij} + 2\,\bar\mu(p)\,\bar\varepsilon_{ij}(x_r, p),$$

where

$$\bar f(x_r, p) = \int_0^\infty e^{-pt} f(x_r, t)\,dt.$$

We have introduced the following notation

$$(1.14)\qquad \begin{cases} \bar\lambda(p) = \dfrac{P_1(p)\,P_4(p) - P_2(p)\,P_3(p)}{3\,P_1(p)\,P_3(p)}, \\[2ex] \bar\mu(p) = \dfrac{P_2(p)}{2P_1(p)}, \quad \bar\gamma(p) = [3\,\bar\lambda(p) + 2\,\bar\mu(p)]\,a_t, \end{cases}$$

for a viscoelastic body, the stress-strain law of which is given by (1.1), and

$$(1.15)\qquad \bar\lambda(p) = p\bar b(p), \quad \bar\mu(p) = p\bar a(p), \quad \bar\gamma(p) = [3\,\bar\lambda(p) + 2\,\bar\mu(p)]\,a_t,$$

for a body obeying the stress-strain law (1.2).

Let us now apply the Laplace transform to the displacement equations (1.11) and (1.12). The first system of equations

$$(1.16)\quad [\bar\lambda(p)+\bar\mu(p)]\frac{\partial\bar e^*(x_r, p)}{\partial x_i} + \bar\mu(p)\,\nabla^2\bar u_i^*(x_r, p) = \bar\gamma(p)\frac{\partial\bar T(x_r, p)}{\partial x_i} \\ (i = 1, 2, 3),$$

concerns the quasi-statical problem. The second system

$$(1.17)\quad [\bar{\lambda}(p)+\bar{\mu}(p)]\frac{\partial \bar{e}^{**}(x_r,p)}{\partial x_i}+\bar{\mu}(p)\,\nabla^2\bar{u}_i^{**}(x_r,p)= \\ =\beta p^2\,\bar{u}_i^{**}(x_r,p)+\bar{\Gamma}_i(x_r,p)\quad (i=1,2,3),$$

where

$$(1.18)\quad \bar{\Gamma}(x_r,p)=\beta\,[p^2\,\bar{u}_i^*(x_r,p)-pu_i^*(x_r,0+)-\dot{u}_i^*(x_r,0+)- \\ -pu_i^{**}(x_r,0+)-\dot{u}_i^{**}(x_r,0+)],$$

takes into account the inertia terms. The final solution $u_i(x_r,t)$ is the sum of $u_i^*(x_r,t)$ and $u_i^{**}(x_r,t)$.

We have still to consider the boundary conditions for the system (1.16) and (1.17). If the boundary of the body is free of tractions, we have at our disposal the following boundary conditions for the system (1.16):

$$(1.19)\qquad -\bar{\gamma}\,Tn_i+\sum_{k=1}^{3}\bar{\tau}_{ik}^*\,n_k=0\qquad (i=1,2,3).$$

For the system (1.17), the boundary conditions have the form

$$(1.20)\qquad \sum_{k=1}^{3}\bar{\tau}_{ik}^{**}\,n_k=0,$$

the following notation being introduced

$$\bar{\tau}_{ik}^*=\bar{\lambda}\bar{e}^*\,\delta_{ik}+2\,\bar{\mu}\,\bar{\varepsilon}_{ik}^*,\qquad \bar{\tau}_{ik}^{**}=\bar{\lambda}\,\bar{e}^{**}\,\delta_{ik}+2\,\bar{\mu}\,\bar{\varepsilon}_{ik}^{**};$$

n_k denotes the cosine of the angle between the normal to the surface of the body and the x_k-axis.

If the boundary of the body is rigidly clamped, the boundary conditions for the equations (1.16) are given by $\bar{u}_i^*=0$, and for (1.17) by $\bar{u}_i^{**}=0$.

Let us consider a perfectly elastic body. The stress-strain law has the form:

$$(1.21)\quad \sigma_{ij}^0(x_r,t)=[\lambda_0\,e^0(x_r,t)-\gamma_0\,T(x_r,t)]\,\delta_{ij}+2\,\mu_0\,\varepsilon_{ij}^0(x_r,t),\quad \gamma_0=(3\,\lambda_0+2\,\mu_0)\,a_t.$$

Here λ_0, μ_0 and γ_0 are independent of time Lamé constants for a perfectly elastic body.

The displacement equations for a perfectly elastic body may be written thus:

$$(1.22)\quad (\lambda_0+\mu_0)\frac{\partial e^0(x_r,t)}{\partial x_i}+\mu_0\,\nabla^2 u_i^0(x_r,t)=\gamma_0\frac{\partial T(x_r,t)}{\partial x_i}+\beta\frac{\partial^2 u_i^0(x_r,t)}{\partial t^2}.$$

The corresponding boundary conditions

$$(1.23)\qquad -\gamma_0\,Tn_i+\sum_{k=1}^{3}\tau_{ij}^0\,n_k,\qquad \tau_{ij}^0=\lambda_0\,e^0\,\delta_{ij}+2\,\mu\varepsilon_{ij}^0,$$

for a body the surface of which is free of tractions, and

$$u_i^0 = 0, \tag{1.24}$$

for a body rigidly clamped. Eq. (1.22) may be split up into two systems:

$$(\lambda_0 + \mu_0) \frac{\partial e^{0*}(x_r, t)}{\partial x_i} + \mu_0 \nabla^2 u_i^{0*}(x_r, t) = \gamma_0 \frac{\partial T(x_r, t)}{\partial x_i}, \tag{1.25}$$

$$(\lambda_0 + \mu_0) \frac{\partial e^{0**}(x_r, t)}{\partial x_i} + \mu_0 \nabla^2 u_i^{0**}(x_r, t) = \beta \frac{\partial^2 u_i^{0**}(x_r, t)}{\partial t^2} + \beta \frac{\partial^2 u_i^{0*}(x_r, t)}{\partial t^2} \tag{1.26}$$

$$(i = 1, 2, 3).$$

If the boundary of the body is free of tractions, the boundary conditions for the equation (1.25) are the following:

$$-\gamma_0 T n_i + \sum_{k=1}^{3} \tau_{ik}^{0*} n_k = 0, \qquad \tau_{ik}^{0*} = \lambda_0 e_{ik}^{0*} + 2\mu_0 \varepsilon_{ik}^{0*}. \tag{1.27}$$

For the system (1.26) they have the form:

$$\sum_{k=1}^{3} \tau_{ik}^{0**} n_k = 0, \qquad \tau_{ik}^{0**} = \lambda_0 e_{ik}^{0**} + 2\mu_0 \varepsilon_{ij}^{0**}. \tag{1.28}$$

In the case of a rigidly clamped body, we have: $u_i^{0*} = 0$ for the system (1.25), and $u_i^{0**} = 0$ for the system (1.26).

Let us apply the Laplace transform to the equations (1.25) and (1.26). Then:

$$(\lambda_0 + \mu_0) \frac{\partial \bar{e}^{0*}(x_r, p)}{\partial x_i} + \mu_0 \nabla^2 \bar{u}_i^{0*}(x_r, p) = \gamma_0 \frac{\partial \bar{\Gamma}(x_r, p)}{\partial x_i}, \tag{1.25.1}$$

$$(\lambda_0 + \mu_0) \frac{\partial \bar{e}^{0**}(x_r, p)}{\partial x_i} + \mu_0 \nabla^2 \bar{u}_i^{0**}(x_r, p) = \beta p^2 \bar{u}_i^{0**}(x_r, p) + \bar{\Gamma}_0(x_r, p), \tag{1.26.1}$$

where

$$\bar{\Gamma}_0(x_r, p) = \beta \left[p^2 \bar{u}_i^{0*}(x_r, p) - p u_i^{0*}(x_r, 0+) - \dot{u}_i^{0*}(x_r, 0+) - \right.$$
$$\left. - p u_i^{0**}(x_r, 0+) - \dot{u}_i^{0**}(x_r, 0+) \right].$$

The boundary conditions are given by

$$-\gamma_0 \bar{T} n_i + \sum_{k=1}^{3} \bar{\tau}_{ik}^{0*} n_k = 0, \tag{1.27.1}$$

for the system (1.25.1), and the boundary free of tractions, and

$$\sum_{k=1}^{3} \bar{\tau}_{ik}^{0**} n_k = 0, \tag{1.28.1}$$

for the system (1.26.1), and the boundary free of tractions.

For a rigidly clamped body: $\bar{u}_i^{0*} = 0$ for the system (1.25.1), and $\bar{u}_i^{0**} = 0$ for (1.26.1).

It follows, from the comparison of the transformed equations and boundary conditions for a perfectly elastic and viscoelastic body, that for the construction of equations for a viscoelastic body the constant quantities μ_0, λ_0 in the equations (1.25.1), (1.26.1) and (1.27.1) should be replaced by the functions of the parameter $p - \bar{\mu}(p)$ and $\bar{\lambda}(p)$.

2. Solution of the Displacement Equations

We shall solve the displacement equations of a viscoelastic body with the help of the Green's function $U_i^{(k)}(x_r, \xi_r, t)$, $(i, k = 1, 2, 3)$. By $U_i^{(k)}(x_r, \xi_r, t)$ we denote the displacement of the point (x_r) (in the direction of the x_i-axis) due to the action of a concentrated force in the viscoelastic body at the point (ξ_r), the direction of which is parallel to the x_k-axis.

The functions $\bar{U}_i^{(k)}(x_r, \xi_r, t)$ ought to satisfy the following system of equations

$$(2.1)\quad [\bar{\lambda}(p)+\bar{u}(p)]\frac{\partial \bar{e}^{(k)}(x_r, \xi_r, p)}{\partial x_i} + \bar{\mu}(p)\nabla^2 \bar{U}_i^{(k)}(x_r, \xi_r, p) + \delta(x_r - \xi_r)\,\delta_{ik} = 0$$

$$(i, k = 1, 2, 3),$$

with the boundary conditions

$$(2.2.1)\qquad \sum_{r=1}^{3} \bar{\tau}_{ir}^{(k)} n_r = 0 \ (i, r = 1, 2, 3), \qquad \bar{\tau}_{ir}^{(k)} = \bar{\lambda}\bar{e}^{(k)} + 2\bar{\mu}\bar{\varepsilon}_{ir}^{(k)},$$

for a body free of tractions, and

$$(2.2.2)\qquad \bar{U}_i^{(k)} = 0 \qquad (i, k = 1, 2, 3),$$

for a body rigidly clamped.

$\bar{e}^{(k)}$, $\bar{\varepsilon}_{ir}^{(k)}$ denote the dilatation and the components of the strain tensor corresponding to the displacement $\bar{U}_i^{(k)}$.

From the solution of the three systems of equations (each contains three equations) with appropriate boundary conditions, nine functions $\bar{U}_i^{(k)}(i, k = 1, 2, 3)$, will be obtained. They will serve for the representation of the solution of the equation (1.16) with the boundary conditions (1.19) in the following integral form:

$$(2.3)\quad u_i^*(x_r, p) = -\gamma(p)\int_V \sum_{k=1}^{3} \bar{U}_i^{(k)}(x_r, \xi_r, p)\frac{\partial \bar{T}(\xi_r, p)}{\partial \xi_k}\,dV +$$

$$+\bar{\gamma}(p)\int_\Omega \bar{T}(\xi_r, p)\sum_{k=1}^{3}\bar{U}_i^{(k)}(x_r, \xi_r, p)\,n_k\,d\Omega \qquad (i = 1, 2, 3).$$

Let us denote by $\bar{U}_k^{(i)}(\xi_r, x_r, p)$ the transform of the displacement of the point (ξ_r) in the direction of the x_k-axis due to the action of a unit concentrated force at the point (x_r) in the direction x_i.

Making use of Maxwell's reciprocal theorem for displacements

$$\bar{U}_i^{(k)}(x_r, \xi_r, p) = U_{(k)}^i(\xi_r, x_r, p), \tag{2.4}$$

and taking into account the Green's formula

$$\int_V \sum_{k=1}^{3} \bar{U}_k^{(i)}(\xi_r, x_r, p) \frac{\partial \bar{T}(\xi_r, p)}{\partial \xi_k} dV = \int_\Omega \bar{T}(\xi_r, p) \sum_{k=1}^{3} \bar{U}_k^{(i)}(\xi_r, x_r, p) n_k d\Omega - \int_V \bar{T}(\xi_r, p) \sum_{k=1}^{3} \frac{\partial \bar{U}_k^{(i)}(\xi_r, x_r, p)}{\partial \xi_k} dV, \tag{2.5}$$

we can transform the integral expression, so that:

$$\bar{u}_i^*(x_r, p) = \bar{\gamma}(p) \int_V \bar{T}(\xi_r, p) \sum_{k=1}^{3} \frac{\partial \bar{U}_k^{(i)}(\xi_r, x_r, p)}{\partial \xi_r} dV, \qquad dV = d\xi_r^3. \tag{2.6}$$

The function $U_k^{(i)}(\xi_r, x_r, p)$ entering (2.6) is a solution of the system of equations:

$$[\bar{\lambda}(p) + \bar{\mu}(p)] \frac{\partial \bar{e}^{(i)}(\xi_r, x_r, p)}{\partial \xi_i} + \bar{\mu}(p) \nabla^2 U_k^{(i)}(\xi_r, x_r, p) + \delta_{ik}\, \delta(\xi_r - x_r) = 0, \tag{2.7}$$

all the operations being performed with respect to the variable ξ.

Let us observe that the sum

$$\sum_{k=1}^{3} \frac{\partial \bar{U}_k^{(i)}(\xi_r, x_r, p)}{\partial \xi_k} = \sum_{k=1}^{3} \bar{\varepsilon}_{kk}^{(i)} = \bar{M}_i(\xi_r, x_r, p), \tag{2.8}$$

may be regarded as a transform of a dilatation at the point (ξ_r) due to the action of a concentrated force at (x_r) in the direction of the x_i-axis.

On the other hand, on account of

$$\frac{\partial \bar{U}_k^{(i)}(\xi_r, x_r, p)}{\partial \xi_k} = \frac{\partial \bar{U}_i^{(k)}(x_r, \xi_r, p)}{\partial x_k}, \tag{2.9}$$

we obtain:

$$\sum_{k=1}^{3} \frac{\partial \bar{U}_i^{(k)}(x_r, \xi_r, p)}{\partial x_k} = \bar{U}_i(x_r, \xi_r, p). \tag{2.10}$$

The function $\bar{U}_i(x_r, \xi_r, p)$ should be regarded as the transform of the displacement at the point (x_r) in the direction x_i due to a centre of pressure situated at the point (ξ_r). Thus the transform of the displacement $\bar{u}_i^*$ can be represented in two forms similar to Maysel's representations for perfectly elastic bodies [1]

$$\bar{u}_i^*(x_r, p) = \bar{\gamma}(p) \int_V \bar{T}(\xi_r, p) \bar{M}_i(\xi_r, x_r, p) dV, \tag{2.11}$$

[1] These formulae are also valid for the boundary conditions (2.2.2). In (2.3) and (2.5), the surface integrals vanish and (2.6) holds.

or

$$\bar{u}_i^*(x_r, p) = \bar{\gamma}(p) \int_V \bar{T}(\xi_r, p)\, \bar{U}_i(x_r, \xi_r, p)\, dV, \quad \bar{M}_i(\xi_r, x_r, p) = \bar{U}_i(x_r, \xi_r, p). \tag{2.12}$$

By making use of the generalized E. Betti's theorem (for thermoelastic problems), W. M. Maysel obtained the following relations for a perfectly elastic body:

$$\bar{u}_i^{0*}(x_r, p) = \gamma_0 \int_V \bar{T}(\xi_r, p)\, M_i^0(\xi_r, x_r)\, dV, \tag{2.13}$$

or

$$\bar{u}_i^{0*}(x_r, p) = \gamma_0 \int_V \bar{T}(\xi_r, p)\, U_i^0(\xi_r, x_r)\, dV. \tag{2.14}$$

Comparing the formulae for the transforms of the displacements in perfectly elastic and viscoelastic bodies, the possibility of application of Maysel's formulae becomes evident. The formulae (2.11) and (2.12) can be obtained from (2.13) and (2.14) replacing in the latter γ_0 by $\bar{\gamma}(p)$ and replacing the quantities μ_0, λ_0, appearing in the functions M_i^0, U_i^0 by $\bar{\mu}(p)$, $\bar{\lambda}(p)$.

Since

$$\bar{\varepsilon}_{ij}^* = \frac{1}{2}\left(\frac{\partial \bar{u}_i^*}{\partial x_j} + \frac{\partial \bar{u}_j^*}{\partial x_i}\right), \tag{2.15}$$

we can obtain from (2.11) and (2.12) expressions for strains and dilatation:

$$\bar{\varepsilon}_{ij}^*(x_r, p) = \bar{\gamma}(p) \int_V \bar{T}(\xi_r, p)\, \bar{M}_{ij}(\xi_r, x_r, p)\, dV \qquad (i \neq j), \tag{2.16.1}$$

$$\bar{\varepsilon}_{ii}^*(x_r, p) = \bar{\gamma}(p) \int_V \bar{T}(\xi_r, p)\, \bar{M}_{ii}(\xi_r, x_r, p)\, dV, \tag{2.16.2}$$

$$\bar{e}^*(x_r, p) = \bar{\gamma}(p) \int_V \bar{T}(\xi_r, p)\, \bar{M}(\xi_r, x_r, p)\, dV. \tag{2.16.3}$$

$\bar{M}_{ij}$ denotes the transform of the dilatation at the point (ξ_r) due to the action of a cross of shear forces at the point (x_r), its vector being perpendicular to the plane $x_i x_j$ and intensity equal to 1/2. Further, $\bar{M}_{ii}$ denotes the transform of the dilatation at the point (ξ_r) due to the action of a double concentrated force at the point (x_r) in the direction x_i. Finally $\bar{M}$ denotes the transform of the dilatation at the point (ξ_r) due to the action of a centre of pressure situated at the point (x_r).

In view of the second form of the displacement (2.14) we have

$$\bar{\varepsilon}_{ij}^*(x_r, p) = \bar{\gamma}(p) \int_V \bar{T}(\xi_r, p)\, \bar{K}_{ij}(x_r, \xi_r, p)\, dV, \tag{2.17}$$

$$\bar{e}^*(x_r, p) = \bar{\gamma}(p) \int_V \bar{T}(\xi_r, p)\, \bar{K}(x_r, \xi_r, p)\, dV. \tag{2.18}$$

By $\overline{K}_{ij}$ we denote the transform of the shear strain (its vector being perpendicular to the plane $x_i x_j$) at the point (x_r) due to the action of a centre of pressure at the point (ξ_r). $\overline{K}$ is the transform of the dilatation at the point (x_r) due to the action of a centre of pressure situated at the point (ξ_r).

In the particular case $i=j$, the function $\overline{K}_{ii}$ represents the axial strain at the point (x_r) (its vector being parallel to the x_i-axis) due to the action of a centre of pressure situated at the point (ξ_r). Let us observe that:

$$\overline{M}(\xi_r, x_r, \mathrm{p}) = \overline{K}(x_r, \xi_r, \mathrm{p}), \qquad \overline{M}_{ij}(\xi_r, x_r, \mathrm{p}) = \overline{K}_{ij}(x_r, \xi_r, \mathrm{p}),$$
$$\overline{M}_{ii}(\xi_r, x_r, \mathrm{p}) = \overline{K}_{ii}(x_r, \xi_r, \mathrm{p}). \tag{2.19}$$

Further, let us consider the solution of the system of equations (1.17). Making use of the Green's function, it can be represented in the form:

$$\bar{u}_i^{**}(x_r, \mathrm{p}) = -\beta \mathrm{p}^2 \int_V \sum_{k=1}^{3} \bar{u}_k^{**}(\xi_r, \mathrm{p})\, \overline{U}_i^{(k)}(x_r, \xi_r, \mathrm{p})\, dV - \int_V \sum_{k=1}^{3} \overline{\Gamma}_k(\xi_r, \mathrm{p})\, \overline{U}_i^{(k)}(x_r, \xi_r, \mathrm{p})\, dV, \tag{2.20}$$

or

$$\bar{u}_i^{**}(x_r, \mathrm{p}) = -\beta \mathrm{p}^2 \int_V \sum_{k=1}^{3} \bar{u}_k^{**}(\xi_r, \mathrm{p})\, \overline{U}_k^{(i)}(\xi_r, x_r, \mathrm{p})\, dV - \int_V \sum_{k=1}^{3} \overline{\Gamma}_k(\xi_r, \mathrm{p})\, \overline{U}_k^{(i)}(\xi_r, x_r, \mathrm{p})\, dV. \tag{2.21}$$

We have obtained a system of three Fredholm's integral equations of the second order for the determination of the function $\bar{u}_i^{**}$. The strain tensor and the dilatation can be represented in the dual form:

$$\bar{\varepsilon}_{ij}^{**}(x_r, \mathrm{p}) = -\int_V \sum_{k=1}^{3} [\beta^2 \mathrm{p}^2 \bar{u}_k^{**}(\xi_r, \mathrm{p}) + \overline{\Gamma}_k(\xi_r, \mathrm{p})]\, \overline{W}_{ij}^{(k)}(\xi_r, x_r, \mathrm{p})\, dV, \tag{2.22}$$

$$\bar{e}^{**}(x_r, \mathrm{p}) = -\int_V \sum_{k=1}^{3} [\beta \mathrm{p}^2 \bar{u}_k^{**}(\xi_r, \mathrm{p}) + \overline{\Gamma}_k(\xi_r, \mathrm{p})]\, \overline{W}^{(k)}(\xi_r, x_r, \mathrm{p})\, dV, \tag{2.23}$$

or

$$\bar{\varepsilon}_{ij}^{**}(x_r, \mathrm{p}) = -\int_V \sum_{k=1}^{3} [\beta \mathrm{p}^2 \bar{u}_k^{**}(\xi_r, \mathrm{p}) + \overline{\Gamma}_k(\xi_r, \mathrm{p})]\, \overline{E}_{ij}(x_r, \xi_r, \mathrm{p})\, dV, \tag{2.24}$$

$$\bar{e}^{**}(x_r, \mathrm{p}) = -\int_V \sum_{k=1}^{3} [\beta \mathrm{p}^2 \bar{u}_k^{**}(\xi_r, \mathrm{p}) + \overline{\Gamma}_k(\xi_r, \mathrm{p})]\, \overline{E}^{(k)}(x_r, \xi_r, \mathrm{p})\, dV. \tag{2.25}$$

$\overline{W}_{ij}^{(k)}$ $(i \neq j)$ denotes the transform of the displacement of the point (ξ_r) in the direction of the x_k-axis due to the action of a cross of shear forces at the point (x_r), the vector of which is perpendicular to the plane $x_i\, x_j$. In the case $i = j$, a double force will act at the point (x_r) in place of the cross, its direction being parallel to the x_i-axis. $\overline{W}^{(k)}$ is the transform of the displacement of the point (ξ_r) in the direction of the x_k-axis due to the action of a centre of pressure situated at the point (x_r). By $\overline{E}_{ij}^{(k)}$ we denote the shear strain at the point (x_r) due to the action of a concentrated force applied at the point (ξ_r). $\overline{E}_{ii}^{(k)}$ and $\overline{E}^{(k)}$ denote the normal strain in the direction of the x_i-axis and the dilatation at the point (x_r) due to the action of a concentrated force situated at the point (ξ_r). We have:

$$(2.26) \qquad \overline{W}_{ij}^{(k)}(\xi_r, x_r, p) = \overline{E}_{ij}^{(k)}(x_r, \xi_r, p), \qquad \overline{W}^{(k)}(\xi_r, x_r, p) = \overline{E}^{(k)}(x_r, \xi_r, p).$$

If the displacements and strains are known, the stresses $\overline{\sigma}_{ij}(x_r, t)$ can be determined from the formula (1.13). For the quasi-statical problem we obtain:

$$(2.27) \qquad \overline{\sigma}_{ij}^{*}(x_r, p) = [\overline{\lambda}(p)\, \overline{e}^{*}(x_r, p) - \overline{\gamma}(p)\, \overline{T}(x_r, p)]\, \delta_{ij} + 2\, \overline{\mu}(p)\, \overline{\varepsilon}_{ij}^{*}(x_r, p).$$

The additional stresses due to the inertia forces can be found from the formula:

$$(2.28) \qquad \overline{\sigma}_{ij}^{**}(x_r, p) = \overline{\lambda}(p)\, \overline{e}^{**}(x_r, p)\, \delta_{ij} + 2\, \overline{\mu}(p)\, \overline{\varepsilon}_{ij}^{**}(x_r, p).$$

By performing the inverse Laplace transform, the stresses $\sigma_{ij}^{*}(x_r, t)$ and $\sigma_{ij}^{**}(x_r, t)$ may be calculated and their sum yields $\sigma_{ij}(x_r, t)$.

3. Quasi-Statical Problems

If the variation of temperature in time is sufficiently slow, the inertia forces may be neglected and the problem considered as a quasi-statical one. The determination of the displacements can be greatly simplified in this case, since the function u_i^{**}, which requires the solution of a system of Fredholm integral equations of the second kind, drops out.

For the determination of the displacement $\overline{u}_i^{*}$, the relation (2.11) or (2.12) may be employed.

Before presenting a few simple examples of the application of the foregoing formulae for displacements and stresses in viscoelastic bodies, a few general conclusions will be considered.

Let us consider the dilatation for a free viscoelastic body:

$$(3.1) \qquad e^{*}(x_r, p) = \overline{\gamma}(p) \int_V \overline{T}(\xi_r, p)\, \overline{M}(\xi_r, x_r, p)\, dV,$$

or

$$(3.2) \qquad e^{*}(x_r, p) = \alpha_t \int_V \overline{T}(\xi_r, p)\, \overline{S}(\xi_r, x_r, p)\, dV,$$

where

$$\bar{S}(x_r, x_r, p) = [3\bar{\lambda}(p) + 2\bar{\mu}(p)]\bar{M}(\xi_r, x_r, p)\, dV.$$

The function $\bar{S}(\xi_r, x_r, p)$ may be regarded as the sum of normal stresses at the point (ξ_r) due to the action of a centre of pressure situated at the point (x_r).

Observe that:

$$\bar{S}_0(\xi_r, p) = \int_{V_1} \bar{S}(\xi_r, x_r, p)\, dV_1 = 3, \qquad dV_1 = dx_1\, dx_2\, dx_3. \tag{3.3}$$

This integration can be interpreted as follows. Let us fill the volume of the body by centres of pressure and let us seek the sum of the normal stresses σ_0 at the point (ξ_r). If the domain under consideration is single--connected, the state of stress is identical with homogeneous tension of the body and $\bar{S}_0(\xi_r, p) = 3$.

The transform of the increment of volume of the viscoelastic body can be expressed by the integral:

$$\Delta\bar{V} = \int_{V_1} \bar{e}^*(x_r, p)\, dV_1 = \alpha_t \int_V \bar{T}(\xi_r, p)\, dV \int_{V_1} \bar{S}(\xi_r, x_r, p)\, dV_1. \tag{3.4}$$

In view of (3.3) we arrive at:

$$\Delta\bar{V} = 3\,\alpha_t \int_V \bar{T}(\xi_r, p)\, dV. \tag{3.5}$$

Hence:

$$\Delta V = 3\,\alpha_t \int_V \bar{T}(\xi_r, t)\, dV. \tag{3.6}$$

It is evident that the change in the volume of the body is independent of the rheological properties of the material.

Making use of the formula (1.13), and computing the sum of the normal stresses, we have:

$$\bar{s}^*(x_r, p) = (3\bar{\lambda} + 2\bar{\mu})\,[\bar{e}^*(x_r, p) - 3\,\alpha_t\,\bar{T}(x_r, p)]. \tag{3.7}$$

Integrating $\bar{s}^*$ over the volume of the body, and taking into account the relations (3.4) and (3.5), we obtain:

$$\int_{V_1} \bar{s}^*(x_r, p)\, dV_1 = 0. \tag{3.8}$$

The change of the volume of the body due to the stresses vanishes, the only change existing being that caused directly by the temperature.

The determination of the state of displacement and stress is greatly simplified in the case of an infinite viscoelastic space. For, the following relation takes place

$$\bar{u}_i^*(x_r, p) = h(p)\,\bar{u}_i^{0*}(x_r, p), \tag{3.9}$$

since for a perfectly elastic body the following is true

$$\bar{u}_i^{0*}(x_r, p) = \gamma_0 \int_V \bar{T}(\xi_r, p)\, U_i^0(x_r, \xi_r)\, dV, \tag{3.10}$$

where $U_i^0(x_r, \xi_r)$ is the displacement at the point (x_r) in the direction of the x_i-axis due to a centre of pressure at the point (ξ_r). It is known from the theory of elasticity that

$$\bar{U}_i^{0*}(x_r, \xi_r) = -\frac{1}{4\pi(\lambda_0 + 2\mu_0)} \frac{\partial}{\partial x_i}\left(\frac{1}{R}\right), \tag{3.11}$$

where

$$R = [(x_1 - \xi_1)^2 + (x_2 - \xi_2)^2 + (x_3 - \xi_2)^2]^{1/2}.$$

For a viscoelastic body:

$$\bar{U}_i(x_r, \xi_r, p) = -\frac{1}{4\pi[\bar{\lambda}(p) + \bar{\mu}(p)]} \frac{\partial}{\partial x_i}\left(\frac{1}{R}\right). \tag{3.12}$$

Comparing (2.12) and (2.14), and taking into account (3.11) and (3.12), we have:

$$\bar{u}_i^*(x_r, p) = \frac{\vartheta(p)}{\vartheta_0} \bar{u}_i^{0*}(x_r, p), \tag{3.13}$$

where

$$\vartheta(p) = \frac{3\bar{\lambda} + 2\bar{\mu}}{\bar{\lambda} + 2\bar{\mu}} \alpha_t, \qquad \vartheta_0 = \frac{3\lambda_0 + 2\mu_0}{\lambda_0 + 2\mu_6} \alpha_t.$$

It was shown by E. Sternberg that the relation (3.9) occurs also in the problem of a cavity in an infinite space, the temperature field being quasi-statical and spherically symmetrical. The relation (3.9) is also valid for an infinite space containing a cylindrical cavity of circular cross-section; similarly, for the particular cases of a uniaxial state of stress in rods.

The following cases are distinguished by a special simplicity: the symmetry of the temperature field with respect to a point, a line and a plane, and simple shapes of the body (sphere, cylinder, layer). In these cases, only one unknown displacement exists. Some of the problems of this type will be considered below in greater detail.

3.1. Layer in the plane $x = \pm a/2$ heated suddenly to the temperature T_0. Let the origin of the coordinate system be situated in the plane of symmetry. The temperature field is symmetrical with respect to the plane $x = 0$, and the surfaces $x = \pm a/2$ are free of tractions. The Laplace transform of the temperature field has the form:

$$\bar{T}(x_r, p) = \frac{T_0}{p} \frac{\operatorname{ch} x\sqrt{q}}{\operatorname{ch}\frac{a}{2}\sqrt{q}}, \qquad q = p/\varkappa. \tag{3.14}$$

The displacement $\bar{u}_x^*$ is given by the integral (2.11):

$$\bar{u}_x^*(x, p) = \bar{\gamma}(p) \int_V \bar{T}(\xi, p) \bar{M}_x(\xi, x, p) \, d\xi, \tag{3.15}$$

or

$$\bar{u}_x^*(\xi, p) = \bar{\gamma}(p) \int_V \bar{T}(x, p) \bar{M}_x(x, \xi, p) \, dx. \tag{3.16}$$

Here $\bar{M}_x(x, \xi, p)$ denotes the transform of the dilatation due to the action of unit forces uniformly distributed in the planes $x = \pm \xi$ of the visco-elastic layer. In order to determine the function $\bar{M}_x(x, \xi, p)$, the following differential equation has to be solved:

$$(\bar{\lambda} + 2\bar{\mu}) \frac{\partial^2 \bar{U}_x}{\partial x^2} + \delta(x - \xi) - \delta(x + \xi) = 0, \tag{3.17}$$

with the boundary conditions

$$\bar{\Sigma}_x(x, p) = 0 \quad \text{for} \quad x = \pm a/2. \tag{3.18}$$

We have denoted by $\bar{\Sigma}_x(x, p)$ the stress corresponding to $\bar{U}_x$. Further,

$$\bar{M}_x = \frac{\partial \bar{U}_x}{\partial x}. \tag{3.19}$$

Since

$$\bar{\Sigma}_x(x, p) = (\bar{\lambda} + 2\bar{\mu}) \frac{\partial \bar{U}_x}{\partial x},$$

the condition (3.18) assumes the form: $\partial U_x / \partial x = 0$ for $x = \pm a/2$.

From the solution of the equation (3.17) with the conditions (3.18) we obtain:

$$\bar{U}_x = \frac{2}{a(\bar{\lambda} + 2\bar{\mu})} \sum_{n=1, 3, \ldots}^{\infty} \frac{1}{\alpha_n^2} \sin \alpha_n \xi \sin \alpha_n x, \qquad \alpha_n = \frac{n\pi}{a}, \tag{3.20}$$

$$\bar{\Sigma}_x = \frac{2}{a} \sum_{n=1, 3, \ldots}^{\infty} \frac{1}{\alpha_n} \sin \alpha_n \xi \cos \alpha_n x, \tag{3.21}$$

or

$$\begin{cases} \bar{U}_x = \dfrac{x}{\bar{\lambda} + 2\bar{\mu}} & \text{for} \quad 0 < x < \xi, \\ \bar{U}_x = \dfrac{\xi}{\bar{\lambda} + 2\bar{\mu}} & \text{for} \quad \xi < x < a/2, \end{cases} \tag{3.22}$$

and

$$\begin{cases} \bar{\Sigma}_x = 1 & \text{for} \quad 0 < x < \xi, \\ \bar{\Sigma}_x = 0 & \text{for} \quad \xi < x < a. \end{cases} \tag{3.23}$$

Taking into account that

$$(3.24)\qquad \begin{cases} \overline{M}_x = \dfrac{1}{\overline{\lambda} + 2\overline{\mu}} & \text{for} \quad 0 < x < \xi, \\ \overline{M}_x = 0 & \text{for} \quad \xi < x < a/2, \end{cases}$$

the formula (3.16) may be written in the form:

$$(3.25)\qquad \bar{u}_x^*(\xi, p) = \bar{\vartheta}(p) \int_0^{\xi} \overline{T}(x, p)\, dx, \qquad \vartheta(p) = \frac{3\overline{\lambda} + 2\overline{\mu}}{\overline{\lambda} + 2\overline{\mu}} \alpha_t,$$

or

$$\bar{u}_x^*(\xi, p) = T_0 \bar{g}(p) \bar{f}(\xi, p), \qquad \bar{g}(p) = \frac{\vartheta(p)}{p}, \qquad f(\xi, p) = \frac{\operatorname{sh} \xi \sqrt{p/\varkappa}}{\sqrt{p/\varkappa} \operatorname{ch} \frac{a}{2} \sqrt{p/\varkappa}}.$$

Thus:

$$(3.26)\qquad u_x(\xi, t) = T_0 \int_0^t g(t - \tau) f(\xi, \tau)\, d\tau.$$

For the Kelvin body:

$P_1(p) = 1, \quad P_2(p) = 2\mu_0(1 + \varepsilon p), \quad P_3(p) = p, \quad P_4(p) = \eta_0 p, \quad \eta_0 = 3\lambda_0 + 2\mu_0,$ ε being the time of retardation. Therefore:

$$\bar{g}(p) = \vartheta_0 \left(\frac{1}{p} - \frac{1}{p + \varkappa_1}\right), \qquad \vartheta_0 = \frac{3\lambda_0 + 2\mu_0}{\lambda_0 + 2\mu_0}, \qquad \varkappa_1 = \frac{3}{4} \frac{\mu_0 + 2\lambda_0}{\mu_0 \varepsilon}.$$

Performing the reverse Laplace transform, we find that:

$$(3.27)\qquad g(t) = \vartheta_0 (1 - e^{-\varkappa_1 t}),$$

$$f(\xi, t) = -\frac{\varkappa}{a} \theta_1 \left(\frac{\xi}{a} \Big/ \frac{4\varkappa i \pi t}{a^2}\right) = -\sqrt{\frac{\varkappa}{4\pi t}} \sum_{n=-\infty}^{n=\infty} (-1)^n e^{-\frac{a^2}{4\varkappa t}\left[\frac{\xi}{a} - \frac{1}{2} + n\right]^2}$$

where θ_1 is the theta function, [10].

Having found the transform of the displacement $\bar{u}_x^*(\xi, p)$, the transform of the stress can easily be determined. Namely, we have:

$$\bar{\sigma}_{11}^* = \overline{\lambda} \frac{\partial \overline{U}_x}{\partial \xi} - \bar{\gamma} \overline{T} = 0, \qquad \bar{\sigma}_{22}^* = \bar{\sigma}_{33}^* = \overline{\lambda} \frac{\partial \overline{U}_x}{\partial \xi} - \bar{\gamma} \overline{T} = -2\bar{\mu}(p)\, \vartheta(p)\, \overline{T}(x, p).$$

Hence:

$$(3.28)\qquad \bar{\sigma}_{22}^* = T_0 \mu_0 \vartheta_0 \left(\frac{1}{p} - \frac{\eta}{p + \varkappa_1}\right) \frac{\operatorname{ch} \xi \sqrt{p/\varkappa}}{\operatorname{ch} \frac{a}{2} \sqrt{p/\varkappa}}.$$

Finally:

$$(3.29)\qquad \sigma_{22}^*(x, t) = T_0 \mu_0 \vartheta_0 \int_0^t g_1(t - \tau) f_1(\xi, \tau)\, d\tau,$$

where

$$g_1(t) = 1 - e^{-\varkappa_1 t}$$

$$f_1(\xi, t) = -\frac{2\varkappa}{a}\frac{\partial}{\partial \xi}\,\theta_1\left(\frac{\xi}{a}\Big/\frac{4\varkappa i\pi t}{a^2}\right).$$

3.2. **Infinite hollow cylinder of circular cross-section.** Let the temperature field depend on time and radius. We assume that the sides of the cylinder are free of tractions. The transform of the radial displacement can be found from:

$$\bar{u}_r^*(\varrho, p) = \frac{\bar{\gamma}(p)}{\varrho}\int_a^b \bar{T}(r, p)\,\overline{\boldsymbol{M}}_r(r, \varrho, p)\,r dr. \tag{3.30}$$

By $\overline{\boldsymbol{M}}_r(r, \varrho, p)$, we have denoted the transform of the dilatation on the surface of the cylinder of the radius r, due to the action of unit forces uniformly distributed on the surface $r = \varrho$. The displacement belonging to this state we denote by U_r. It can be obtained by solving the equation

$$(\bar{\lambda} + 2\bar{\mu})\left(\frac{\partial^2 \bar{U}_r}{dr^2} + \frac{1}{r}\frac{d\bar{U}_r}{dr} - \frac{\bar{U}_r}{r^2}\right) + \delta(r - \varrho) = 0, \tag{3.31}$$

with the boundary conditions:

$$\bar{\Sigma}_r(r, p) = 0 \quad \text{for } r = a \quad \text{and } r = b. \tag{3.32}$$

Here $\bar{\Sigma}_r$ denotes the transform of the stress:

$$\bar{\Sigma}_r = \bar{\lambda}\,\overline{\boldsymbol{M}}_r + 2\bar{\mu}\frac{\partial \bar{U}_r}{dr}, \quad \overline{\boldsymbol{M}}_r = \frac{d\bar{U}_r}{dr} + \frac{\bar{U}_r}{r}. \tag{3.33}$$

From the solution of the equation (3.31) we have:

$$\begin{cases} \bar{U}_r = A_1 r + A_2 r^{-1}, & \overline{\boldsymbol{M}}_r = 2A_1, \quad a < r < \varrho, \\ \bar{U}_r = B_1 r + B_2 r^{-1}, & \overline{\boldsymbol{M}}_r = 2B_1, \quad \varrho < r < b, \end{cases} \tag{3.34}$$

and

$$\begin{cases} A_1 = \dfrac{b^2}{2(b^2 - a^2)(\bar{\lambda} + 2\bar{\mu})}\left(1 + \dfrac{\bar{\mu}\varrho^2}{b^2(\bar{\lambda} + \bar{\mu})}\right), \\ B_1 = \dfrac{a^2}{2(b^2 - a^2)(\bar{\lambda} + 2\bar{\mu})}\left(1 + \dfrac{\bar{\mu}\varrho^2}{a^2(\bar{\lambda} + \bar{\mu})}\right), \\ A_2 = \dfrac{a^2}{\bar{\mu}}(\bar{\lambda} + \bar{\mu})A_1, \quad B_2 = \dfrac{b^2}{\bar{\mu}}(\bar{\lambda} + \bar{\mu})B_1. \end{cases} \tag{3.35}$$

Inserting $\overline{\boldsymbol{M}}_r$ from (3.34) into (3.30), we obtain:

$$\bar{u}_r^*(\varrho, p) = \frac{\bar{\gamma}(p)}{\varrho}\left[A_1\int_a^\varrho \bar{T}(r, p)\,r dr + B_1\int_\varrho^b \bar{T}(r, p)\,r dr\right]. \tag{3.36}$$

The transform of the displacement for a cylindrical cavity in an infinite space is of particular simplicity ($a \neq 0$, $b = \infty$). For, in this case $B_1 = 0$ and hence:

$$\bar{u}_r^* (\varrho, p) = \frac{\vartheta (p)}{\vartheta_0 p} \int_a^\varrho \bar{T} (r, p) \, r dr, \quad \vartheta (p) = \frac{3 \bar{\lambda} + 2 \bar{\mu}}{\bar{\lambda} + 2 \bar{\mu}} \alpha_t. \tag{3.37}$$

Observe that for $\varrho = a$ is $\bar{u}_r^* (a, p) = 0$, and therefore also $u_r^* (a, t) = 0$. It is evident that in this particular case the formula (3.13) is true, namely

$$u_r^* (\varrho, p) = \frac{\vartheta (p)}{\vartheta_0} \bar{u}_r^{0*} (\varrho, p), \tag{3.38}$$

where $\bar{u}_r^{0*} (\varrho, p)$ is the radial displacement on the surface of the cylinder of the radius ϱ, in a perfectly elastic body.

For the Kelvin material:

$$\vartheta (p) = \vartheta_0 \left(1 - \frac{p}{p + \varkappa_1}\right), \qquad \varkappa_1 = \frac{3}{4} \frac{\mu_0 + 2 \lambda_0}{\mu_0 \varepsilon}.$$

The displacement is expressed by the formula:

$$u_r^* (\varrho, t) = \int_0^t (1 - e^{-\varkappa_1 (t - \tau)}) \frac{\partial}{\partial \tau} u^{0*} (\varrho, \tau) \, d\tau. \tag{3.39}$$

For the Maxwell model of a viscoelastic body

$$P_1(p) = p + \varepsilon_*^{-1}, \quad P_2(p) = 2 \mu_0 p, \quad P_3(p) = p, \quad P_4(p) = (3 \lambda_0 + 2 \mu_0) p, \tag{3.40}$$

where ε_*^{-1} is the time of relaxation. We have in this case:

$$\vartheta (p) = \vartheta_0 \frac{p + \varepsilon^{-1}}{p + \varkappa_2}, \qquad \vartheta_0 = \frac{3 \lambda_0 + 2 \mu_0}{\lambda_0 + 2 \mu_0} \alpha_t, \qquad \varkappa_2 = \frac{3 \lambda_0 + 2 \mu_0}{3 (\lambda_0 + 2 \mu_0)}.$$

In view of (3.38):

$$u^* (\varrho, t) = \int_0^t [1 - (\varkappa_2 - \varepsilon_*^{-1}) \, e^{-\varkappa_2 (t - \tau)}] \frac{\partial}{\partial \tau} u_r^{0*} (\varrho, \tau) \, d\tau. \tag{3.41}$$

Finally, for the Biot model of a viscoelastic body, in which

$$\bar{\lambda} (p) = \lambda_0 \frac{p}{p + \varepsilon}, \quad \bar{\mu} (p) = \mu_0 \frac{p}{p + \varepsilon}, \quad \vartheta (p) = \vartheta_0, \tag{3.42}$$

we arrive at the relation:

$$u_r^* (\varrho, t) = u_r^{0*} (\varrho, t). \tag{3.43}$$

The stresses can be determined from the formulae:

$$\bar{\sigma}_{rr}^* = \bar{\lambda} \bar{e}^* + 2 \bar{\mu} \frac{\partial \bar{u}_r^*}{\partial \varrho} - \bar{\gamma} \bar{T}, \quad \bar{\sigma}_{\varphi\varphi}^* = \bar{\lambda} \bar{e}^* + 2 \bar{\mu} \frac{\bar{u}_r^*}{\varrho} - \bar{\gamma} \bar{T}.$$

For the Biot model:

$$(3.44)\quad \begin{cases} \sigma^{*}_{rr}(\varrho,t)=\int\limits_0^t e^{-\varepsilon(t-\tau)}\left[\lambda_0\dfrac{\partial e^{0*}(\varrho,\tau)}{\partial\tau}+2\mu_0\dfrac{\partial^2 u^{0*}(\varrho,\tau)}{\partial\tau\,\partial\varrho}-\gamma_0\dfrac{\partial T(\varrho,\tau)}{\partial\tau}\right]d\tau,\\[2ex] \sigma^{*}_{\varphi\varphi}(\varrho,t)=\int\limits_0^t e^{-\varepsilon(t-\tau)}\left[\lambda_0\dfrac{\partial e^{0*}(\varrho,\tau)}{\partial\tau}+\dfrac{2\mu_0}{\varrho}\dfrac{\partial u^{0*}(\varrho,\tau)}{\partial\tau}-\gamma_0\dfrac{\partial T(\varrho,\tau)}{\partial\tau}\right]d\tau. \end{cases}$$

The functions e^{0*}, u^{0*} correspond to a perfectly elastic body. It remains to determine the function $u^{0*}(\varrho,t)$ from

$$(3.45)\qquad \bar{u}^{0*}(\varrho,p)=\vartheta_0\int\limits_0^{\varrho}\bar{T}(r,p)\,r\,dr.$$

Let the surface $r=a$ be heated suddenly from the zero temperature to T_0. In this case the transform of the function T has the form [11]:

$$\bar{T}(r,p)=\frac{T_0}{p}\,\frac{K_0(r\sqrt{q})}{K_0(a\sqrt{q})},\qquad q=p/\varkappa.$$

Substituting (3.46) in (3.45) and integrating, we obtain:

$$(3.46)\qquad \bar{u}^{0*}(\varrho,p)=\frac{T_0\vartheta_0\sqrt{\varkappa}}{\varrho\sqrt{p}\,pK_0(a\sqrt{p/\varkappa})}\left[aK_1(a\sqrt{p/\varkappa})-\varrho K_1(\varrho\sqrt{p/\varkappa})\right].$$

Performing the reverse transform we arrive at the relation

$$(3.47)\qquad u^{0*}(\varrho,t)=T_0\vartheta_0\sqrt{\varkappa}\left[\frac{a}{\varrho}\,\Phi(a,t)-\Phi(\varrho,t)\right],$$

where

$$\Phi(a,t)=\int\limits_0^t\frac{1}{\sqrt{\pi(t-\tau)}}\,\varphi(a,\tau)\,d\tau,\qquad \Phi(\varrho,t)=\int\limits_0^t\frac{1}{\sqrt{\pi(t-\tau)}}\,\varphi(\varrho,\tau)\,d\tau,$$

and the function $\varphi(a,t)$, $\varphi(\varrho,t)$ are given by:

$$\varphi(a,t)=\frac{2}{\pi}\int\limits_0^\infty e^{-\varkappa\xi^2 t}\,\frac{J_1(a\xi)\,Y_0(a\xi)-Y_1(a\xi)\,J_0(a\xi)}{J_0^2(a\xi)+Y_0^2(a\xi)}\,\frac{d\xi}{\xi},$$

$$\varphi(\varrho,t)=\frac{2}{\pi}\int\limits_0^\infty e^{-\varkappa\xi^2 t}\,\frac{J_1(\varrho\xi)\,Y_0(a\xi)-Y_1(\varrho\xi)\,J_0(a\xi)}{J_0^2(a\xi)+Y_0^2(a\xi)}\,\frac{d\xi}{\xi}.$$

3.3. Hollow sphere. Let there be a spherically symmetrical temperature field in a hollow sphere of inner radius a and the outer radius b. Denoting by $\bar{M}_R(R,\varrho;p)$ the dilatation on the surface of the sphere of the radius R due to the action of radial unit forces uniformly distributed on the sphere of the diameter ϱ, we obtain the formula for $\bar{u}_R^*$:

$$\bar{u}_R^*(\varrho, p) = \frac{\bar{\gamma}(p)}{\varrho^2} \int_a^b \bar{T}(R, p)\, \bar{\boldsymbol{M}}_R(R, \varrho, p)\, R^2\, dR. \tag{3.48}$$

The displacement $\bar{U}_R(R, \varrho, p)$ is determined by solving the differential equation

$$(\bar{\lambda} + 2\bar{\mu}) \frac{d}{dR}\left[\frac{1}{R^2} \frac{d}{dR}(R^2 \bar{U}_R)\right] + \delta(R - \varrho) = 0, \tag{3.49}$$

with the boundary conditions

$$\bar{\Sigma}_R(R, p) = 0 \quad \text{for} \quad R = 0 \quad \text{and} \quad R = b. \tag{3.50}$$

The equation (3.50) may be replaced by two homogeneous equations

$$\begin{cases} \dfrac{d}{dR}\left[\dfrac{1}{R^2} \dfrac{d}{dR}(R^2 \bar{U}_R')\right] = 0, & a < R < \varrho, \\ \dfrac{d}{dR}\left[\dfrac{1}{R^2} \dfrac{d}{dR}(R^2 \bar{U}_R'')\right] = 0, & \varrho < R < b, \end{cases} \tag{3.51}$$

possessing the solutions:

$$\begin{cases} \bar{U}_R' = A_1 R + A_2 R^{-2}, & a < R < \varrho, \\ \bar{U}_R'' = B_1 R + B_2 R^{-2}, & \varrho < R < b. \end{cases} \tag{3.52}$$

The constants A_1, A_2, B_1, B_2 can be determined from the following boundary conditions:

$$\begin{cases} \bar{U}_R'(\varrho, \varrho, p) = \bar{U}_R''(\varrho, \varrho, p), \qquad \bar{\Sigma}_R'(a, \varrho, p) = 0, \qquad \bar{\Sigma}_R''(b, \varrho, p) = 0, \\ \bar{\Sigma}_R'(\varrho, \varrho, p) - \bar{\Sigma}_R''(\varrho, \varrho, p) = 1. \end{cases} \tag{3.53}$$

Taking into account that

$$\bar{\Sigma}_R = \bar{\lambda}\, \bar{\boldsymbol{M}}_R + 2\bar{\mu} \frac{\partial \bar{U}_R}{\partial R}, \qquad \bar{\boldsymbol{M}}_R = \frac{\partial \bar{U}_R}{\partial R} + \frac{2}{R} \bar{U}_R, \tag{3.54}$$

the constants from (3.53) are given by:

$$\begin{cases} A_1 = \dfrac{b^3}{3(b^3 - a^3)(\bar{\lambda} + 2\bar{\mu})}\left(1 + \dfrac{2\mu}{3\bar{\lambda} + \bar{\mu}} \dfrac{\varrho^3}{b^3}\right), \\ B_1 = \dfrac{a^3}{3(b^3 - a^3)(\bar{\lambda} + 2\bar{\mu})}\left(1 + \dfrac{2\bar{\mu}}{3\bar{\lambda} + \bar{\mu}} \dfrac{\varrho^3}{a^3}\right), \\ A_2 = \dfrac{a^3}{2\bar{\mu}}(3\bar{\lambda} + \bar{\mu})\, A_1, \\ B_2 = \dfrac{b^3}{2\bar{\mu}}(3\bar{\lambda} + \bar{\mu})\, B_1. \end{cases} \tag{3.55}$$

Since

$$\begin{cases} \bar{M}_R(R, \varrho, p) = 3A_1 & \text{for} \quad a < R < \varrho, \\ \bar{M}_R(R, \varrho, p) = 3B_1 & \text{for} \quad \varrho < R < b, \end{cases} \tag{3.56}$$

we have:

$$u_R^*(\varrho, p) = \frac{3\gamma}{\varrho^2}\left[A_1 \int_a^\varrho \bar{T}(R, p) R^2\, dR + B_1 \int_\varrho^b \bar{T}(R, p) R^2\, dR\right], \tag{3.57}$$

or

$$\bar{u}_R^*(\varrho, p) = \frac{\vartheta(p)}{\varrho^2}\, \frac{b^3}{b^3 - a^3}\left\{\int_a^\varrho \bar{T}(R, p) R^2\, dR + \frac{a^3}{b^3}\int_\varrho^b \bar{T}(R, p) R^2\, dR + \right.$$
$$\left. + \frac{2\bar{\mu}}{3\bar{\lambda} + \bar{\mu}}\, \frac{\varrho^3}{b^3}\int_a^b \bar{T}(R, p) R^2\, dR\right\}. \tag{3.58}$$

In the particular case $b \to \infty$ (i.e. in an infinite space with a spherical cavity) considered by E. Sternberg, [6], we have:

$$\bar{u}_R^*(\varrho, p) = \frac{\vartheta(p)}{\varrho^2}\int_a^\varrho \bar{T}(R_1 p) R^2\, dR. \tag{3.59}$$

Only in this particular case $\bar{u}_R^* = 0$ for $R = a$. For a solid sphere $(a = 0)$, considered by a different method by M. Sokołowski, [12], we obtain:

$$u_R^*(\varrho, p) = \frac{\vartheta(p)}{\varrho^2}\left[\int_0^\varrho \bar{T}(R, p) R^2\, dR + \frac{2\bar{\mu}}{3\bar{\lambda} + \bar{\mu}}\, \frac{\varrho^3}{b^3}\int_0^b \bar{T}(R, p) R^2\, dR\right]. \tag{3.60}$$

In the case of the Biot model of a viscoelastic body:

$$\vartheta(p) = \vartheta_0 = \text{const}, \qquad \frac{2\bar{\mu}}{3\bar{\lambda} + \bar{\mu}} = \frac{2\mu_0}{3\lambda_0 + \mu_0} = \text{const}.$$

Therefore the displacements $\bar{u}_R^*(\varrho, p)$ for the Biot model are identical with the displacements $\bar{u}^{0*}(\varrho, p)$ for a perfectly elastic body:

$$\bar{u}_R^*(\varrho, p) = \bar{u}_R^{0*}(\varrho, p). \tag{3.61}$$

This result is true for a spherical cavity as well as for a hollow or solid sphere.

The stresses are determined from the formulae

$$\bar{\sigma}_{RR}^* = \bar{\lambda}\bar{e}^* + 2\bar{\mu}\bar{\varepsilon}_{RR}^*, \qquad \bar{\sigma}_{\varphi\varphi}^* = \bar{\sigma}_{\vartheta\vartheta}^* = \bar{\lambda}\bar{e}^* + 2\bar{\mu}\bar{\varepsilon}_{\varphi\varphi}^*, \tag{3.62}$$

where

$$\bar{\varepsilon}_{RR}^* = \frac{\partial \bar{u}_R^*}{\partial \varrho}, \qquad \bar{\varepsilon}_{\varphi\varphi}^* = \frac{\bar{u}_R^*}{\varrho}, \qquad \bar{e}^* = \frac{\partial \bar{u}_R^*}{\partial \varrho} + \frac{2}{\varrho}\bar{u}_R^*.$$

Since for the Biot model

$$\bar{\mu}(\mathrm{p}) = \mu_0 \frac{\mathrm{p}}{\mathrm{p}+\varepsilon}, \qquad \bar{\lambda}(\mathrm{p}) = \lambda_0 \frac{\mathrm{p}}{\mathrm{p}+\varepsilon},$$

we obtain:

$$\tag{3.63} \begin{cases} \sigma^*_{RR}(\varrho, t) = \int\limits_0^t e^{-\varepsilon(t-\tau)} \dfrac{\partial}{\partial \tau} \sigma^{0*}_{RR}(R, \tau)\, d\tau, \\ \sigma^*_{\varphi\varphi}(\varrho, t) = \int\limits_0^t e^{-\varepsilon(t-\tau)} \dfrac{\partial}{\partial \tau} \sigma^{0*}_{\varphi\varphi}(R, \tau)\, d\tau, \end{cases}$$

where σ^{0*}_{RR}, $\sigma^{0*}_{\varphi\varphi}$, $\sigma^{0*}_{\vartheta\vartheta}$ denote stresses due to the action of the temperature field in a perfectly elastic body.

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Streszczenie

NIEUSTALONE NAPRĘŻENIA TERMICZNE W OŚRODKU LEPKO-SPRĘŻYSTYM (I)

Naprężenia występujące w ciałach doskonale sprężystych wywołane działaniem pola temperatury wyznaczyć można na dwu drogach; bądź przez bezpośrednie rozwiązanie równań różniczkowych, bądź też przez rozwiązanie równoważnego mu układu równań całkowych.

Druga z tych metod, opierająca się na wykorzystaniu twierdzenia o wzajemności przemieszczeń, obmyślona została i z powodzeniem stosowana przez W. M. Majziela.

Przedmiotem niniejszej pracy jest rozszerzenie metody W. M. Majziela na ciała lepko-sprężyste. W pracy przedstawiono dwuetapowy sposób rozwiązywania zagadnień dynamicznych, wywołanych nieustalonym polem temperatury w ciałach lepko-sprężystych. W pierwszym etapie uzyskuje się rozwiązanie zagadnienia quasi-statycznego, w drugim uwzględnia się wpływ sił inercyjnych. W niniejszej pierwszej części pracy omawia się w sposób szczegółowy zagadnienie quasi-statyczne. Sposób postępowania ilustrują trzy proste przykłady.

Резюме

НЕСТАЦИОНАРНЫЕ ТЕРМИЧЕСКИЕ НАПРЯЖЕНИЯ
В ВЯЗКО-УПРУГОЙ СРЕДЕ (I)

Напряжения в идеально упругих телах, вызванные действием температурного поля могут быть определены двумя методами — или непосредственным решением дифференциальных уравнений, или-же решением эквивалентной системы интегральных уравнений.

Второй метод, использующий существенным образом теорему о взаимности перемещений, был разработан и успешно применен В. М. Майзелем.

Целью настоящей работы является расширение метода Майзеля на вязко-упругие тела. В работе дан двухэтапный метод решения динамических проблем погрешности и перемещений, вызванных в вязко-упругих телах нестационарным температурным полем. В первом этапе решается квази-статическая проблема, во втором учитывается влияние инерционных сил. В первой части работы рассматривается подробно квази-статическая проблема. Метод иллюстрируется тремя простыми примерами.

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