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TWO STEADY-STATE THERMOELASTIC PROBLEMS

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The object of this paper is to determine, in an elastic space and semi-space, the state of stress due to a temperature field. Two problems will be considered, the difference between them being that of thermal boundary conditions in the $z = 0$ plane. The solution obtained for an elastic space will serve for the construction of a more complex problem — that of determining the state of stress in an elastic semi-space free from stress in the $z = 0$ plane.

1. Consider an infinite elastic space in which heat is generated over the surface of the circle of radius a in the $z = 0$ plane, the heat flow being constant and equal to Q. As a result a temperature and stress field is established. The temperature field is described by the differential equation

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} = 0, \tag{1.1}$$

with the boundary conditions

$$\begin{cases} -2\lambda\left[\dfrac{\partial T}{\partial z}\right]_{z=0} = Q & \text{for} \quad 0 < r < a, \\ \left[\dfrac{\partial T}{\partial z}\right]_{z=0} = 0, & \text{for} \quad r > a \end{cases} \tag{1.2}$$

and $T = 0$ at infinity; λ denotes the coefficient of heat conduction.

The differential equation (1.1), and the boundary conditions, are satisfied by the function, [1],

$$T = \frac{Qa}{2\lambda}\int_0^\infty \alpha^{-1} e^{-\alpha|z|} J_1(\alpha a) J_0(\alpha r)\, d\alpha. \tag{1.3}$$

For $z = 0$ we have

$$-2\lambda\left[\frac{\partial T}{\partial z}\right]_{z=0} = Qa\int_0^\infty J_1(\alpha a) J_0(\alpha r)\, d\alpha = \begin{cases} Q & \text{for} \quad 0 < r < a, \\ 0 & \text{for} \quad r > a. \end{cases} \tag{1.4}$$

To determine the state of stress $(\bar{\sigma}_{ij})$, we shall use the potential of thermoelastic displacement Φ, [2]. It is related to the displacements u, v, w by the equations

$$u = \frac{\partial \Phi}{\partial x}, \qquad v = \frac{\partial \Phi}{\partial y}, \qquad w = \frac{\partial \Phi}{\partial z}. \tag{1.5}$$

Introducing the Eqs. (1.5) into the three displacement equations of the theory of elasticity, we can reduce them to the unique equation, [2],

$$\nabla^2 \Phi = \vartheta_0 T, \qquad \vartheta_0 = \frac{1+\nu}{1-\nu} a_t, \tag{1.6}$$

where ν is Poisson's ratio and a_t is the coefficient of thermal dilatation. The knowledge of the function Φ enables the determination of the stress components $(\bar{\sigma}_{ij})$ from the equations

$$\bar{\sigma}_{ij} = 2G\left(\frac{\partial^2 \Phi}{\partial i \partial j} - \nabla^2 \Phi \delta_{ij}\right), \tag{1.7}$$

where δ_{ij} is Kronecker's delta.

Bearing in mind that

$$e^{-\alpha|z|} = \frac{2}{\pi} \alpha \int\limits_0^\infty \frac{\cos \gamma z \, d\gamma}{\alpha^2 + \gamma^2},$$

we represent the function (1.3) in the form

$$T = \frac{Qa}{\pi\lambda} \int\limits_0^\infty \int\limits_0^\infty (\alpha^2 + \gamma^2)^{-1} J_1(\alpha a) J_0(\alpha r) \cos \gamma z \, d\alpha \, d\gamma. \tag{1.8}$$

A particular integral of the Eq. (1.6) is

$$\Phi = -\frac{Qa\vartheta_0}{\pi\lambda} \int\limits_0^\infty \int\limits_0^\infty (\alpha^2 + \gamma^2)^{-2} J_1(\alpha a) J_0(\alpha r) \cos \gamma z \, d\alpha \, d\gamma, \tag{1.9}$$

or

$$\Phi = -\frac{Qa\vartheta_0}{4\lambda} \int\limits_0^\infty \frac{(1 + \alpha z) e^{-\alpha|z|}}{\alpha^3} J_0(\alpha r) J_1(\alpha a) \, d\alpha. \tag{1.10}$$

Using the Eq. (1.7), we determine the components of the state of stress $(\bar{\sigma}_{ij})$

$$\begin{cases} \bar{\sigma}_{rr} = 2G\left(\dfrac{\partial^2 \Phi}{\partial r^2} - \nabla^2 \Phi\right) = \\ \qquad = A \displaystyle\int\limits_0^\infty e^{-\alpha|z|} \alpha^{-1} J_1(\alpha a) \left[(\alpha z - 1) J_0(\alpha r) - (1 + \alpha z) \frac{J_1(\alpha r)}{\alpha r}\right] d\alpha, \end{cases}$$

$$
(1.11)\quad \begin{cases}
\bar{\sigma}_{\varphi\varphi} = 2\,G\left(\dfrac{1}{r}\,\dfrac{\partial\,\Phi}{\partial\,r} - \nabla^2\Phi\right) = \\
\qquad = A\displaystyle\int_0^\infty e^{-\alpha|z|}\,\alpha^{-1}\,J_1(\alpha\,a)\left[(1+\alpha\,z)\,\frac{J_1(\alpha r)}{\alpha\,r} - 2\,J_0(\alpha\,r)\right]d\,\alpha, \\
\bar{\sigma}_{zz} = 2\,G\left(\dfrac{\partial^2\,\Phi}{\partial\,z^2} - \nabla^2\Phi\right) = -A\displaystyle\int_0^\infty e^{-\alpha|z|}\,\alpha^{-1}(1+\alpha z)\,J_1(\alpha a)\,J_0(\alpha r)\,d\,\alpha, \\
\bar{\sigma}_{rz} = 2\,G\,\dfrac{\partial^2\Phi}{\partial r\,\partial z} = -A z\displaystyle\int_0^\infty e^{-\alpha|z|}\,J_1(\alpha\,a)\,J_1(\alpha\,r)\,d\,\alpha, \\
\bar{\sigma}_{r\varphi} = 0,
\end{cases}
$$

where

$$
A = \frac{G\,Q\,a\,\vartheta_0}{2\,\lambda}.
$$

After integration, we obtain

$$
(1.12)\quad \begin{cases}
\bar{\sigma}_{rr} = A\displaystyle\sum_{m=0}^\infty \frac{\Gamma(1+2\,m)}{m!\,\Gamma(2+m)}\left(\frac{-a^2}{4\,z^2}\right)^m \times \\
\qquad \times\left\{[(2\,m+1)\,z+1]\;{}_2F_1\left(-m,-1-m;1;\dfrac{r^2}{a^2}\right) - \right. \\
\qquad \left. -\dfrac{1}{r}\,[1+(2\,m+1)\,z]\;{}_2F_1\left(-m,-1-m;2;\dfrac{r^2}{a^2}\right)\right\}, \\
\bar{\sigma}_{\varphi\varphi} = -A\displaystyle\sum_{m=0}^\infty \frac{\Gamma(1+2\,m)}{m!\,\Gamma(2+m)}\left(\frac{-a^2}{4\,z^2}\right)^m \left\{{}_2F_1\left(-m,-1-m;1;\frac{r^2}{a^2}\right)2 - \right. \\
\qquad \left. -\dfrac{1}{r}\,[z(1+2\,m)+1]\;{}_2F_1\left(-m,-1-m;2;\dfrac{r^2}{a^2}\right)\right\}, \\
\sigma_{zz} = -A\displaystyle\sum_{m=0}^\infty \frac{\Gamma(1+2\,m)}{m!\,\Gamma(2+m)}\left(\frac{-a^2}{4\,z^2}\right)^m \times \\
\qquad \times\left\{[1+(1+2\,m)\,z]\;{}_2F_1\left(-m,-1-m;1;\dfrac{r^2}{a^2}\right)\right\}, \\
\bar{\sigma}_{rz} = -A z\displaystyle\sum_{m=0}^\infty \frac{\Gamma(3+2\,m)}{m!\,\Gamma(2+m)}\left(\frac{-a^2}{4\,z^2}\right)^m\;{}_2F_1\left(-m,-1-m;2;\frac{r^2}{a^2}\right),
\end{cases}
$$

where

$$
{}_2F_1(a_1,a_2;\gamma_1;\eta) = \sum_{k=0}^\infty \frac{(a_1)_k\,(a_2)_k\,\eta^k}{(\gamma_1)_k\,k!}
$$

is a Gaussian hypergeometric series.

The solution of this fundamental problem enables us to solve the more complicated case of stress in an elastic semi-space. The boundary conditions in the $z=0$ plane (bounding the semi-space) are (1.2), and it is assumed in addition that there are no stresses in that plane.

In order to solve this problem such a state of stress $(\bar{\bar{\sigma}}_{ij})$ should be superposed over $(\bar{\sigma}_{ij})$, that the following conditions are satisfied in the $z=0$ plane:

$$\bar{\sigma}_{rz}+\bar{\bar{\sigma}}_{rz}=0, \qquad \bar{\sigma}_{zz}+\bar{\bar{\sigma}}_{zz}=0. \tag{1.13}$$

To determine the state of stress $(\bar{\bar{\sigma}}_{ij})$, Love's function φ, [3], will be used. This function satisfies the biharmonic equation

$$\nabla^2\nabla^2\varphi=0. \tag{1.14}$$

The stress components $(\bar{\bar{\sigma}}_{ij})$ are related to the function φ by the equations

$$\left\{\begin{aligned} \bar{\bar{\sigma}}_{rr}&=\frac{2G}{1-2\nu}\frac{\partial}{\partial z}\left(\nu\nabla^2\varphi-\frac{\partial^2\varphi}{\partial r^2}\right), & \bar{\bar{\sigma}}_{\varphi\varphi}&=\frac{2G}{1-2\nu}\frac{\partial}{\partial z}\left(\nu\nabla^2\varphi-\frac{1}{r}\frac{\partial\varphi}{\partial r}\right),\\ \bar{\bar{\sigma}}_{zz}&=\frac{2G}{1-2\nu}\frac{\partial}{\partial z}\left[(2-\nu)\nabla^2\varphi-\frac{\partial^2\varphi}{\partial z^2}\right], & \bar{\bar{\sigma}}_{rz}&=\frac{2G}{1-2\nu}\frac{\partial}{\partial r}\left[(1-\nu)\nabla^2\varphi-\frac{\partial^2\varphi}{\partial z^2}\right]. \end{aligned}\right. \tag{1.15}$$

The function φ will be assumed in the form

$$\varphi=\int_0^\infty Z(a,z)\,J_0(ar)\,da, \tag{1.16}$$

where $Z(a,z)=(C+azD)\,e^{-az}$.

In view of the first of the conditions (1.13), we have $C=2\nu D$. Introducing the function φ in the Eqs. (1.15), we obtain

$$\left\{\begin{aligned} \bar{\bar{\sigma}}_{rr}&=\frac{2G}{1-2\nu}\int_0^\infty D(a)\,a^3e^{-az}\left[(1-az)J_0(ar)+(2\nu-1+az)\frac{J_1(ar)}{ar}\right]da,\\ \bar{\bar{\sigma}}_{\varphi\varphi}&=\frac{2G}{1-2\nu}\int_0^\infty D(a)\,a^3e^{-az}\left[2\nu J_0(ar)-(2\nu-1+az)\frac{J_1(ar)}{ar}\right]da,\\ \bar{\bar{\sigma}}_{zz}&=\frac{2G}{1-2\nu}\int_0^\infty D(a)\,a^3e^{-az}(1+az)\,J_0(ar)\,da,\\ \bar{\bar{\sigma}}_{rz}&=\frac{2G}{1-2\nu}z\int_0^\infty D(a)\,a^4e^{-az}J_1(ar)\,da. \end{aligned}\right. \tag{1.17}$$

From the second of the boundary conditions (1.13), we find that

$$D(\alpha)=\frac{A}{2G}\,\frac{J_1(\alpha a)}{\alpha^4}(1-2\nu). \tag{1.18}$$

Substituting $D(\alpha)$ in the Eqs. (1.17), and bearing in mind that the resultant stress is obtained by adding $(\bar{\sigma}_{ij})$ and $(\bar{\bar{\sigma}}_{ij})$, we obtain finally

$$\left\{\begin{aligned}
\sigma_{rr}&=-\frac{GQa}{\lambda r}(1+\nu)\,\alpha_t\int\limits_0^\infty e^{-\alpha z}\alpha^{-2}J_1(\alpha a)\,J_1(\alpha r)\,d\alpha=\\
&=-\frac{GQa}{r\lambda}(1+\nu)\,\alpha_t\sum_{m=0}^{\infty}\frac{\Gamma(1+2m)}{m!\,\Gamma(2+m)}\left(\frac{-a^2}{4z^2}\right)^m{}_2F_1\left(-m,-1-m;2;\frac{r^2}{a^2}\right),\\
\sigma_{\varphi\varphi}&=-\frac{GQa}{\lambda}(1+\nu)\,\alpha_t\int\limits_0^\infty e^{-\alpha z}\alpha^{-1}J_1(\alpha a)\left[J_0(\alpha r)-\frac{J_1(\alpha r)}{\alpha r}\right]d\alpha=\\
&=-\frac{GQa}{\lambda}(1+\nu)\,\alpha_t\sum_{m=0}^{\infty}\frac{\Gamma(1+2m)}{m!\,\Gamma(2+m)}\left(\frac{-a^2}{4z^2}\right)^m\times\\
&\quad\times\left[{}_2F_1\left(-m,-1-m;1;\frac{r^2}{a^2}\right)-\frac{1}{r}\,{}_2F_1\left(-m,-1-m;2;\frac{r^2}{a^2}\right)\right],\\
\sigma_{zz}&=0,\qquad \sigma_{rz}=0.
\end{aligned}\right. \tag{1.19}$$

Observe that in the case of an elastic semi-space the stresses σ_{zz} and σ_{rz} vanish.

For $z=0$ we have

$$\left\{\begin{aligned}
\sigma_{rr}(r,0)&=-\frac{GQa}{r\lambda\pi}(1+\nu)\,\alpha_t(r+a)\left[E\left(\frac{2i\sqrt{ra}}{|r-a|}\right)-K\left(\frac{2i\sqrt{ra}}{|r-a|}\right)\right],\\
\sigma_{\varphi\varphi}(r,0)&=-\frac{2GQa}{\pi\lambda}(1+\nu)\,\alpha_t\left\{E\left(\frac{r}{a}\right)-\frac{r+a}{2r}\left[E\left(\frac{2i\sqrt{ra}}{|r-a|}\right)-K\left(\frac{2i\sqrt{ra}}{|r-a|}\right)\right]\right\}\\
&\qquad\text{for}\quad 0<r<a,\\
\sigma_{\varphi\varphi}(r,0)&=-\frac{2GQa}{\pi\lambda}(1+\nu)\,\alpha_t\left\{\frac{r}{a}\left[K\left(\frac{a}{r}\right)-\left(1-\frac{a^2}{r^2}\right)E\left(\frac{a}{r}\right)\right]-\right.\\
&\qquad\left.-\frac{r+a}{2r}\left[E\left(\frac{2i\sqrt{ra}}{|r-a|}\right)-K\left(\frac{2i\sqrt{ra}}{|r-a|}\right)\right]\right\}\quad\text{for}\quad r>a,
\end{aligned}\right. \tag{1.20}$$

where

$$E(\eta)=\int\limits_0^{\pi/2}(1-\eta^2\sin^2\Phi)^{1/2}\,d\Phi=\frac{\pi}{2}\,{}_2F_1\left(-\frac{1}{2},\frac{1}{2};1;\eta^2\right),$$

$$K(\eta)=\int\limits_0^{\pi/2}(1-\eta^2\sin^2\Phi)^{-1/2}\,d\Phi=\frac{\pi}{2}\,{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\eta^2\right)$$

are complete elliptic integrals.

It will be shown below that the stresses σ_{zz} and σ_{rz} are equal to zero also in the case in which the condition $-2\lambda(\partial T/\partial z)=Q$ is satisfied over any region Γ in the $z=0$ plane, and we have $\partial T/\partial z=0$ outside this region.

For this purpose, let us solve the auxiliary problem of determining the Green's function for heat conduction, with the boundary conditions

$$-2\lambda\left[\frac{\partial G}{\partial z}\right]_{z=0}=\delta(x-\xi)\,\delta(y-\eta) \tag{1.21}$$

and $G=0$ at infinity; δ is the symbol of the Dirac function.

The solution of the heat equation

$$\nabla^2 G=0, \tag{1.22}$$

with the boundary conditions (1.21) is

$$\left\{\begin{aligned} G&=\frac{1}{2\lambda\pi^2}\int_0^\infty\int_0^\infty\frac{e^{-\vartheta z}}{\vartheta}\cos\alpha(x-\xi)\cos\beta(y-\eta)\,d\alpha\,d\beta,\\ \vartheta&=(\alpha^2+\beta^2)^{1/2},\end{aligned}\right. \tag{1.23}$$

or

$$G=\frac{1}{4\pi\lambda R},$$

where

$$R=[(x-\xi)^2+(y-\eta)^2+z^2]^{1/2}.$$

It can easily be verified that the particular integral of the equation

$$\nabla^2\Phi^*=\vartheta_0 G \tag{1.24}$$

is

$$\Phi^*=-\frac{\vartheta_0}{4\lambda\pi^2}\int_0^\infty\int_0^\infty\frac{e^{-\vartheta z}}{\vartheta^3}(1+\vartheta z)\cos\alpha(x-\xi)\cos\beta(y-\eta)\,d\alpha\,d\beta. \tag{1.25}$$

Using the Eqs. (1.7), the stress components (σ_{ij}^*) will be found:

(1.26)

$$\left\{\begin{aligned} \bar\sigma_{xx}^*&=-K_0\int_0^\infty\int_0^\infty\frac{e^{-\vartheta z}}{\vartheta^3}[(1+\vartheta z)\beta^2+(1-\vartheta z)\vartheta^2]\cos\alpha(x-\xi)\cos\beta(y-\eta)\,d\alpha\,d\beta,\\ \bar\sigma_{yy}^*&=-K_0\int_0^\infty\int_0^\infty\frac{e^{-\vartheta z}}{\vartheta^3}[(1+\vartheta z)\alpha^2+(1-\vartheta z)\vartheta^2]\cos\alpha(x-\xi)\cos\beta(y-\eta)\,d\alpha\,d\beta,\end{aligned}\right.$$

$$\left\{\begin{aligned} \bar{\sigma}^*_{zz} &= -K_0 \int_0^\infty \int_0^\infty \frac{e^{-\vartheta z}}{\vartheta} (1+\vartheta z) \cos\alpha(x-\xi) \cos\beta(y-\eta)\, d\alpha\, d\beta, \\ \bar{\sigma}^*_{xy} &= -K_0 \int_0^\infty \int_0^\infty \frac{e^{-\vartheta z}}{\vartheta^3} (1+\vartheta z)\, \alpha\beta \sin\alpha(x-\xi) \sin\beta(y-\eta)\, d\alpha\, d\beta, \\ \bar{\sigma}^*_{xz} &= -K_0 z \int_0^\infty \int_0^\infty \frac{e^{-\vartheta z}}{\vartheta}\, \alpha \sin\alpha(x-\xi) \cos\beta(y-\eta)\, d\alpha\, d\beta, \\ \bar{\sigma}^*_{yz} &= -K_0 z \int_0^\infty \int_0^\infty \frac{e^{-\vartheta z}}{\vartheta}\, \beta \cos\alpha(x-\xi) \sin\beta(y-\eta)\, d\alpha\, d\beta, \\ K_0 &= \frac{G\vartheta_0}{2\lambda\pi^2}. \end{aligned}\right.$$

Not all boundary conditions are satisfied by the state of stress $(\bar{\sigma}^*_{ij})$, however. It is true that for $z=0$ we have $\bar{\sigma}^*_{xz}=0$, $\bar{\sigma}^*_{yz}=0$, but $\bar{\sigma}^*_{zz}\neq 0$. A state of stress $(\bar{\bar{\sigma}}^*_{ij})$ should be superposed over $(\bar{\sigma}^*_{ij})$ such that

$$\bar{\bar{\sigma}}^*_{xz}=0, \qquad \bar{\bar{\sigma}}^*_{yz}=0, \qquad \bar{\sigma}^*_{zz}+\bar{\bar{\sigma}}^*_{zz}=0 \tag{1.27}$$

in the $z=0$ plane.

To determine the stress components $(\bar{\bar{\sigma}}^*_{ij})$, B. G. Galerkin's displacement function χ, [4], will be used. It reduces the displacement equations of the theory of elasticity (assuming an isothermal state) to a single biharmonic equation

$$\nabla^2\nabla^2\chi=0, \tag{1.28}$$

where the stress components $(\bar{\bar{\sigma}}^*_{ij})$ are expressed by the relations

$$\left\{\begin{aligned} \bar{\bar{\sigma}}^*_{xx} &= \frac{\partial}{\partial z}\left(\nu\nabla^2\chi-\frac{\partial^2\chi}{\partial x^2}\right), & \bar{\bar{\sigma}}^*_{xz} &= \frac{\partial}{\partial x}\left(\frac{\partial^2\chi}{\partial x^2}+\frac{\partial^2\chi}{\partial y^2}-\nu\nabla^2\chi\right), \\ \bar{\bar{\sigma}}^*_{yy} &= \frac{\partial}{\partial z}\left(\nu\nabla^2\chi-\frac{\partial^2\chi}{\partial y^2}\right), & \bar{\bar{\sigma}}^*_{yz} &= \frac{\partial}{\partial y}\left(\frac{\partial^2\chi}{\partial x^2}+\frac{\partial^2\chi}{\partial y^2}-\nu\nabla^2\chi\right), \\ \bar{\bar{\sigma}}^*_{zz} &= \frac{\partial}{\partial z}\left[(1-\nu)\nabla^2\chi+\frac{\partial^2\chi}{\partial x^2}+\frac{\partial^2\chi}{\partial y^2}\right], & \bar{\bar{\sigma}}^*_{xy} &= -\frac{\partial^3\chi}{\partial x\,\partial y\,\partial z}. \end{aligned}\right. \tag{1.29}$$

The function χ will be assumed in the form of a Fourier integral

$$\chi=\int_0^\infty Z(\alpha,\beta,z)\cos\alpha(x-\xi)\cos\beta(y-\eta)\,d\alpha\,d\beta, \tag{1.30}$$

where

$$Z=(A+B\vartheta z)\,e^{-\vartheta z}$$

and A and B are functions of the parameters α and β. The first two of the boundary conditions (1.27) lead to the relation $A = 2\nu B$. From the third, we obtain $B = K_0/\vartheta^4$.

Using the Eqs. (1.29) we determine the stress components $(\bar{\bar{\sigma}}{}_{ij}^*)$. Adding these to the stress components $(\bar{\sigma}_{ij}^*)$, we obtain finally the stress components (σ_{ij}^*).

We find that

$$
(1.31)\quad \left\{
\begin{aligned}
\sigma_{xx}^* &= -2K_0(1-\nu)\int_0^\infty\int_0^\infty \frac{e^{-\vartheta z}}{\vartheta^3}\alpha^2\cos\alpha(x-\xi)\cos\beta(y-\eta)\,d\alpha\,d\beta = \\
&= -\frac{E\alpha_t}{2\pi\lambda}\frac{1}{r^2(R+z)}\left[(x-\xi)^2+\frac{z}{R}(y-\eta)^2\right],\\
\sigma_{yy}^* &= -2K_0(1-\nu)\int_0^\infty\int_0^\infty \frac{e^{-\vartheta z}}{\vartheta^3}\beta^2\cos\alpha(x-\xi)\cos\beta(y-\eta)\,d\alpha\,d\beta = \\
&= -\frac{E\alpha_t}{2\pi\lambda}\frac{1}{r^2(R+z)}\left[(y-\eta)^2+\frac{z}{R}(x-\xi)^2\right],\\
\sigma_{xy}^* &= -2K_0(1-\nu)\int_0^\infty\int_0^\infty \frac{e^{-\vartheta z}}{\vartheta^3}\alpha\beta\sin\alpha(x-\xi)\sin\beta(y-\eta)\,d\alpha\,d\beta = \\
&= -\frac{E\alpha_t}{2\pi\lambda}\frac{(x-\xi)(y-\eta)}{R(R^2+z)^2},\\
\sigma_{zz}^* &= 0, \qquad \sigma_{xz}^* = 0, \qquad \sigma_{yz}^* = 0,
\end{aligned}
\right.
$$

where

$$
r = [(x-\xi)^2+(x-\eta)^2]^{1/2}, \qquad E = 2G(1+\nu)
$$

If the following conditions are valid for a region Γ in the $z = 0$ plane:

$$
(1.32)\quad \left\{
\begin{aligned}
-2\lambda\left[\frac{\partial T}{\partial z}\right]_{z=0} &= Q(x,y) \text{ over the region } \Gamma,\\
\left[\frac{\partial T}{\partial z}\right]_{z=0} &= 0 \text{ outside the region } \Gamma,
\end{aligned}
\right.
$$

then the temperature field is determined by the integral

$$
(1.33)\qquad T(x,y,z) = \iint_{(\Gamma)} Q(\xi,\eta)\,G(x,y,z;\xi,\eta,0)\,d\xi\,d\eta,
$$

and the stress components are

$$
(1.34)\qquad \sigma_{ij}(x,y,z) = \iint_{(\Gamma)} Q(\xi,\eta)\,\sigma_{ij}^*(x,y,z;\xi,\eta,0)\,d\xi\,d\eta, \qquad i,j = x,y,z.
$$

It is seen that in view of the Eqs. (1.31), the stresses σ_{xz}, σ_{yz} and σ_{zz} vanish at every point of the elastic semi-space.

2. Let the temperature V over a circle of radius a in the $z=0$ plane be constant. The temperature field is described by the differential equation

$$\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}+\frac{\partial^2 T}{\partial z^2}=0, \tag{2.1}$$

with the boundary conditions

$$\begin{cases} T=V & \text{for} \quad z=0 \quad \text{and} \quad 0<r>a, \\ \dfrac{\partial T}{\partial z}=0 & \text{for} \quad z=0 \quad \text{and} \quad r>a, \\ T=0 & \text{at infinity.} \end{cases} \tag{2.2}$$

The differential equation (2.1) together with the conditions (2.2) is satisfied by the function

$$T=\frac{2V}{\pi}\int_0^\infty e^{-\alpha|z|}\,\alpha^{-1}\sin\alpha a\,J_0(\alpha r)\,d\alpha. \tag{2.3}$$

For $z=0$, we find that

$$\begin{cases} T(r,0)=\dfrac{2V}{\pi}\displaystyle\int_0^\infty \alpha^{-1}\sin\alpha a\,J_0(\alpha r)\,d\alpha=\begin{cases} V & \text{for } 0<r<a. \\ \dfrac{2V}{\pi}\arcsin\dfrac{a}{r} & \text{for } r>a, \end{cases} \\ \left[\dfrac{\partial T}{\partial z}\right]_{z=0}=-\dfrac{2V}{\pi}\displaystyle\int_0^\infty \sin\alpha a\,J_0(\alpha r)\,d\alpha=\begin{cases} -\dfrac{2V}{\pi}(a^2-r^2)^{-1/2} & \text{for } 0<r<a, \\ 0 & \text{for } r>a. \end{cases} \end{cases} \tag{2.4}$$

From the Eq. (1.6), we find the particular integral

$$\Phi=-\frac{V\vartheta_0}{\pi^2}\int_0^\infty\int_0^\infty\frac{\sin\alpha a\,J_0(\alpha r)\cos\gamma z\,d\alpha\,d\gamma}{(\alpha^2+\gamma^2)}= \tag{2.5}$$

$$=-\frac{V\vartheta_0}{\pi}\int_0^\infty\frac{(1+\alpha z)\,e^{-\alpha|z|}}{\alpha^3}\sin\alpha a\,J_0(\alpha r)\,d\alpha.$$

The knowledge of the function Φ enables us to determine the stress components $(\bar{\sigma}_{ij})$. Using the Eqs. (1.7) we find that

$$
(2.6)\qquad \left\{\begin{aligned}
\bar{\sigma}_{rr} &= N\int_0^\infty e^{-\alpha|z|}\alpha^{-1}\sin\alpha a\left[(\alpha z-1)J_0(\alpha r)-(1+\alpha z)\frac{J_1(\alpha r)}{\alpha r}\right]d\alpha,\\
\bar{\sigma}_{\varphi\varphi} &= N\int_0^\infty e^{-\alpha|z|}\alpha^{-1}\sin\alpha a\left[(\alpha z+1)\frac{J_1(\alpha r)}{\alpha r}-2J_0(\alpha r)\right]d\alpha,\\
\bar{\sigma}_{rz} &= -Nz\int_0^\infty e^{-\alpha|z|}\sin\alpha a\,J_1(\alpha r)\,d\alpha,\\
\bar{\sigma}_{zz} &= -N\int_0^\infty e^{-\alpha|z|}\alpha^{-1}(1+\alpha z)\sin\alpha a\,J_0(\alpha r)\,d\alpha,
\end{aligned}\right.
$$

where $N=2\,GV\vartheta_0/\pi$.

These equations determine the stress components due to the action of a temperature field in an infinite elastic space.

Consider now an elastic semi-space, for which the thermal boundary conditions (2.2) are satisfied in the $z=0$ plane, and for which it is assumed that $\sigma_{zz}=0$ and $\sigma_{rz}=0$ in that plane. The state of stress (σ_{ij}) will be obtained by superposing over the state $(\bar{\sigma}_{ij})$, the components of which are expressed by the Eqs. (2.6), the state of stress $(\bar{\bar{\sigma}}_{ij})$ so chosen that the following boundary conditions are satisfied in the $z=0$ plane:

$$
(2.7)\qquad [\bar{\bar{\sigma}}_{rz}]_{z=0}=0,\qquad [\bar{\sigma}_{zz}+\bar{\bar{\sigma}}_{zz}]_{z=0}=0.
$$

Using the Eqs. (1.17) we find, from the second of the conditions (2.7), the quantity $D(\alpha)$

$$
(2.8)\qquad D(\alpha)=\frac{V\vartheta_0}{\pi}(1-2\nu)\frac{\sin\alpha a}{\alpha^4}.
$$

Substituting this in the Eqs. (1.17), we obtain the stress components $(\bar{\bar{\sigma}}_{ij})$ and — adding the stress components $(\bar{\sigma}_{ij})$ from the Eqs. (2.6) — the stress components (σ_{ij}). We find that, [5],

$$
\left|\begin{aligned}
\sigma_{rr} &= -\frac{4VG}{\pi r}(1+\nu)\,\alpha_t\int_0^\infty\frac{e^{-\alpha z}\sin\alpha a}{\alpha^2}J_1(\alpha r)\,d\alpha=\\
&= -\frac{4VG}{\pi r^2}(1+\nu)\,\alpha_t\int_0^r\eta^{1/2}\arcsin\left[\frac{2a}{\sqrt{z^2+(a+\eta)^2}+\sqrt{z^2+(a-\eta)^2}}\right]d\eta,
\end{aligned}\right.
$$

$$(2.9)\left\{\begin{aligned}\sigma_{\varphi\varphi} &= -\frac{4VG}{\pi}(1+\nu)\alpha_t\int\limits_0^\infty e^{-\alpha z}\alpha^{-1}\sin\alpha a\left[J_0(\alpha r)-\frac{J_1(\alpha r)}{\alpha r}\right]d\alpha = \\ &= -\frac{4VG}{\pi}(1+\nu)\alpha_t\left\{\arcsin\left[\frac{2a}{\sqrt{z^2+(a+r)^2}+\sqrt{z^2+(a-r)^2}}\right]-\right. \\ &\qquad \left.-\frac{1}{r^2}\int\limits_0^r \eta^{1/2}\arcsin\left[\frac{2a}{\sqrt{z^2+(a+\eta)^2}+\sqrt{z^2+(a-\eta)^2}}\right]d\eta\right\}, \\ \sigma_{r\varphi} &= 0.\end{aligned}\right.$$

Also in this complicated case with discontinuous thermal boundary conditions in the $z=0$ plane, the stresses σ_{rz} and σ_{zz} vanish.

Consider finally a case with the following boundary conditions in the $z = 0$ plane bounding the elastic semi-space considered:

$$(2.10)\quad \left\{\begin{array}{lll} T = V(x,y) & \text{for} \quad z=0 & \text{over any region } \Gamma, \\ \dfrac{\partial T}{\partial z} = 0 & \text{for} \quad z=0 & \text{outside the region } \Gamma, \\ T = 0 & & \text{at infinity.} \end{array}\right.$$

The solution of the heat equation (2.1) with the boundary conditions (2.10) will be sought in the form of an integral equation, [6],

$$(2.11)\qquad T(x,y,z) = -2\lambda\iint\limits_{(\Gamma)}\Psi(\xi,\eta)\,G(x,y,z;\xi,\eta,0)\,d\xi\,d\eta,$$

where the G r e e n' s function G is expressed by the Eq. (1.23). $\Psi(\xi,\eta)$ denotes the temperature gradient $\partial T/\partial z$ in the region Γ. The function $\Psi(\xi,\eta)$ will be assumed as unknown, and will be found by using the first of the conditions (2.10). For $z=0$, we obtain from the Eq. (2.11)

$$(2.12)\qquad V(x,y) = -2\lambda\iint\limits_{(\Gamma)}\Psi(\xi,\eta)\,G(x,y,0;\xi,\eta,0)\,d\xi\,d\eta,$$

or

$$-2\pi V(x,y) = \iint\limits_{(\Gamma)}\Psi(\xi,\eta)\left[(x-\xi)^2+(y-\eta)^2\right]^{-1/2}d\xi\,d\eta.$$

Solving the integral equation (2.12), which is F r e d h o l m' s equation of the first type, we find the unknown function $\Psi(\xi,\eta)$. The knowledge of this function enables us to determine, on the basis of the Eq. (2.11), the temperature field T. The stress components (σ_{ij}) will be found from the equations

$$(2.13)\qquad \sigma_{ij}(x,y,z) = -2\lambda\iint\limits_{(\Gamma)}\Psi(\xi,\eta)\,\sigma^*_{ij}(x,y,z;\xi,\eta,0)\,d\xi\,d\eta,$$

where the Green's function for the stresses σ^*_{ij} will be taken from the Eqs. (1.31). The stress components σ^*_{zz}, σ^*_{xz} and σ^*_{yz} being equal to zero, the stress components σ_{zz}, σ_{xz} ane σ_{yz} will also be equal to zero.

It has been proved in both of the above problems of thermoelasticity concerning the state of stress in an elastic semi-space, that for given thermal boundary conditions, (1.32) and (2.10), the stresses σ_{zz}, σ_{zx} and σ_{zy} vanish.

This is also valid in the case of an elastic plate, of thickness h, extending to infinity both in the x- and y-directions, with the boundary conditions (1.32) or (2.10) in one or both faces of the plate.

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Streszczenie

O DWU USTALONYCH ZAGADNIENIACH TERMOSPRĘŻYSTOŚCI

Pierwsze zagadnienie dotyczy wyznaczenia stanu naprężenia wywołanego w przestrzeni nieograniczonej oraz w półprzestrzeni sprężystej działaniem pola temperatury określonego równaniem (1.1) wraz z warunkami brzegowymi (1.2).

Rozwiązanie zagadnienia następuje przy użyciu funkcji potencjału sprężystego Φ. Całka szczególna (1.10) równania spełnia warunki brzegowe dla nieograniczonej przestrzeni sprężystej, tak że składowe stanu naprężenia $(\bar{\sigma}_{ij})$ otrzymuje się ze wzorów (1.7).

Dla półprzestrzeni sprężystej należy do stanu naprężenia $(\bar{\sigma}_{ij})$ dodać stan naprężenia $(\bar{\bar{\sigma}}_{ij})$ tak dobrany, aby spełnił równania Love'a (1.14) wraz z warunkami brzegowymi (1.13), charakteryzującymi brak naprężeń w płaszczyźnie $z = 0$. Okazuje się, że w przypadku półprzestrzeni sprężystej równe zeru są naprężenia σ_{zz}, σ_{rz}. Przy użyciu funkcji Gree-

na wykazano, że naprężenia $\sigma_{xz}, \sigma_{yz}, \sigma_{zz}$ są równe zeru również w przypadku ogólniejszym, mianowicie gdy warunki (1.32) odnoszą się do dowolnego obszaru Γ leżącego w płaszczyźnie $z=0$.

W drugiej części pracy wyznaczono stan naprężenia w przestrzeni i półprzestrzeni sprężystej, wywołany działaniem pola temperatury, określonego równaniem przewodnictwa cieplnego (2.1) oraz nieciągłymi warunkami brzegowymi termicznymi (2.2) w płaszczyźnie $z=0$. W przypadku szczególnym półprzestrzeni wykazano dla warunków brzegowych (2.2) oraz ogólniej sformułowanych (2.10), że naprężenia $\sigma_{zz}, \sigma_{xz}, \sigma_{yz}$ są równe zeru.

Powyższe twierdzenia są słuszne w przypadku słoja sprężystego (płyty o grubości h rozciągającej się nieograniczenie w kierunku osi x i y) o warunkach brzegowych (1.32) lub (2.10) występujących w jednej lub w obu płaszczyznach ograniczających słój.

Резюме

О ДВУХ СТАЦИОНАРНЫХ ЗАДАЧАХ ТЕРМОУПРУГОСТИ

Первая задача касается определения напряженного состояния, вызванного в бесконечном пространстве, а также в упругом полупространстве, действием температурного поля определенного уравнением (1.1) и краевыми условиями (1.2).

Задача решается при использовании функции упругого потенциала Φ. Частный интеграл (1.10) уравнения удовлетворяет краевым условиям для бесконечного упругого пространства так, что составляющие напряженного состояния $(\bar{\sigma}_{ij})$ получаются из формул (1.7).

Для упругого пространства следует сложить напряженное состояние $(\bar{\sigma}_{ij})$ с так подобранным напряженным состоянием $(\bar{\bar{\sigma}}_{ij})$, чтобы они удовлетворяли уравнению Лява (1.14), вместе с краевыми условиями (1.13), характеризующими отсутствие напряжений в плоскости $z=0$. Оказывается, что в случае упругого полупространства σ_{zz} и σ_{rz} равняются нулю. Доказывается, что при использовании функции Грина, напряжения σ_{xz}, σ_{yz} и σ_{zz} равняются нулю также в более общем случае, а именно когда условия (1.32) касаются произвольной области Γ плоскости $z=0$.

Во второй части работы определяется напряженное состояние в упругом пространстве и упругом полупространстве, вызванное действием температурного поля, выраженного уравнением теплопроводности (2.1), а также разрывными термическими краевыми условиями (2.2) в плоскости $z=0$. В частном случае полупространства, доказано

для краевых условий (2.2), а также сформулированных более общим образом условий (2.10), что напряжения σ_{zz}, σ_{xz} и σ_{yz} равняются нулю.

Приведенные выше утверждения справедливы в случае упругого слоя (пластинка толщиной h, распространяющаяся неограниченно по направлению осей x и y), с краевыми условиями (1.32) или (2.10) на одной или в обеих плоскостях, ограничивающих слой.

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