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PAŃSTWOWE WYDAWNICTWO NAUKOWE

A DYNAMICAL PROBLEM OF THERMOELASTICITY

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1. An Instantaneous Source of Heat

Let an instantaneous source of heat act in an infinite elastic space of initial temperature $T = 0$. The action of this source will result in a temperature and stress field in the elastic space. We assume that thermal and elastic properties of the medium are constants, independent of the coordinates and the temperature. The action of the instantaneous source will provoke dynamical effects. In the displacement equations of the theory of elasticity the inertia terms will therefore be taken into consideration. The problem thus stated is characterized by spherical symmetry, the temperature and stress being dependent on the distance R from the source and time t.

The heat equation has, in the case under consideration, the form

$$\left\{\begin{array}{l} \nabla^2 T - \dfrac{1}{k}\dfrac{\partial T}{\partial t} = -\dfrac{W}{k}\delta(R)\,\delta(t)\,, \qquad \nabla^2 = \dfrac{\partial^2}{\partial R^2} + \dfrac{2}{R}\dfrac{\partial}{\partial R}, \\ [T(R,t)]_{t=0} = 0\,, \end{array}\right. \tag{1.1}$$

where δ denotes the Dirac function; $k = \lambda_0/\varrho c$, where λ_0 is the coefficient of heat conduction, ϱ density and c specific heat. $W = Q/\varrho c$, where Q denotes the heat quantity emitted by the heat source per unit time. The solution of the equation is, [1],

$$T = \frac{W}{(\pi\vartheta)^{3/2}}\, e^{-R^2/\vartheta}\,, \tag{1.2}$$

where

$$R = (x^2+y^2+z^2)^{1/2} \qquad \text{and} \qquad \vartheta = 4kt.$$

The displacement equations of the theory of elasticity have the form, [2]:

$$\left\{\begin{array}{l} (\lambda+\mu)\dfrac{\partial\Theta}{\partial x} + \mu\nabla^2 u - \beta\dfrac{\partial^2 u}{\partial t^2} = (3\lambda+2\mu)\,a_t\,\dfrac{\partial T}{\partial x}, \\ (\lambda+\mu)\dfrac{\partial\Theta}{\partial y} + \mu\nabla^2 v - \beta\dfrac{\partial^2 v}{\partial t^2} = (3\lambda+2\mu)\,a_t\,\dfrac{\partial T}{\partial y}, \\ (\lambda+\mu)\dfrac{\partial\Theta}{\partial z} + \mu\nabla^2 w - \beta\dfrac{\partial^2 w}{\partial t^2} = (3\lambda+2\mu)\,a_t\,\dfrac{\partial T}{\partial z}, \end{array}\right. \tag{1.3}$$

where u, v, w are the displacement components and

$$\Theta = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$

the dilatation. The quantities $\mu = G, \lambda = 2G\nu/1-2\nu$ are Lamé's constants, where ν is Poisson's ratio and G the shear modulus. Finally, β denotes the mass per unit volume and a_t the coefficient of thermal expansion.

Introducing the potential of thermoelastic displacements Φ, where

$$u = \frac{\partial \Phi}{\partial x}, \qquad v = \frac{\partial \Phi}{\partial y}, \qquad w = \frac{\partial \Phi}{\partial z}, \tag{1.4}$$

which is equivalent to the assumption of an irrotational stress field, we reduce the system of equations (1.3) to the unique equation

$$\nabla^2 \Phi - \sigma^2 \frac{\partial^2 \Phi}{\partial t^2} = \vartheta_0 T, \tag{1.5}$$

where

$$\sigma^2 = \frac{1}{c^2}, \qquad c = \left(\frac{1-\nu}{1-2\nu} \, \frac{2G}{\beta} \right)^{1/2}$$

being the velocity of the dilatation wave, and

$$\vartheta_0 = \frac{1+\nu}{1-\nu} a_t .$$

The knowledge of the function Φ enables us to determine the stress components from the Eqs., [3]:

$$\sigma_{ij} = 2G \left[\frac{\partial^2 \Phi}{\partial i \, \partial j} + \left(\frac{\nu}{1-2\nu} \nabla^2 \Phi - \frac{1+\nu}{1-2\nu} a_t T \right) \delta_{ij} \right], \tag{1.6.1}$$

or, bearing in mind the Eq. (1.5),

$$\sigma_{ij} = 2G \left(\frac{\partial^2 \Phi}{\partial i \, \partial j} - \nabla^2 \Phi \delta_{ij} \right) + \beta \frac{\partial^2 \Phi}{\partial t^2} \delta_{ij}, \tag{1.6.2}$$

where δ_{ij} is Kronecker's delta.

In the present case of an instantaneous concentrated source of heat, the system of equations (1.3) is reduced, in view of the spherical symmetry of the temperature and stress field, to the unique equation

$$\frac{\partial}{\partial R} \left[\frac{1}{R^2} \frac{\partial}{\partial R} (R^2 u_R) \right] - \sigma^2 \frac{\partial^2 u_R}{\partial t^2} = \vartheta_0 \frac{\partial T}{\partial R}, \tag{1.7}$$

with the initial conditions

$$u_R = 0\,, \qquad \frac{\partial u_R}{\partial t} = 0 \quad \text{for} \quad t = 0\,.$$

In the Eq. (1.7), u_R denotes the displacement in the direction of the radius R. Introducing the potential of thermoelastic displacement Φ, where

$$u_R = \frac{\partial \Phi}{\partial R} \tag{1.8}$$

the Eq. (1.7) can be expressed in the form

$$\nabla^2 \Phi - \sigma^2 \frac{\partial^2 \Phi}{\partial t^2} = \vartheta_0 T\,, \quad \text{where} \quad \nabla^2 = \frac{\partial^2}{\partial R^2} + \frac{2}{R}\frac{\partial}{\partial R}. \tag{1.9}$$

Applying to the Eqs. (1.1) and (1.9) the Laplace transformation,

$$L(T) = T^* = \int_0^\infty e^{-pt}\, T(R,t)\, dt\,, \qquad L(\Phi) = \Phi^* = \int_0^\infty e^{-pt}\, \Phi(R,t)\, dt\,, \tag{1.10}$$

and taking the initial conditions into consideration, we have

$$\nabla^2 T^* - \frac{p}{k} T^* = -\frac{W}{k}\delta(R)\,, \qquad \nabla^2 \Phi^* - \sigma^2 p^2 \Phi^* = \vartheta_0 T^*\,. \tag{1.11}$$

The transform

$$T^* = \frac{W}{4\pi k}\frac{1}{R} e^{-R(p/k)^{1/2}} \tag{1.12}$$

may be expressed by means of the Fourier-Hankel integral as follows:

$$T^* = \frac{W}{4\pi k}\int_0^\infty a J_0(ar)\left(a^2 + \frac{p}{k}\right)^{-1/2} \exp\left[-z\left(a^2+\frac{p}{k}\right)^{1/2}\right] da = \tag{1.13}$$

$$= \frac{W}{2\pi^2 k}\int_0^\infty\int_0^\infty a J_0(ar)\left(a^2+\gamma^2+\frac{p}{k}\right)^{-1} \cos\gamma z\, da\, d\gamma\,,$$

$$r = (x^2+y^2)^{1/2}.$$

Assuming the function Φ^* in the form

$$\Phi^* = \int_0^\infty\int_0^\infty C(a,\gamma,p)\, J_0(ar)\cos\gamma z\, da\, d\gamma\,, \tag{1.14}$$

we obtain from the second equation of the group (1.11)

$$(1.15)\quad \Phi^{*}=-\frac{\vartheta_0 W}{2\pi^2 k}\int_0^{\infty}\int_0^{\infty}\alpha J_0(\alpha r)\left(\alpha^2+\gamma^2+\frac{p}{k}\right)^{-1}(\alpha^2+\gamma^2+p\sigma^2)^{-1}\times$$

$$\times\cos\gamma z\,d\alpha\,d\gamma=\frac{\vartheta_0 W}{4\pi k\sigma^2}\,p^{-1}\left(p-\frac{1}{k\sigma^2}\right)^{-1}R^{-1}\left(e^{-R\sigma p}-e^{-R(p/k)^{1/2}}\right).$$

Performing the inverse transformation, we obtain:

$$(1.16.1)\quad \Phi=KR^{-1}\left\{\operatorname{Erfc}\left(\frac{R}{\sqrt{\vartheta}}\right)-\frac{1}{2}e^{\vartheta/4k^2\sigma^2}\left[e^{R/\sigma k}\operatorname{Erfc}\left(\frac{R}{\sqrt{\vartheta}}-\frac{\sqrt{\vartheta}}{2k\sigma}\right)+\right.\right.$$

$$\left.\left.+e^{-R/k\sigma}\operatorname{Erfc}\left(\frac{R}{\sqrt{\vartheta}}+\frac{\sqrt{\vartheta}}{2k\sigma}\right)\right]\right\},$$

for $0<t<R\sigma$ or $0<\vartheta<4Rk\sigma$.

$$(1.16.2)\quad \Phi=KR^{-1}\left\{e^{\frac{1}{4k^2\sigma^2}(\vartheta-4Rk\sigma)}-1+\operatorname{Erfc}\left(\frac{R}{\sqrt{\vartheta}}\right)-\right.$$

$$\left.-\frac{1}{2}e^{\vartheta/4k^2\sigma^2}\left[e^{R/k\sigma}\operatorname{Erfc}\left(\frac{R}{\sqrt{\vartheta}}-\frac{\sqrt{\vartheta}}{2k\sigma}\right)+e^{-R/k\sigma}\operatorname{Erfc}\left(\frac{R}{\sqrt{\vartheta}}+\frac{\sqrt{\vartheta}}{2k\sigma}\right)\right]\right\},$$

for $\vartheta>4Rk\sigma$, where

$$K=\frac{W\vartheta_0}{4\pi},\qquad \vartheta=4kt.$$

The knowledge of the function Φ enables us to determine the stress components (σ_{ij}) from the Eq. (1.6).

In spherical coordinates, we have

$$(1.17)\quad \begin{cases}\sigma_{RR}=-2G\dfrac{2}{R}\dfrac{\partial\Phi}{\partial R}+\beta\dfrac{\partial^2\Phi}{\partial t^2},\\[2ex] \sigma_{\varphi\varphi}=\sigma_{\vartheta\vartheta}=2G\left(\dfrac{1}{R}\dfrac{\partial\Phi}{\partial R}+\dfrac{\partial^2\Phi}{\partial R^2}\right)+\beta\dfrac{\partial^2\Phi}{\partial t^2},\\[2ex] \sigma_{R\varphi}=0,\qquad \sigma_{\varphi\vartheta}=0,\qquad \sigma_{R\vartheta}=0.\end{cases}$$

Introducing the notations

$$\sigma_{RR}=\begin{cases}\sigma'_{RR} & \text{for}\quad 0<\vartheta<4Rk\sigma,\\ \sigma''_{RR} & \text{for}\quad \vartheta>4Rk\sigma,\end{cases}$$

and

$$\sigma_{\varphi\varphi}=\begin{cases}\sigma'_{\varphi\varphi} & \text{for}\quad 0<\vartheta<4Rk\sigma,\\ \sigma''_{\varphi\varphi} & \text{for}\quad \vartheta>4Rk\sigma,\end{cases}$$

we obtain therefore the following equations for stresses:

$$
(1.18)\quad
\begin{cases}
\sigma'_{RR} = 4\,GKR^{-3}\Big\{\operatorname{Erfc}\left(\dfrac{R}{\sqrt{\vartheta}}\right) - \\
\qquad -\dfrac{1}{2}\,e^{\vartheta/4k^2\sigma^2}\left[\left(1-\dfrac{R}{k\sigma}-\dfrac{R^2\eta}{k^2\sigma^2}\right)e^{R/k\sigma}\operatorname{Erfc}\left(\dfrac{R}{\sqrt{\vartheta}}-\dfrac{\sqrt{\vartheta}}{2k\sigma}\right)+\right. \\
\qquad \left. +\left(1+\dfrac{R}{k\sigma}-\dfrac{R^2\eta}{k^2\sigma^2}\right)e^{-R/k\sigma}\operatorname{Erfc}\left(\dfrac{R}{\sqrt{\vartheta}}-\dfrac{\sqrt{\vartheta}}{2k\sigma}\right)\right]+\dfrac{4R^3\eta\,e^{-R^2/\vartheta}}{\sqrt{\pi\vartheta^3}}\Big\}, \\
\sigma''_{RR} = 4\,GKR^{-3}\left[\left(1+\dfrac{R}{k\sigma}+\dfrac{\eta R^2}{k^2\sigma^2}\right)e^{\frac{1}{4k^2\sigma^2}(\vartheta-4Rk\sigma)}-1\right]+\sigma'_{RR}, \\
\sigma'_{\varphi\varphi} = \sigma'_{\vartheta\vartheta} = -2\,GKR^{-3}\Big\{\operatorname{Erfc}\left(\dfrac{R}{\sqrt{\vartheta}}\right)-\dfrac{2R^2}{k\sigma\sqrt{\pi\vartheta}}\,e^{-R^2/\vartheta} - \\
\qquad -\dfrac{e^{\vartheta/4\sigma^2k^2}}{2}\left[\left(1-\dfrac{R}{k\sigma}-\dfrac{R^2}{k^2\sigma^2}(1-2\eta)\right)e^{R/k\sigma}\operatorname{Erfc}\left(\dfrac{R}{\sqrt{\vartheta}}-\dfrac{\sqrt{\vartheta}}{2k\sigma}\right)+\right. \\
\qquad \left.+\left(1+\dfrac{R}{k\sigma}-\dfrac{R^2}{k^2\sigma^2}(1-2\eta)\right)e^{-R/k\sigma}\operatorname{Erfc}\left(\dfrac{R}{\sqrt{\vartheta}}+\dfrac{\sqrt{\vartheta}}{2k\sigma}\right)\right]-\dfrac{8R^3\eta}{\sqrt{\pi\vartheta^3}}\,e^{-R^2/\vartheta}\Big\}, \\
\sigma''_{\varphi\varphi} = \sigma''_{\vartheta\vartheta} = -2\,GKR^{-3}\Big\{\Big[1+\dfrac{R}{k\sigma}+ \\
\qquad +\dfrac{R^2}{k^2\sigma^2}(1-2\eta)\Big]e^{(1/4\sigma^2k^2)(\vartheta-4Rk\sigma)}-1\Big\}+\sigma'_{\varphi\varphi}, \\
\sigma_{\varphi\vartheta}=0, \qquad \sigma_{R\vartheta}=0, \qquad \sigma_{R\varphi}=0,
\end{cases}
$$

where $\eta = \beta/4G\sigma^2$.

2. A Continuous Source of Heat

Let a continuous source of heat of constant intensity W act in an infinite elastic space of initial temperature $T = 0$ starting from the time $t = 0$.

Applying the Laplace transformation to the heat equation, we obtain

$$
(2.1)\qquad \nabla^2 T^* - \frac{p}{k}T^* = -\frac{W}{k}\,\delta(R)\,p^{-1}.
$$

The solution of this equation is

$$
(2.2)\quad T^* = \frac{W}{4\pi kR}p^{-1}e^{-R(p/k)^{1/2}} =
$$

$$
= \frac{W}{2\pi^2 k}p^{-1}\int_0^\infty\int_0^\infty \alpha J_0(\alpha r)(\alpha^2+\gamma^2+p/k)^{-1}\cos\gamma z\,d\alpha\,d\gamma.
$$

Performing the inverse transformation, we find

$$T = \frac{W}{4\pi k R} \operatorname{Erfc}\left(\frac{R}{\sqrt{\vartheta}}\right). \tag{2.3}$$

The solution of the second of the Eqs. (1.11) is

$$\Phi^* = \frac{W \vartheta_0}{4\pi k \sigma^2} p^{-2} \left(p - \frac{1}{k\sigma^3}\right)^{-1} R^{-1} \left[e^{-R\sigma p} - e^{-R(p/k)^{1/2}}\right]. \tag{2.4}$$

Performing the inverse transformation, we have

$$\tag{2.5} \left\{ \begin{aligned} \Phi = &-KR^{-1}\left\{k\sigma^2\left[1 - \frac{1}{4k^2\sigma^2}(\vartheta + 2R^2)\right]\operatorname{Erfc}\frac{R}{\sqrt{\vartheta}} - \right. \\ &-\frac{k\sigma^2}{2} e^{\vartheta/4k^2\sigma^2}\left[e^{Rk\sigma}\operatorname{Erfc}\left(\frac{R}{\sqrt{\vartheta}} + \frac{\sqrt{\vartheta}}{2k\sigma}\right) + e^{-Rk\sigma}\operatorname{Erfc}\left(\frac{R}{\sqrt{\vartheta}} - \frac{\sqrt{\vartheta}}{2k\sigma}\right)\right] + \\ &\left. + \frac{R\sqrt{\vartheta}}{2k\sqrt{\pi}} e^{-R^2/\vartheta}\right\} \text{ for } 0 < \vartheta < 4Rk\sigma; \\ \Phi = &-KR^{-1}\left\{\frac{1}{4k}(\vartheta - 4Rk\sigma) + k\sigma\left[e^{\frac{1}{4k^2\sigma^2}(\vartheta - 4Rk\sigma)} - 1\right] + k\sigma^2\left[1 - \right.\right. \\ &\left. - \frac{1}{4k^2\sigma^2}(\vartheta + 2R^2)\right]\operatorname{Erfc}\left(\frac{R}{\sqrt{\vartheta}}\right) - \frac{k\sigma^2}{2} e^{\vartheta/4k^2\sigma^2}\left[e^{Rk\sigma}\operatorname{Erfc}\left(\frac{R}{\sqrt{\vartheta}} + \frac{\sqrt{\vartheta}}{2k\sigma}\right) + \right. \\ &\left.\left. + e^{-Rk\sigma}\operatorname{Erfc}\left(\frac{R}{\sqrt{\vartheta}} - \frac{\sqrt{\vartheta}}{2k\sigma}\right)\right] + \frac{R\sqrt{\vartheta}}{2k\sqrt{\pi}} e^{-R^2/\vartheta}\right\} \text{ for } \vartheta > 4Rk\sigma. \end{aligned} \right.$$

From the Eqs. (1.17) we obtain

$$\tag{2.6} \left\{ \begin{aligned} \sigma'_{RR} = &-4KGR^{-3}\left\{k\sigma^2\left[1 + \frac{1}{4k^2\sigma^2}(\vartheta - 2R^2)\right]\operatorname{Erfc}\frac{R}{\sqrt{\vartheta}} + \right. \\ &+ \frac{R}{2k}\sqrt{\frac{\vartheta'}{\pi}} e^{-R^2/\vartheta} - \frac{k\sigma^2}{2} e^{\vartheta/4k^2\sigma^2}\left[\left(1 - \frac{R}{k\sigma} + \frac{R^2\eta}{k^2\sigma^2}\right)e^{Rk\sigma}\operatorname{Erfc}\times \right. \\ &\left.\left. \times\left(\frac{R}{\sqrt{\vartheta}} - \frac{\sqrt{\vartheta}}{2k\sigma}\right) + \left(1 + \frac{R}{k\sigma} + \frac{R^2\eta}{k^2\sigma^2}\right)e^{-Rk\sigma}\operatorname{Erfc}\left(\frac{R}{\sqrt{\vartheta}} + \frac{\sqrt{\vartheta}}{2k\sigma}\right)\right]\right\}, \\ \sigma''_{RR} = &-4GKR^{-3}k\sigma^2\left[\left(1 + \frac{R}{k\sigma} - \frac{R^2\eta}{k^2\sigma^2}\right)e^{\frac{1}{4k^2\sigma^2}(\vartheta - 4Rk\sigma)} - \right. \\ &\left. - 1 - \frac{\vartheta}{4k^2\sigma^2}\right] + \sigma'_{RR}, \end{aligned} \right.$$

$$\left|\begin{array}{l}\sigma'_{\varphi\varphi}=\sigma'_{\vartheta\vartheta}=-2\,GKR^{-3}\left\{\dfrac{k\,\sigma^2}{2}\,e^{\vartheta/4k^2\sigma^2}\left[\left(1-\dfrac{R}{k\,\sigma}-\right.\right.\right.\\ \qquad\qquad \left.-\dfrac{R^2}{k^2\,\sigma^2}\,(1+2\,\eta)\right)e^{R/k\,\sigma}\,\mathrm{Erfc}\left(\dfrac{R}{\sqrt{\vartheta}}-\dfrac{\sqrt{\vartheta}}{2\,k\,\sigma}\right)+\\ \qquad\qquad \left.+\left(1+\dfrac{R}{k\,\sigma}-\dfrac{R^2}{k^2\,\sigma^2}\,(1+2\,\eta)\right)e^{-R/k\,\sigma}\mathrm{Erfc}\left(\dfrac{R}{\sqrt{\vartheta}}+\dfrac{\sqrt{\vartheta}}{2\,k\,\sigma}\right)\right]+\\ \qquad \left.+\dfrac{R\,e^{-R^2/\vartheta}}{2\,k\,\sqrt{\pi\vartheta}}\,(4\,R\,k\,\sigma-\vartheta)-k\,\sigma^2\left[1+\dfrac{1}{4\,k^2\,\sigma^2}\,(\vartheta+2\,R^2)\right]\mathrm{Erfc}\left(\dfrac{R}{\sqrt{\vartheta}}\right)\right\},\\ \sigma''_{\varphi\varphi}=2\,GKR^{-3}\,k\,\sigma^2\left[\left(1+\dfrac{R}{k\,\sigma}+\dfrac{R^2}{k^2\,\sigma^2}\,(1+2\,\eta)\right)e^{\frac{1}{4k^2\sigma^2}(\vartheta-4Rk\sigma)}-\right.\\ \qquad\qquad \left.-\dfrac{\vartheta}{4\,k^2\,\sigma^2}-1\right]+\sigma'_{\varphi\varphi}.\end{array}\right.$$

3. A Heat Source Varying Harmonically in Function of Time

Let a concentrated source of heat varying harmonically in function of time act in an infinite elastic space:

$$W(t)=W_0\,e^{i\omega t}.$$

Assuming $T(R,t)=e^{i\omega t}\,U(R)$, we reduce the heat equation to the form

$$\nabla^2 U-\frac{\omega\,i}{k}\,U=-\frac{W_0}{k}\,\delta(R)\,. \tag{3.1}$$

Assuming in an analogous way that $\Phi(R,t)=e^{i\omega t}\,F(R)$, we reduce the Eq. (1.9) to the form

$$\nabla^2 F+\sigma^2\,\omega^2\,F=\vartheta_0\,U\,. \tag{3.2}$$

The analogy between the Eqs. (3.1), (3.2) and (1.11) is evident. We see that p should be replaced by $i\,\omega$, the function T^* by U, and the function Φ^* by F. Assuming the same initial conditions as in the Art. 1, we obtain the solution of the Eq. (3.1) and (3.2) in the form

$$T=\frac{W_0}{4\,\pi\,kR}\,e^{i\omega t-R\sqrt{i\omega k}}, \tag{3.3}$$

$$\Phi=\frac{\vartheta_0\,W_0}{4\,\pi\,k\,\sigma^2}\,R^{-1}\,(\omega\,i)^{-1}\left(\omega\,i-\frac{1}{k\,\sigma^2}\right)^{-1}\left(e^{-R\sigma i\omega}-e^{-R\sqrt{i\omega k}}\right)e^{i\omega t}. \tag{3.4}$$

In the particular case $W(t)=W_0\cos\omega\,t$, we should take the real part of the functions (3.3) and (3.4), and for $W(t)=W_0\sin\omega\,t$, the imaginary parts of the functions Φ and T. The case of $W(t)=W_0 \cos \omega t$ alone will be considered in this paper. We have

$$(3.5) \quad T=\frac{W_0}{4\pi kR} e^{-R\sqrt{\omega/2k}} \cos\left(\omega t - R\sqrt{\frac{\omega}{2k}}\right),$$

$$(3.6) \quad \Phi=\frac{\vartheta_0 W_0 R^{-1}}{4\pi\omega(1+k^2\sigma^4\omega^2)}\left\{k\sigma^2\omega\left[\cos\omega(t-R\sigma)-\right.\right.$$
$$\left.-e^{-R\sqrt{\omega/2k}}\cos\left(\omega t-R\sqrt{\frac{\omega}{2k}}\right)\right]-\left[\sin\omega(t-R\sigma)-\right.$$
$$\left.\left.-e^{-R\sqrt{\omega/2k}}\sin\left(\omega t-R\sqrt{\frac{\omega}{2k}}\right)\right]\right\}.$$

The stress components will be found from the Eqs. (1.17). We have

$$(3.7) \quad \left\{\begin{aligned}
\sigma_{RR}=&-2AGR^{-3}\left\{\omega\sigma^2 k\left(1+\frac{R}{\sigma k}-R^2\eta_0\right)\cos\omega(t-R\sigma)+\right.\\
&+(1-R\omega^2\sigma^3 k-R^2\eta_0)\sin\omega(t-R\sigma)-\\
&-e^{-R\sqrt{\omega/2k}}\left[\omega\sigma^2 k\left(1+R\sqrt{\frac{\omega}{2k}}+\frac{R}{\sqrt{2k\omega}}\frac{1}{\sigma^2 k}-\right.\right.\\
&\left.-R^2\eta_0\right)\cos\omega\left(t-\frac{R}{\sqrt{2k\omega}}\right)+\left(1+R\sqrt{\frac{\omega}{2k}}-R\omega\sigma^2 k\sqrt{\frac{\omega}{2k}}-\right.\\
&\left.\left.\left.-R^2\eta_0\right)\sin\omega\left(t-\frac{R}{\sqrt{2k\omega}}\right)\right]\right\},\\
\sigma_{\vartheta\vartheta}=\sigma_{\varphi\varphi}=&-AGR^{-3}\left\{e^{-R\sqrt{\omega/2k}}\left[\omega\sigma^2 k\left(1+R\sqrt{\frac{\omega}{2k}}+\right.\right.\right.\\
&\left.\left.+\frac{R}{\sqrt{2k\omega}}\frac{1}{k\sigma^2}+\frac{R^2}{2k^2\sigma^2}-2R^2\eta_0\right)\right]\cos\omega\left(t-\frac{R}{\sqrt{2\omega k}}\right)-\\
&-\left[1+R\sqrt{\frac{\omega}{2k}}+R\omega\sigma^2 k\sqrt{\frac{\omega}{2k}}\frac{R^2}{2\sigma^2k^2}-2R^2\eta_0\right]\sin\omega\left(t-\frac{R}{\sqrt{2\omega k}}\right)-\\
&-\omega\sigma^2 k\left[1+\frac{R}{\sigma k}-R^2\omega^2\sigma^2+2R^2\eta_0\right]\cos\omega(t-R\sigma)-\\
&\left.-[1-R\omega^2\sigma^3 k-R^2\omega^2\sigma^2+2R^2\eta_0]\sin\omega(t-R\sigma)\right\},\\
\sigma_{R\varphi}=0,\quad &\sigma_{R\vartheta}=0,\quad \sigma_{\varphi\vartheta}=0,
\end{aligned}\right.$$

where

$$A=\frac{W_0\vartheta_0}{2\pi\omega(\omega^2\sigma^4k^2+1)}, \qquad \eta_0=\frac{\omega^2\beta}{4G}.$$

The solution for a harmonically variable heat source thus obtained can be used for constructing solutions for sources varying with time in a periodic manner. Expanding the function $W(t)$ in a Fourier series

$$W(t) = \sum_{n=0}^{\infty} A_n \cos(n\,\omega\,t - \varepsilon_n), \tag{3.8}$$

we obtain the temperature and stress field by superposing their harmonic components.

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Streszczenie

O PEWNYM DYNAMICZNYM ZAGADNIENIU TERMOSPRĘŻYSTOŚCI

Celem pracy jest wyznaczenie stanu naprężenia wywołanego w nieograniczonej przestrzeni sprężystej działaniem zmiennego w czasie skupionego źródła ciepła. Działanie to wywołuje efekty dynamiczne. Uwzględniono zatem w równaniach przemieszczeniowych teorii sprężystości człony inercyjne. Rozpatrywane zagadnienie charakteryzuje się symetrią sferyczną; zarówno temperatura jak i naprężenia są funkcjami odległości R od źródła ciepła i czasu t. Dlatego też równania przemieszczeniowe (1.3) sprowadzają się do jednego równania (1.7), lub też po wprowadzeniu potencjału termosprężystego przemieszczenia Φ do równania (1.9). Znajomość funkcji Φ zezwala już przy użyciu wzorów (1.6) wyznaczyć składowe stanu naprężenia σ_{ij}. Rozwiązanie równań (1.1) i (1.9) opisujących pole temperatury i stan naprężenia dokonano przy użyciu transformacji Laplace'a i Fourier-Hankela. Szczegółowo rozpatrzono działanie chwilowego źródła ciepła, ciągłego źródła ciepła o stałej intensywności oraz działanie źródła zmieniającego się w czasie w sposób harmoniczny.

Резюме

О НЕКОТОРОЙ ДИНАМИЧЕСКОЙ ЗАДАЧЕ ТЕРМОУПРУГОСТИ

Целью работы является определение напряженного состояния, вызванного в неограниченном упругом пространстве действием изменяющегося во времени, сосредоточенного источника тепла. Это действие вызывает динамические эффекты. Таким образом, в уравнениях для

перемещений учитываются инерционные члены. Рассматриваемая задача характеризуется сферической симметрией: температура, равно как и напряжения, являются функциями расстояния R от источника тепла и времени t. Поэтому уравнения для перемещений (1.3) сводятся к одному уравнению (1.7) или же, вводя термоупругий потенциал перемещений Φ, к уравнению (1.9). Знание функции Φ дает возможность определить, при использовании формул (1.6), компоненты напряженного состояния σ_{ij}. Решение уравнений (1.1) и (1.9), определяющих поле температурное и напряженное состояние, выполнено при помощи преобразования Лапласа и Фурье-Ганкеля. Подробно рассматривается действие временного источника тепла, непрерывного источника тепла постоянной интенсивности, а также действие источника гармонически изменяющегося во времени.

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