APPLIED MECHANICS

PROCEEDINGS OF
THE TENTH INTERNATIONAL CONGRESS OF
APPLIED MECHANICS
Stresa (Italy) 1960

Edited by

F. ROLLA

National Research Council, Rome (Italy)

and

W. T. KOITER

Laboratory of Applied Mechanics, Delft (The Netherlands)

ELSEVIER PUBLISHING COMPANY

AMSTERDAM — NEW YORK

1962

THERMAL STRESS PROPAGATION IN VISCO-ELASTIC BODIES*

W. NOWACKI

Polish Academy of Sciences, Warsaw (Poland)

The paper deals with the computation of stresses due to the action of a non-stationary temperature field in a visco-elastic body. It was assumed that temperature range is not very large and that therefore the material constants are independent of temperature, and depend only on time. The problem is regarded as dynamic.

The natural state of the body was assumed to be the initial state; homogeneous initial conditions for displacements, stresses, and temperature were taken.

The stress-strain law for various models of visco-elastic bodies, after an application of the Laplace transform, can be written thus

$$\bar{\sigma}_{ij} = 2\,\bar{\mu}\,\bar{\varepsilon}_{ij} + \delta_{ij}\,(\bar{\lambda}\,\bar{\varepsilon}_{kk} - \bar{\gamma}\,\bar{T}) \qquad i, j = 1, 2, 3$$
$$\bar{f}(x_r, p) = \int_0^\infty \mathrm{e}^{-pt} f(x_r, t)\,\mathrm{d}t \qquad \bar{\gamma} = \alpha_t(3\bar{\lambda} + 2\bar{\mu}) \tag{1}$$

i.e. in a unified form. Here $\bar{\sigma}_{ij}(x_r, p)$, $\bar{\varepsilon}_{ij}(x_r, p)$ are the transforms of the components of the stress and strain tensors, respectively. The quantities $\bar{\mu}(p)$, $\bar{\lambda}(p)$ for a perfectly elastic body represent the Lamé constants μ_0, λ_0; α_t is the coefficient of thermal expansion and $\bar{T}$ the transform of the temperature.

Introducing Equation (1) into the transformed equations of motion

$$\bar{\sigma}_{ij,j} = p^2\,\varrho\,\bar{u}_i \qquad i = 1, 2, 3 \tag{2}$$

we obtain the following transformed system of displacement equations

$$\bar{\mu}\,\bar{u}_{i,kk} + (\bar{\lambda} + \bar{\mu})\,\bar{u}_{k,ki} + \bar{\gamma}\,\bar{T}_{,i} = \varrho p^2\,\bar{u}_i \qquad i = 1, 2, 3 \tag{3}$$

The transforms of the strain ε_{ij} were here expressed by the transforms of the dis-

* To be published in *Österreichisches Ingenieur-Archiv.*

placements, $\bar{\varepsilon}_{ij} = \frac{1}{2}(\bar{u}_{i,j} + \bar{u}_{j,i})$. If the body is not loaded the boundary conditions have the form

$$\bar{\sigma}_{ij}\, \eta_j = 0$$

In the case of an infinite visco-elastic space it is sufficient to find the particular solution of Equation (3).

The change in time of the temperature field will result in the infinite space in the propagation of longitudinal waves only. Putting $\bar{u}_i = \bar{\Phi}_{,i}$ we reduce Equation (3) to the form

$$\bar{\Phi}_{,kk} - p^2 \bar{\sigma} \bar{\Phi} = \bar{m}\, \bar{T} \tag{5}$$

where

$$\bar{\sigma} = \frac{\varrho}{\lambda + 2\mu} \qquad \bar{m} = \frac{\bar{\gamma}}{\lambda + 2\mu}$$

The deformations and stresses are obtained by performing the inverse Laplace transform in the relations

$$\begin{aligned} \bar{\varepsilon}_{ij} &= \bar{\Phi}_{,ij} \\ \bar{\sigma}_{ij} &= 2\bar{\mu}(\bar{\Phi}_{,ij} - \delta_{ij}\, \bar{\Phi}_{,kk}) + \varrho p^2 \bar{\Phi} \end{aligned} \tag{6}$$

The stresses are given by the formulae

$$\sigma_{ij} = \int_0^t \mathrm{d}\tau \int_V \mathrm{d}V(\xi_r)\; T(\xi_r, \tau)\, \sigma_{ij}{}^* \,(x_r, \xi_r, t - \tau) \tag{7}$$

where $\sigma_{ij}{}^*(x_r, \xi_r, t)$ is the stress due to an instantaneous thermal inclusion $\delta(x_r - \xi_r)\; \delta(t)$. The potential $\varphi^*(x_r,\, t)$ corresponding to this inclusion should satisfy the equation

$$\bar{\Phi}_{,}{}^*{}_{kk} - p^2 \bar{\sigma} \bar{\Phi} \;=\; \bar{m}\; \delta(x_r - \xi_r) \tag{8}$$

The re-transformed stresses are given by

$$\sigma_{ij}{}^* = 2\bar{\mu}(\bar{\Phi}_{,}{}^*{}_{ij} - \delta_{ij}\, \bar{\Phi}_{,}{}^*{}_{kk}) + p^2\, \varrho\, \bar{\Phi}^* \tag{9}$$

If in Equation (5) the term $-p^2\sigma\, \bar{\Phi}$ be neglected and in Equations (6) the term $\varrho p^2\, \bar{\Phi}$, then we arrive at the equations of the quasistatic case.

For a bounded body the solution of Equation (3) with the conditions (4) can be represented in the form

$$\bar{u}_i = \bar{\gamma} \int_V \bar{T}(\xi_r, p)\;\; \bar{\theta}_i(\xi_r, x_r, p)\, \mathrm{d}V(\xi_r) + \varrho p^2 \int_V \bar{u}_j(x_r, p)\; U_i{}^{(j)}\,(x_r, \xi_r, p)\; \mathrm{d}V\,(\xi_r) \tag{10}$$

Replacing the system (3) by the appropriate system of Fredholm integral equations of the second kind we make it possible to employ all known methods of solution of these equations. The Green function $U_i{}^{(j)}(x_r,\, p)$ has been introduced above for the displacements.

It satisfies the system of differential equations

$$\bar{\mu}\; \bar{U}^{(j)}{}_{i,kk} + (\bar{\lambda} + \bar{\mu})\;\; \bar{U}^{(j)}{}_{k,ki} + \delta(x_k - \xi_k)\; \delta_{ik} = 0 \tag{11}$$

with the boundary conditions

$$\bar{\tau}_{ij}\,\eta_j = 0 \qquad \bar{\tau}_{ij} = 2\bar{\mu}\,\bar{\varepsilon}_{ij} + \delta_{ij}\,\bar{\lambda}\,\bar{\varepsilon}_{kk} \tag{12}$$

$U^{(j)}{}_i(x_r, \xi_r, t)$ denotes the displacement of the point x_r in the direction of the x_i-axis, due to the action of an instantaneous concentrated force at the point ξ_r in the direction x_j and $\theta_i(\xi_r x_r, t)$ is the dilatation caused at the point ξ_r by an instantaneous concentrated force at the point x_r acting in the direction x_i. Equations (10) constitute an extension of the equations of Maysel for visco-elastic bodies. If in the right-hand side of Equations (10) the second term is neglected we arrive at the equations of the quasi-static problem. The solution of the system of Equations (3) can also be represented as follows:

$$\bar{u}_i = \bar{\gamma} \int_V \bar{T}(\xi_r, p)\,\bar{\theta}_i(\xi_r, x_r, p)\,\mathrm{d}V(\xi_r) \tag{13}$$

where θ_i is the dilatation connected with the displacements $V^{(i)}{}_i$ satisfying the system of equations

$$\bar{\mu}\,\bar{V}^{(j)}{}_{i,kk} + (\bar{\lambda} + \bar{\mu})\,\bar{V}^{(j)}{}_{k,ki} - \varrho p^2\,\bar{V}_i{}^{(j)} + \delta(x_k - \xi_k)\,\sigma_{ij} = 0 \tag{14}$$

with the boundary conditions (12).

In the particular case of an infinite space we have $\bar{\gamma}\,\bar{\theta}_i = \bar{\Phi}^*{}_{,i}$.

A detailed investigation has been made of the propagation of stress waves in a visco-elastic space, due to the action of a concentrated, linear, and plane source of heat. The case of non-stationary stresses due to a sudden heating of the surface of a semi-space is also dealt with.