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DISTORTION IN MICROPOLAR ELASTICITY

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1. Fundamental relations and equations

Consider a micropolar, isotropic, homogeneous and centro-symmetric elastic body, subject to initial strain $\gamma_{ji}^{0}, \varkappa_{ji}^{0}$ depending on position $\underline{x}$. This strain can arise in metals in exceeding the yield limit or during changes occuring in a heat working. A special case of distortions is the temperature strain $\gamma_{ji}^{0} = \alpha_t \delta_{ji} \theta, \varkappa_{ji}^{0} = 0$ where α_t denotes the coefficient of linear thermal expansion and $\theta = T - T_0$ is the temperature increase. $T(\underline{x})$ is the absolute temperature at point $\underline{x}$ and $T_0 =$ const is the temperature of the natural state.

We assume that the strain $\gamma_{ji}^{0}, \varkappa_{ji}^{0}$ is of the same order as the elastic strain. The introduction of the initial strain $\gamma_{ji}^{0}, \varkappa_{ji}^{0}$ into the body produces a state of elastic strain $\gamma_{ji}', \varkappa_{ji}'$ and the state of stress and couple stress σ_{ji}, μ_{ji}. The total strain $\gamma_{ji}, \varkappa_{ji}$ consists of two parts: the initial strain $\gamma_{ji}^{0}, \varkappa_{ji}^{0}$ and the elastic strain $\gamma_{ji}', \varkappa_{ji}'$, i.e.

$$\gamma_{ji} = \gamma_{ji}^{0} + \gamma_{ji}', \quad \varkappa_{ji} = \varkappa_{ji}^{0} + \varkappa_{ji}' \tag{1.1}$$

The elastic strain $\gamma_{ji}', \varkappa_{ji}'$ is a linear function of the stress [1]

$$\begin{aligned} \gamma_{ji}' &= (\mu' + \alpha')\,\sigma_{ji} + (\mu' - \alpha')\,\sigma_{ij} + \lambda' \delta_{ij} \sigma_{kk} \,, \\ \varkappa_{ji}' &= (\gamma' + \epsilon')\,\mu_{ji} + (\gamma' - \epsilon')\,\mu_{ij} + \beta' \delta_{ij} \mu_{kk} \,, \end{aligned} \tag{1.2}$$

where $\mu', \lambda', \alpha', \beta', \gamma', \epsilon'$ are material constants.

Substituting (1.2) into (1.1) and solving the latter equations for stresses we obtain

$$\begin{aligned} \sigma_{ji} &= (\mu + \alpha)(\gamma_{ji} - \gamma_{ji}^{0}) + (\mu - \alpha)(\gamma_{ij} - \gamma_{ij}^{0}) + \lambda \delta_{ij} (\gamma_{kk} - \gamma_{kk}^{0}) \\ \mu_{ji} &= (\gamma + \epsilon)(\varkappa_{ji} - \varkappa_{ji}^{0}) + (\gamma - \epsilon)(\varkappa_{ij} - \varkappa_{ij}^{0}) + \beta \delta_{ij} (\varkappa_{kk} - \varkappa_{kk}^{0}) \end{aligned} \tag{1.3}$$

where $\mu, \lambda, \alpha, \beta, \gamma, \epsilon$ are the material constants of the Cosserat medium and

$$2\mu' = \frac{1}{2\mu}, \quad 2\alpha' = \frac{1}{2\alpha}, \quad 2\gamma' = \frac{1}{2\gamma}, \quad 2\epsilon' = \frac{1}{2\epsilon},$$

$$\lambda' = -\frac{\lambda}{2\mu(3\lambda+2\mu)}, \quad \beta' = -\frac{\beta}{2\gamma(3\beta+2\gamma)}.$$

The total strain $\gamma_{ji}, \varkappa_{ji}$ can be expressed in terms of the displacement vector $\underline{u}$ and the rotation vector $\underline{\varphi}$ as follows [1]:

$$\gamma_{ji} = u_{i,j} - \epsilon_{kji}\varphi_k, \quad \varkappa_{ji} = \varphi_{i,j} \tag{1.4}$$

If the stress σ_{ji}, μ_{ji} from the formulae (1.3) is introduced into the equilibrium equations

$$\sigma_{ji,j} = 0, \quad \epsilon_{ijk}\sigma_{jk} + \mu_{ji,j} = 0 \tag{1.5}$$

and the relations (1.4) are taken into account, then we arrive at a system of six equations in displacements and rotations

$$\begin{cases} (\mu+\alpha)\nabla^2\underline{u} + (\lambda+\mu-\alpha)\,\text{grad div}\,\underline{u} + 2\alpha\,\text{rot}\,\underline{\varphi} + \underline{X} = 0, \\ ((\gamma+\epsilon)\,\nabla^2 - 4\alpha)\underline{\varphi} + (\beta+\gamma-\epsilon)\,\text{grad div}\,\underline{\varphi} + 2\alpha\,\text{rot}\,\underline{u} + \underline{Y} = 0, \end{cases} \tag{1.6}$$

We have introduced here fictitious body forces

$$X_j = -\sigma^0_{ji,j}, \quad Y_i = -\epsilon_{ijk}\sigma^0_{jk} - \mu^0_{ji,j}, \tag{1.7}$$

where

$$\begin{aligned} \sigma^0_{ji} &= (\mu+\alpha)\,\gamma^0_{ji} + (\mu-\alpha)\,\gamma^0_{ij} + \lambda\delta_{ij}\gamma^0_{kk}, \\ \mu^0_{ji} &= (\gamma+\epsilon)\,\varkappa^0_{ji} + (\gamma-\epsilon)\,\varkappa^0_{ij} + \beta\delta_{ij}\varkappa^0_{kk}. \end{aligned} \tag{1.8}$$

Eqs (1.6) should be completed by boundary conditions which may be given in displacements and rotations or in surface forces and moments on the surface A bounding the body.

Solving the differential equations (1.6) we obtain displacement $\underline{u}$ and rotation $\underline{\varphi}$. Eqs (1.4) serve then for the determination of the strain $\gamma_{ji}, \varkappa_{ji}$ and (1.3) makes it possible to calculate the stress σ_{ji}, μ_{ji}.

Eqs (1.6) are particularly simple if we are faced with thermal distortions, namely

$$\gamma^0_{ji} = \alpha_t\delta_{ij}\theta(\underline{x}), \quad \varkappa^0_{ji} = 0 \tag{1.9}$$

Then they have the form

$$\begin{cases} (\mu+\alpha)\ \nabla^2\underline{u}+(\lambda+\mu-\alpha)\ \text{grad div}\ \underline{u}+2\alpha\ \text{rot}\ \varphi=\nu\ \text{grad}\ \theta\ , \\ ((\gamma+\epsilon)\ \nabla^2-4\alpha)\ \underline{\varphi}+(\beta+\gamma-\epsilon)\ \text{grad div}\ \underline{\varphi}+2\alpha\ \text{rot}\ \underline{u}=0\ ,\ \nu=(3\lambda+2\mu)\ \alpha_t\ . \end{cases} \tag{1.10}$$

A different method of determination of the state of stress due to the action of a distortion was given by K. H. Anthony [2] and W. D. Claus and A. C. Eringen [3].

In view of relations (1.1) and (1.4) we have

$$u_{i,j}-\epsilon_{kji}\varphi_k=\overset{0}{\gamma}_{ji}+\gamma'_{ji}\ ,\ \ \varphi_{i,j}=\overset{0}{\varkappa}_{ji}+\varkappa'_{ji}\ . \tag{1.11}$$

Eliminating from Eqs (1.11) the quantities φ_i and u_i we arrive at the compatibility equations

$$\begin{cases} \epsilon_{jhl}\gamma'_{li,h}-\varkappa'_{ij}+\delta_{ij}\varkappa'_{kk}=\alpha_{ji}\ , \\ \epsilon_{jhl}\varkappa_{li,h}=\theta_{ji}\ , \end{cases} \tag{1.12}$$

where

$$\begin{cases} \alpha_{ji}=-\epsilon_{jhl}\overset{0}{\gamma}_{li,h}+\overset{0}{\varkappa}_{ij}-\delta_{ij}\overset{0}{\varkappa}_{kk}\ , \\ \theta_{ji}=-\epsilon_{jhl}\overset{0}{\varkappa}_{li,h}\ . \end{cases}$$

The quantities α_{ji} and θ_{ji} are known functions and constitute the distortion densities. Representing the elastic strain γ'_{ji}, $\varkappa'_{ji}$ in terms of the stress σ_{ji}, μ_{ji} by means of relations (1.9) we obtain the compatibility equations in stresses. Making use of the equations of equilibrium (1.5) we arrive at equations in stresses constituting the counterpart of the Beltrami-Michell equations of classical elasticity. Eqs (1.12) are particularly convenient in the case of plane state of strain.

In the particular case of thermal distortion $\overset{0}{\gamma}_{ji}=\alpha_t\theta\,\delta_{ji}$, $\overset{0}{\varkappa}_{ji}=0$ we have $\theta_{ji}=0$, $\alpha_{ji}=\epsilon_{jih}\theta_{,h}$ and the compatibility equations take the form

$$\begin{cases} \epsilon_{jhl}\gamma'_{li,h}-\varkappa'_{ij}+\delta_{ij}\varkappa'_{kk}=\alpha_t\epsilon_{jih}\theta_{,h}\ , \\ \epsilon_{jhl}\varkappa'_{li,h}=0\ . \end{cases} \tag{1.13}$$

2. Principle of virtual work

Consider a micropolar body in equilibrium, subject to external loading (body forces and moments $\underline{X}$, $\underline{Y}$ and surface forces and moments $\underline{p}$, $\underline{m}$) and a field of initial strain γ_{ji}^0, $\varkappa_{ji}^0$. Suppose that on the surface A_σ forces and moments are given, while on A_u displacements and rotations are prescribed. We have $A = A_u + A_\sigma$ where A is the total surface bounding the body.

The principle of virtual work for virtual displacement δu_i and virtual rotation $\delta\varphi_i$ has the form

$$\int_V (X_i \delta u_i + Y_i \delta\varphi_i)\, dV + \int_A (p_i \delta u_i + m_i \delta\varphi_i)\, dA = \int_V (\sigma_{ji}\delta\gamma_{ji} + \mu_{ji}\delta\varkappa_{ji})\, dV \ . \tag{2.1}$$

This principle states that the virtual work of the external forces and moments is equal to the virtual work of the internal forces.

Introducing into (2.1) the constitutive relations (1.3) we obtain the equation

$$\int_A (X_i \delta u_i + Y_i \delta\varphi_i)\, dV + \int_A (p_i \delta u_i + m_i \delta\varphi_i)\, dA = dW - \int_V (\sigma_{ji}^0\, \delta\gamma_{ji} + \mu_{ji}^0 \delta\varkappa_{ji})\, dV \ , \tag{2.2}$$

where

$$W = \int_V (\mu\gamma_{(ij)}\gamma_{(ij)} + \alpha\gamma_{<ij>}\gamma_{<ij>} + \frac{\lambda}{2}\gamma_{kk}\gamma_{nn} + \gamma\varkappa_{(ij)}\varkappa_{(ij)} + \epsilon\varkappa_{<ij>}\varkappa_{<ij>} + \frac{\beta}{2}\varkappa_{kk}\varkappa_{nn})\, dV \ .$$

The symbols ( ) and <> refer to the symmetric and skew-symmetric parts of the tensor, respectively. Since the body forces and moments and the surface forces and moments are not subject to any variation, (2.2) can be written in the form

$$\delta\, [W - \int_V (X_i u_i + Y_i \varphi_i)\, dV - \int_{A_\sigma} (p_i u_i - m_i \varphi_i)\, dA - \int_V (\gamma_{ji}^0 \sigma_{ji} + \varkappa_{ji}^0 \mu_{ji})\, dV] = 0 \ . \tag{2.3}$$

We have made use here of the identity

$$\sigma_{ji}^0 \gamma_{ji} + \mu_{ji}^0 \varkappa_{ji} = \sigma_{ji}\gamma_{ji}^0 + \mu_{ji}\varkappa_{ji}^0 \ . \tag{2.4}$$

The expression in the square brackets in Eq.(2.3) is the potential energy. This energy takes an extremum value. A procedure analogous to that in classical elasticity proves that the potential energy takes the absolute minimum.

Let us now return to Eq.(2.2). Making use of relations (1.4) and the Ostrogradski-Gauss theorem we obtain

$$\int_V [(X_i - \sigma_{ji,j}^0)\, \delta u_i + (Y_i - (\mu_{ji,j}^0 + \epsilon_{ijk}\, \sigma_{jk}^0))\, \delta\varphi_i]\, dV$$

$$+ \int_A [(p_i + \sigma_{ji}^0 n_j)\, \delta u_i + (m_i + \mu_{ji}^0 n_j)\, \delta\varphi_i]\, dA = \delta W \ . \tag{2.5}$$

Consider now the same bounded body, of the same shape and the same material. Assume that it is subject to the external body forces and moments X_i^*, Y_i^*; on A_σ there act forces p_i^* and moments m_i^* and on A_u displacements u_i^* and rotations φ_i^* are prescribed. We assume however that initial strain is absent.

We now ask the question, what should the quantities X_i^* and Y_i^* in the interior of the body be and what quantities p_i^*, m_i^* on A_σ with the same boundary conditions on A_u should be prescribed, in order that in the considered body there occur the same fields of displacement $\underline{u}$ and rotation $\underline{\varphi}$ as in the case of action of distortion $\overset{0}{\gamma}_{ji}$, $\overset{0}{\varkappa}_{ji}$. To answer it we write down the equation of virtual work for the considered body

$$\int_V (X_i^* \delta u_i + Y_i^* \delta\varphi_i)\, dV + \int_A (p_i^* \delta u_i + m_i^* \delta\varphi_i)\, dA = \delta W \ . \tag{2.6}$$

Since displacement $\underline{u}$ and rotation $\underline{\varphi}$ are the same in both cases, the right-hand sides of Eqs (2.5) and (2.6) are identical. Comparing the left-hand sides of (2.5) and (2.6) we obtain the relations

$$\begin{cases} X_i^* = X_i - \overset{0}{\sigma}_{ji,j}\,, \quad Y_i^* = Y_i - (\epsilon_{ijk}\overset{0}{\sigma}_{jk} + \overset{0}{\mu}_{ji,j})\,, & \underline{x}\in V\,, \\ p_i^* = p_i + \overset{0}{\sigma}_{ji} n_j\,, \quad m_i^* = m_i + \overset{0}{\mu}_{ji} n_j\,, & \underline{x}\in A_\sigma\,, \\ u_i^* = u_i\,, \qquad \varphi_i^* = \varphi_i\,, & \underline{x}\in A_u\,. \end{cases} \tag{2.7}$$

The above quantities represent the counterparts of the body forces and moments. This analogy makes it possible to reduce every static distortion problem to a boundary problem of the non-symmetric elasticity theory.

3. Theorem of minimum of the complementary work

Consider the quadratic form

$$W'_\sigma = \left(\frac{\mu'+\alpha'}{2}\right)\sigma_{ji}\sigma_{ji} + \left(\frac{\mu'-\alpha'}{2}\right)\sigma_{ij}\sigma_{ji} + \frac{\lambda'}{2}\sigma_{kk}\sigma_{nn} + \left(\frac{\gamma'+\epsilon'}{2}\right)\mu_{ji}\mu_{ji} + \left(\frac{\gamma'-\epsilon'}{2}\right)\mu_{ji}\mu_{ij} + \frac{\beta}{2}\mu_{kk}\mu_{nn}\ . \tag{3.1}$$

In view of (1.1) and (1.2), we obtain

$$\begin{aligned} \frac{\partial W'_\sigma}{\partial \sigma_{ji}} &= (\mu'+\alpha')\,\sigma_{ji} + (\mu'-\alpha')\,\sigma_{ij} + \lambda'\sigma_{kk}\,\delta_{ij} = \gamma_{ji} - \overset{0}{\gamma}_{ji}\,, \\ \frac{\partial W'_\sigma}{\partial \mu_{ji}} &= (\gamma'+\epsilon')\,\mu_{ji} + (\gamma'-\epsilon')\,\mu_{ij} + \beta'\mu_{kk}\,\delta_{ij} = \varkappa_{ji} - \overset{0}{\varkappa}_{ji} \end{aligned} \tag{3.2}$$

where we have introduced the virtual increments of stress $\delta\sigma_{ji}$, $\delta\mu_{ji}$. We assume that the virtual stress satisfies the equilibrium equations

$$\delta\sigma_{ji,j}=0\,,\quad \epsilon_{ijk}\,\delta\sigma_{jk}+\delta\mu_{ji,j}=0\,, \tag{3.3}$$

and that on surface A_σ we have $\delta p_i=0$, $\delta m_i=0$. On surface $A_u=A-A_\sigma$ the virtual increments δp_i, δm_i are arbitrary. Let us multiply relation $(3.2)_1$ by $\delta\sigma_{ji}$ and $(3.2)_2$ by $\delta\mu_{ji}$. Adding the results and integrating over the volume of the body we obtain

$$\int_V \Big(\frac{\partial W'_\sigma}{\partial\sigma_{ji}}\,\delta\sigma_{ji}+\frac{\partial W'_\sigma}{\partial\mu_{ji}}\,\delta\mu_{ji}\Big)\,dV=\int_V [(\gamma_{ji}-\gamma^0_{ji})\,\delta\sigma_{ji}+(\varkappa_{ji}-\varkappa^0_{ji})\,\delta\mu_{ji}]\,dV \tag{3.4}$$

or

$$\delta W_\sigma+\int_V(\gamma^0_{ji}\delta\sigma_{ji}+\varkappa^0_{ji}\delta\mu_{ji})\,dV=\int_V(\gamma_{ji}\delta\sigma_{ji}+\varkappa_{ji}\delta\mu_{ji})\,dV,\quad W_\sigma=\int_V W'_\sigma\,dV \tag{3.5}$$

On the other hand, transforming the right-hand side of Eq.(3.5) we have

$$\begin{cases}\displaystyle\int_V(\gamma_{ji}\delta\sigma_{ji}+\varkappa_{ji}\delta\mu_{ji})\,dV=\int_A(u_i\delta p_i+\varphi_i\delta m_i)\,dV\\ \displaystyle-\int_V[u_i\delta\sigma_{ji,j}+\varphi_i\,(\epsilon_{ijk}\,\delta\sigma_{jk}+\delta\mu_{ji,j})]\,dV,\quad \delta p_i=\delta\sigma_{ji}n_j\,,\quad \delta m_i=\delta\mu_{ji}n_j\,.\end{cases} \tag{3.6}$$

We have made use here of the definition of strain (1.4). Taking into account the equilibrium equation (3.3) and bearing in mind that $\delta p_i=0$, $\delta m_i=0$ on A_σ we obtain from (3.5) and (3.6)

$$\delta W_\sigma+\int_V(\gamma^0_{ji}\delta\sigma_{ji}+\varkappa^0_{ji}\delta\mu_{ji})\,dV=\int_{A_u}(u_i\delta p_i+\varphi_i\delta m_i)\,dA\,, \tag{3.7}$$

or

$$\delta\Gamma=0\,,$$

where

$$\Gamma=W_\sigma+\int_V(\gamma^0_{ji}\sigma_{ji}+\varkappa^0_{ji}\mu_{ji})\,dV-\int_{A_u}(p_iu_i+m_i\varphi_i)\,dA\,.$$

Here Γ is the complementary work. As in the classical elasticity we can prove that the complementary energy takes the absolute minimum.

4. Reciprocity theorem

In deriving this theorem we make use of the analogy of body forces and moments. The reciprocity theorem for a body without initial strain has the form

$$\int_V (X_i^* u_i' + Y_i^* \varphi_i')\, dV + \int_A (p_i^* u_i' + m_i^* \varphi_i')\, dA$$
$$= \int_V (X_i'^* u_i + Y_i'^* \varphi_i) dV + \int_A (p_i'^* u_i + m_i'^* \varphi_i)\, dA \tag{4.1}$$

where X_i^*, Y_i^*, p_i^*, m_i^* refer to the first system of loadings producing displacement u_i and the rotation φ_i, while $X_i'^*$, $Y_i'^*$, $p_i'^*$, $m_i'^*$ refer to the second system of loadings leading to displacement u_i' and rotation φ_i'.

Consider now the same body with the first system of loadings X_i, Y_i, p_i, m_i and the initial strain γ_{ji}^0, $\varkappa_{ji}^0$ leading to displacement and rotation fields u_i and φ_i, respectively. The second system of loadings, initial strain and rotation will be devoted by primes. Making use of the analogy of body forces and moments (2.7) we obtain

$$\int_V [(X_i - \sigma_{ji,j}^0)\, u_i' + (Y_i - (\epsilon_{ijk}\sigma_{jk}^0 + \mu_{ji,j}^0))\, \varphi_i']\, dV + \int_A [(p_i + \sigma_{ji}^0 n_j)\, u_i' + (m_i + \mu_{ji}^0 n_j)\, \varphi_i']\, dA$$
$$= \int_V [(X_i' - \sigma_{ji,j}'^0)\, u_i + (Y_i' - (\epsilon_{ijk}\sigma_{jk}'^0 + \mu_{ji,j}'^0))\, \varphi_i]\, dV + \int_A [(p_i' + \sigma_{ji}'^0 n_j)\, u_i + (m_i' + \mu_{ji}'^0 n_j)\, \varphi_i]\, dA\,. \tag{4.2}$$

After simple transformations, making use of the Ostrogradski-Gauss theorem, we arrive at the final form of the generalization of the reciprocity theorem to distortion problems:

$$\int_V (X_i u_i' + Y_i \varphi_i')\, dV + \int_A (p_i u_i' + m_i \varphi_i')\, dA + \int_V (\gamma_{ji}^0 \sigma_{ji}' + \varkappa_{ji}^0 \mu_{ji}')\, dV$$
$$= \int_V (X_i' u_i + Y_i' \varphi_i)\, dV + \int_A (p_i' u_i + m_i' \varphi_i)\, dA + \int_V (\gamma_{ji}'^0 \sigma_{ji} + \varkappa_{ji}'^0 \mu_{ji})\, dV\,. \tag{4.3}$$

Consider a bounded body, free of body forces ($X_i = 0$) and body moments ($Y_i = 0$), free of loadings on A_σ, i.e. $p_i = 0$, $m_i = 0$ and clamped on A_u, $\underline{u} = 0$, $\underline{\varphi} = 0$. Our aim is to determine displacement u_i and rotation φ_i at a point $\underline{\xi}$ of the body, due to the action of initial strain γ_{ji}^0, $\varkappa_{ji}^0$.

We take for the second system of loadings a concentrated force $X_i' = \delta(\underline{x} - \underline{\xi})\, \delta_{ik}$ at point $\underline{\xi}$ acting in the direction of the x_k-axis. Thus, $Y_i' = 0$, $\gamma_{ji}'^0 = 0$, $\varkappa_{ji}'^0 = 0$. Moreover, we assume that u_i', φ_i' vanish on A_u and p_i', m_i' vanish on A_σ. The action of the concentrated force $X_i' = \delta(\underline{x} - \underline{\xi})\, \delta_{ik}$ produces in the body displacements $U_i^{(k)}(\underline{x}, \underline{\xi})$ and rotations $\Phi_i^{(k)}(\underline{x}, \underline{\xi})$. By means of the latter functions we determine the strain $\gamma_{ji}^{(k)}$, $\varkappa_{ji}^{(k)}$ and the stress $\sigma_{ji}^{(k)}$, $\mu_{ji}^{(k)}$, making use of the formulae

$$\sigma_{ji}^{(k)} = (\mu + \alpha)\, \gamma_{ji}^{(k)} + (\mu - \alpha)\, \gamma_{ij}^{(k)} + \lambda \delta_{ij} \gamma_{nn}^{(k)}\,,$$
$$\mu_{ji}^{(k)} = (\gamma + \epsilon)\, \varkappa_{ji}^{(k)} + (\gamma - \epsilon)\, \varkappa_{ij}^{(k)} + \beta \delta_{ij} \varkappa_{nn}^{(k)}\,. \tag{4.4}$$

Applying to the two systems of loadings the reciprocity theorem (4.3) we obtain

$$\int_V (\gamma_{ji}^0 \sigma_{ji}^{(k)} + \varkappa_{ji}^0 \mu_{ji}^{(k)})\, dV = \int_V \delta\,(\underline{x}-\underline{\xi})\, \delta_{ik}\, u_i(\underline{x})\, dV(\underline{x}) \,,$$

whence

$$u_k(\underline{\xi}) = \int_V [\gamma_{ji}^0(\underline{x})\, \sigma_{ji}^{(k)}(\underline{x},\underline{\xi}) + \varkappa_{ji}^0(\underline{x})\, \mu_{ji}^{(k)}(\underline{x},\underline{\xi})]\, dV(\underline{x}) \,. \tag{4.5}$$

We take for the second system of loadings (primed) a concentrated body moment $Y_i' = \delta\,(\underline{x}-\underline{\xi})\,\delta_{ik}$. We assume also that $X_i'=0$, $\gamma_{ji}'^0 = \varkappa_{ji}'^0 = 0$ inside the body and $u_i' = \varphi_i' = 0$ on A_u ; furthermore, $p_i'=0$, $m_i'=0$ on A_σ.

The body moment $Y_i' = \delta\,(\underline{x}-\underline{\xi})\,\delta_{ik}$ produces in the body displacement $u_i' = \hat{U}_i^{(k)}(\underline{x},\underline{\xi})$ and rotation $\varphi_i' = \hat{\Phi}^{(k)}(\underline{x},\underline{\xi})$. By means of the formulae (4.4) we determine the stress $\hat{\sigma}_{ji}^{(k)}(\underline{x},\underline{\xi})$ and $\hat{\mu}_{ji}^{(k)}(\underline{x},\underline{\xi})$. Applying to the considered states the reciprocity theorem (4.3) we obtain

$$\varphi_k(\underline{\xi}) = \int_V [\gamma_{ji}^0(\underline{x})\, \hat{\sigma}_{ji}^{(k)}(\underline{x},\underline{\xi}) + \varkappa_{ji}^0(\underline{x})\, \hat{\mu}_{ji}^{(k)}(\underline{x},\underline{\xi})]\, dV(\underline{x}) \,. \tag{4.6}$$

Relations (4.5) and (4.6) constitute a generalization of V.M.Maysel's relations [4], to the distortion problem in micropolar elasticity. They also hold for an infinite elastic space. In this particular case the singular solutions $U_i^{(k)}$, $\Phi_i^{(k)}$, $\hat{U}_i^{(k)}$, $\hat{\Phi}_i^{(k)}$ are known, namely [5]

$$\begin{cases} U_i^{(k)} = \dfrac{1}{8\pi\mu}\left(\delta_{ik}\nabla^2 R - \dfrac{\lambda+\mu}{\lambda+2\mu} R_{,ik}\right) + B\left(l^2\left(\dfrac{e^{-R/l}-1}{R}\right)_{,ik} - \delta_{ik}\dfrac{e^{-R/l}}{R}\right) \,, \\[2ex] \Phi_i^{(k)} = -\dfrac{1}{8\pi\mu}\,\epsilon_{kij}\left(\dfrac{e^{-R/l}-1}{R}\right)_{,j} \,, \end{cases} \tag{4.7}$$

and

$$\begin{cases} \hat{U}_i^{(k)} = -\dfrac{1}{8\pi\mu}\,\epsilon_{kij}\left(\dfrac{e^{-R/l}-1}{R}\right)_{,j} \,, \\[2ex] \hat{\Phi}_i^{(k)} = \dfrac{1}{16\pi\mu}\left(\dfrac{1-e^{-R/l}}{R}\right) + \dfrac{1}{16\pi\alpha}\left(\dfrac{e^{-R/\nu}-e^{-R/l}}{R}\right)_{,ik} + \dfrac{\mu+\alpha}{16\pi\alpha\mu l^2}\,\delta_{ik}\dfrac{e^{-R/l}}{R} \end{cases} \tag{4.8}$$

where

$$R = (\underline{x}-\underline{\xi}) \,,\; \nu^2 = \frac{\beta+2\gamma}{4\alpha} \,,\; l^2 = \frac{(\mu+\alpha)(\gamma+\epsilon)}{4\alpha\mu} \,,\; B = \frac{\alpha}{4\pi\mu(\mu+\alpha)} \,.$$

Let us now consider a simple example. We shall calculate the change of volume of a bounded simply-connected body, due to the action of distortion γ_{ji}^0, $\varkappa_{ji}^0$. We assume that inside the body $X_i=0$, $Y_i=0$ and $p_i=0$, $m_i=0$ on the surface A.

The second system of loadings acting on the body consists only of the surface forces $p_i' = 1 n_i$. We are faced here with a multi-axial extension $\sigma_{ji}' = 1\,\delta_{ji}$, for only in this case

$p'_i = \sigma'_{ji} n_j = 1\, n_i$. Since $X'_i = 0$, $Y'_i = 0$, $m'_i = 0$, $\gamma'^{0}_{ji} = 0$, $\varkappa'^{0}_{ji} = 0$, Eq. (4.3) takes the form

$$\int_A p'_i u_i \, dA = \int_V \gamma^0_{ji} \sigma'_{ji} \, dV \ . \tag{4.9}$$

Now we have

$$\int_A p'_i u_i \, dA = \int_A \sigma'_{ji} n_j u_i \, dA = \int_A u_i n_i \, dA = \Delta V$$

where ΔV is the volume increment of the body. Therefore, (4.9) yields

$$\Delta V = \int_V \gamma^0_{kk} \, dV \ . \tag{4.10}$$

The volume increment ΔV has the form of a very simple integral formula. The constitutive relations (1.6) imply the relation

$$\sigma_{kk} = (3\lambda - 2\mu)(\gamma_{kk} - \gamma^0_{kk}) \ . \tag{4.11}$$

Eliminating γ^0_{kk} from (4.10) and (4.11) and bearing in mind that

$$\Delta V = \int_V u_{k,k} \, dV = \int_A u_i n_i \, dA$$

we arrive at the interesting relation

$$\int_V \sigma_{kk} \, dV = 0 \ . \tag{4.12}$$

Observe that the volume increment ΔV is independent of the material constants and that the integrand contains the sum of normal distortions. In the particular case of thermal distortion we have

$$\Delta V = 3\alpha_t \int_V \theta \, dV \, , \quad \int_V \sigma_{kk} \, dV = 0 \ . \tag{4.13}$$

5. Equations in displacements and rotations

Consider the differential equations in displacements and rotations

$$\begin{aligned} &(\mu+\alpha)\,\nabla^2 \underline{u} + (\lambda+\mu-\alpha)\,\text{grad div}\,\underline{u} + 2\alpha\,\text{rot}\,\underline{\varphi} + \underline{X} = 0 \, , \\ &((\gamma+\epsilon)\,\nabla^2 - 4\alpha)\,\underline{\varphi} + (\beta+\gamma-\epsilon)\,\text{grad div}\,\underline{\varphi} + 2\alpha\,\text{rot}\,\underline{u} + \underline{Y} = 0 \ , \end{aligned} \tag{5.1}$$

where

$$X_i = -\sigma_{ji,j} \; , \quad Y_i = -\epsilon_{ijk}\,\overset{0}{\sigma}_{jk} - \overset{0}{\mu}_{ji,j} \; .$$

Applying in Eqs (5.1) the divergence operator we have

$$\begin{aligned} &\nabla^2 \operatorname{div} \underline{u} = -\frac{1}{\lambda+2\mu} \operatorname{div} \underline{X} \\ &H \operatorname{div} \underline{\varphi} = -\frac{1}{\beta+2\gamma} \operatorname{div} \underline{Y} \end{aligned} \tag{5.2}$$

where

$$H = \nabla^2 - \frac{1}{\gamma^2} \; , \quad \nu^2 = \frac{\beta+2\gamma}{4\alpha} \; .$$

Observe that for $\underline{X}=0$ the dilatation $\operatorname{div} \underline{u}$ is a harmonic function and for $\underline{Y}=0$ the function $\underline{\varphi}$ satisfies the homogeneous Helmholtz equation.

Applying in Eqs (5.1) the rotation operator we obtain the relations

$$\left\{ \begin{aligned} &(\mu+\alpha)\,\nabla^2 \operatorname{rot} u + 2\alpha \operatorname{rot}\operatorname{rot} \underline{\varphi} = -\operatorname{rot} \underline{X} \; , \\ &(\,(\gamma+\epsilon)\,\nabla^2 - 4\alpha)\,\underline{\varphi} + 2\alpha \operatorname{rot}\operatorname{rot} \underline{u} = -\operatorname{rot} \underline{Y} \; . \end{aligned} \right. \tag{5.3}$$

Next we apply to Eq. $(5.1)_1$ the operator $\nabla^2 D$ where $D = (\gamma+\epsilon)\,\nabla^2 - 4\alpha$ and making use of the relations $(5.2)_1$ and $(5.3)_2$, after transformations we arrive at the following equation containing only the displacement vector $\underline{u}$:

$$\left\{ \begin{aligned} &D\,\nabla^2\nabla^2 \underline{u} = \frac{1}{4\alpha\mu l^2}\,[2\alpha\nabla^2 \operatorname{rot} \underline{Y} - (\gamma+\epsilon)\,\nabla^2 G\underline{X} \\ &-\frac{4\alpha^2}{\lambda+2\mu} \operatorname{grad}\operatorname{div} \underline{X} + \frac{(\lambda+\mu-\alpha)(\gamma+\epsilon)}{\lambda+2\mu} \operatorname{grad}\operatorname{div} G\underline{X}] \; . \end{aligned} \right. \tag{5.4}$$

We have introduced here the notations

$$D = \nabla^2 - \frac{1}{l^2} \; , \; G = \nabla^2 - \frac{1}{\varkappa^2} \, , \; l^2 = \frac{(\gamma+\epsilon)(\mu+\alpha)}{4\mu\alpha} \, , \; \varkappa^2 = \frac{\gamma+\epsilon}{4\alpha} \; .$$

Applying in Eq. $(5.2)_2$ the operator $\nabla^2 H$ and making use of relations $(5.2)_2$ and $(5.3)_1$, after elimination of the function $\underline{u}$ we obtain the equation

$$\left\{ \begin{aligned} &DH\nabla^2 \underline{\varphi} = \frac{1}{4\alpha\mu l^2}\,[2\alpha\nabla^2 \operatorname{rot} \underline{X} - (\mu+\alpha)\,\nabla^2 H\underline{Y} \\ &-\frac{4\alpha^2}{\beta+2\gamma} \operatorname{grad}\operatorname{div} \underline{Y} + \frac{(\beta+\gamma-\epsilon)(\mu+\alpha)}{\beta+2\gamma} \operatorname{grad}\operatorname{div} \nabla^2 \underline{Y}] \end{aligned} \right. \tag{5.5}$$

Eqs (5.4) and (5.5) are very useful in determining the functions $\underline{u}$ and $\underline{\varphi}$ due to an action of distortion in the infinite elastic space.

Let us consider some particular examples.

1. Consider the thermal distortion $\gamma_{ji}^0 = \alpha_t \theta\, \delta_{ji}$, $\varkappa_{ji}^0 = 0$. Then

$$X_i = -(2\mu + 3\lambda)\, \alpha_t \theta_{,i}\, , \quad Y_i = 0\ . \tag{5.6}$$

Observe that in this case Eq. (5.4) is reduced to the equation

$$\nabla^2 u_i = m\theta_{,i}\, , \quad m = \frac{(3\lambda + 2\mu)\,\alpha_t}{\lambda + 2\mu}\, , \tag{5.7}$$

and that $\varphi_i = 0$. Introducing the potential of thermoelastic strain $u_i = \Phi_{,i}$ we transform Eq. (5.7) to the form

$$\nabla^2 \Phi = m\theta\, , \tag{5.8}$$

the solution of the latter equation is the function

$$\Phi(\underline{x}) = -\frac{m}{4\pi} \int_V \frac{\theta(\underline{\xi})\, dV(\underline{\xi})}{R(\underline{x} - \underline{\xi})}\ . \tag{5.9}$$

It is interesting to note that $\varphi_i = 0$. Thus, in the infinite space strain $\varkappa_{ji}$ and couple stress μ_{ji} do not appear. The stress σ_{ji} is now given by the formula

$$\sigma_{ji} = 2\mu(\Phi_{,ij} - \delta_{ij}\Phi_{,kk})\ . \tag{5.10}$$

In the particular case of the thermal nucleus $\theta(\underline{x}) = \theta_0 \delta(\underline{x})$ we obtain

$$\Phi(\underline{x}) = -\frac{m\theta_0}{4\pi R(\underline{x}, 0)}\, , \quad u_i = \Phi_{,i}\ . \tag{5.11}$$

2. Assume now that $\gamma_{ji}^0 = 0$ and $\varkappa_{ji}^0 = \varkappa^0(\underline{x})\, \delta_{ji}$. Then

$$X_i = 0\, , \quad Y_i = -(3\beta + 2\gamma)\, \varkappa_{,i}^0\ .$$

Eq. (5.5) yields $u_i = 0$ and Eq. (5.4) is reduced to the simpler Helmholtz equation

$$H\varphi_i = n\varkappa_{,i}^0\, , \quad n = \frac{3\beta + 2\gamma}{\beta + 2\gamma}\, , \quad H = \nabla^2 - \frac{1}{\nu^2}\ . \tag{5.12}$$

Introducing the potential $\varphi_i = \Omega_{,i}$ we reduce Eq. (5.12) to the form

$$H\Omega = n\varkappa^0(\underline{x}) \ . \tag{5.13}$$

The solution of the latter equation is the following:

$$\Omega(\underline{x}) = -\frac{n}{4\pi}\int_V \frac{\varkappa^0(\underline{\xi})\, e^{-R/\nu}}{R(\underline{x}-\underline{\xi})}\, dV(\underline{\xi}) \ . \tag{5.14}$$

In the considered case $u_i = 0$. The rotations φ_i are given by the formula $\varphi_i = \Omega_{,i}$. Besides the couple stress there occurs also the ordinary stress. We have

$$\mu_{ji} = 2\gamma(\Omega_{,ij} - \delta_{ij}\Omega_{,kk}) \ , \quad \sigma_{ji} = -2\alpha e_{kji}\Omega_{,k} \ . \tag{5.15}$$

In the particular case $\varkappa^0(\underline{x}) = \delta(\underline{x})$ formula (5.14) yields

$$\Omega(\underline{x}) = -\frac{n}{4\pi}\,\frac{e^{-R/\nu}}{R} \ , \quad R = (x_1^2 + x_2^2 + x_3^2)^{1/2} \ . \tag{5.16}$$

Consider the case in which the right-hand side of Eq. (5.12) is the stress $n\varkappa^0 H(a-r)$ where $H(z)$ is the Heaviside function. Eq. (5.12)

$$H\Omega(r) = n\varkappa^0 H(a-r) \ , \quad r = (x_1^2 + x_2^2)^{1/2} \ , \quad \varkappa^0 = \text{const} \ , \tag{5.17}$$

is now axisymmetric. Consequently, in an infinite cylinder of radius a and axis $x_3 = z$ the distortion $\varkappa^0 = \text{const}$. Outside of the cylinder the distortion vanishes. Since the problem is axisymmetric, Eq. (5.17) has the form

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{\nu^2}\right)\Omega(r) = n\varkappa^0 H(a-r) \ . \tag{5.18}$$

Applying to the above equation the Hankel integral transform, we obtain for the rotation angle φ_r the formula

$$\varphi_r = \frac{\partial\Omega}{\partial r} = A\int_0^\infty \frac{\xi J_1(\xi a) J_1(\xi r)}{\xi^2 + \frac{1}{\nu^2}}\, d\xi \ , \quad A = na\varkappa^2 \ . \tag{5.19}$$

From $(5.15)_1$ we obtain

$$\mu_{rr} = -\frac{2\gamma}{r}\frac{\partial\Omega}{\partial r} \ , \quad \mu_{\theta\theta} = -2\gamma\frac{\partial^2\Omega}{\partial r^2} \ , \quad \mu_{zz} = \beta\nabla^2\Omega \ . \tag{5.20}$$

Taking into account that

$$\left\{\begin{aligned} P_1 &= \int_0^\infty \frac{\zeta J_1(\zeta a) J_1(\zeta r)}{\zeta^2 + \frac{1}{\nu^2}}\, d\zeta = \begin{cases} I_1(\frac{r}{\nu})\, K_1(\frac{a}{\nu}) & \text{for } 0<r<a \\ I_1(\frac{a}{\nu})\, K_1(\frac{r}{\nu}) & \text{for } a<r<\infty \end{cases} \\ P_2 &= \int_0^\infty \frac{\zeta^2 J_1(\zeta a) J_0(\zeta r)}{\zeta^2 + \frac{1}{\nu^2}}\, d\zeta = \begin{cases} \frac{1}{\nu} I_0(\frac{r}{\nu}) K_1(\frac{a}{\nu}) & \text{for } 0<r<a \\ 0 & \text{for } a<r<\infty \end{cases} \end{aligned}\right. \tag{5.21}$$

we obtain

$$\mu_{rr} = -2\gamma A \frac{P_1}{r}\ , \quad \mu_{\theta\theta} = -2\gamma A\ (P_2 - \frac{P_1}{r})\ , \quad \mu_{zz} = \beta A P_2\ . \tag{5.22}$$

Observe that μ_{rr} is a continuous function in the interval $0<r<\infty$, the function $\mu_{\theta\theta}$ has a discontinuity on the circle $r=a$ and μ_{zz} is different from zero in the interval $0<r<a$.

6. Compatibility equations

Consider the plane state of strain. Assume that all sources and unknown functions depend on the variables x_1, x_2. In this particular case the system of equations (1.12) is decomposed into two independent systems of compatibility equations.

The first system has the form

$$\left\{\begin{aligned} &\partial_1 \gamma'_{21} - \partial_2 \gamma'_{11} - \varkappa'_{13} = \alpha_{31}\ , \\ &\partial_1 \gamma'_{22} - \partial_2 \gamma'_{12} - \varkappa'_{23} = \alpha_{32}\ , \\ &\partial_1 \varkappa'_{23} - \partial_2 \varkappa'_{13} = \theta_{33}\ , \end{aligned}\right. \tag{6.1}$$

where

$$\left\{\begin{aligned} \alpha_{31} &= -\partial_1 \overset{0}{\gamma}_{21} + \partial_2 \overset{0}{\gamma}_{11} + \overset{0}{\varkappa}_{13}\ , \\ \alpha_{32} &= -\partial_1 \overset{0}{\gamma}_{22} + \partial_2 \overset{0}{\gamma}_{12} + \overset{0}{\varkappa}_{23}\ , \\ \theta_{33} &= -\partial_1 \overset{0}{\varkappa}_{23} + \partial_2 \overset{0}{\varkappa}_{13}\ . \end{aligned}\right.$$

It is readily observed that in the considered case the displacement and rotation vectors have the form

$$\underline{u} = (u_1, u_2, 0)\ , \quad \underline{\varphi} = (0, 0, \varphi_3)\ . \tag{6.2}$$

The system of equations (6.1) can be transformed to the form

$$\begin{cases} \partial_2^2\gamma'_{11} + \partial_1^2\gamma'_{22} - \partial_1\partial_2(\gamma'_{12}+\gamma'_{21}) = A_1 \ , \\ \partial_1\partial_2(\gamma'_{11}-\gamma'_{22}) + \partial_2^2\gamma'_{12} - \partial_1^2\gamma'_{21} + \partial_1\varkappa'_{13} + \partial_2\varkappa'_{23} = A_2 \ , \\ \partial_1\varkappa'_{23} - \partial_2\varkappa'_{13} = A_3 \ , \end{cases} \tag{6.3}$$

where

$$A_1 = \theta_{33} + \partial_1\alpha_{32} - \partial_2\alpha_{31} \ , \quad A_2 = -\partial_1\alpha_{31} - \partial_2\alpha_{32} \ , \quad A_3 = \theta_{33} \ .$$

Replacing strain γ'_{ji}, $\varkappa'_{ji}$ by stress σ'_{ji}, μ'_{ji} (making use of relations (1.2)) we arrive at a system of three equations in stresses

$$\partial_2^2\sigma_{11} + \partial_1^2\sigma_{22} - \frac{\lambda}{2(\lambda+\mu)}\nabla_1^2(\sigma_{11}+\sigma_{22}) - \partial_1\partial_2(\sigma_{12}+\sigma_{21}) = 2\mu A_1 \ ,$$

$$(\partial_2^2-\partial_1^2)(\sigma_{12}+\sigma_{21}) + \frac{\mu}{\alpha}\nabla_1^2(\sigma_{12}-\sigma_{21}) + \frac{4\mu}{\gamma+\epsilon}(\partial_1\mu_{13}+\partial_2\mu_{23}) + 2\partial_1\partial_2(\sigma_{11}-\sigma_{22}) = 4\mu A_2 \ ,$$

$$\partial_1\mu_{23} - \partial_2\mu_{13} = (\gamma+\epsilon)A_3 \ . \tag{6.4}$$

The state of stress has now the form

$$\underline{\sigma} = \begin{Vmatrix} \sigma_{11} & \sigma_{12} & 0 \\ \sigma_{21} & \sigma_{22} & 0 \\ 0 & 0 & \sigma_{33} \end{Vmatrix} \qquad \underline{\mu} = \begin{Vmatrix} 0 & 0 & \mu_{13} \\ 0 & 0 & \mu_{23} \\ \mu_{31} & \mu_{32} & 0 \end{Vmatrix} \tag{6.5}$$

Three of the above components, namely the stresses σ_{33}, μ_{31}, μ_{32}, can be expressed by the remaining ones. The system of equations (6.4) contain six unknown components of the state of stress.

The compatibility equations should be completed by the equilibrium equations

$$\partial_1\sigma_{11} + \partial_2\sigma_{21} = 0 \ , \quad \partial_1\sigma_{12} + \partial_2\sigma_{22} = 0 \ , \quad \partial_1\mu_{13} + \partial_2\mu_{23} + \sigma_{12} - \sigma_{21} = 0 \ . \tag{6.6}$$

Then the number of equations is equal to the number of the unknowns.

Let us consider two particular cases.

1. $A_1 \neq 0$, $A_2 \neq 0$, $A_3 = 0$.

We express the stress by the Airy-Mindlin function

$$\begin{cases} \sigma_{11} = \partial_2^2 F - \partial_1 \partial_2 \Psi , & \sigma_{22} = \partial_1^2 F + \partial_1 \partial_2 \Psi , \\ \sigma_{12} = -\partial_1 \partial_2 F - \partial_2^2 \Psi , & \sigma_{21} = -\partial_1 \partial_2 F + \partial_1^2 \Psi , \\ \mu_{13} = \partial_1 \Psi , & \mu_{23} = \partial_2 \Psi , \end{cases} \tag{6.7}$$

then the equilibrium equations and Eq. $(6.4)_3$ are identically satisfied and the remaining equations $(6.4)_{1,2}$ are reduced to the simple differential equations

$$\nabla_1^2 \nabla_1^2 F = \frac{4\mu(\lambda+\mu)}{\lambda+2\mu} A_1 , \quad \nabla_1^2 (l^2 \nabla_1^2 - 1) \Psi = -(\gamma+\epsilon) A_2 . \tag{6.8}$$

Let us consider some particular cases.

a) Assume that at the origin there acts a concentrated distortion $\gamma_{11}^0(x_1, x_2)$ of intensity γ^0: $\gamma_{11}^0(x_1, x_2) = \gamma^0 \delta(x_1)\delta(x_2)$. Then $A_1 = -\gamma^0 \partial_2^2 \delta(x_1)\delta(x_2)$, $A_2 = -\gamma^0 \partial_1 \partial_2 \delta(x_1)\delta(x_2)$. Substituting into (6.8) and solving the equations we obtain

$$\begin{cases} F = \dfrac{\mu(\lambda+\mu)\gamma^0}{2\pi(\lambda+2\mu)} \dfrac{\partial^2}{\partial x_2^2} [r^2(lnr + C)] , \quad C - \text{Euler's constant} \\ \Psi = -\dfrac{(\gamma+\epsilon)\gamma^0}{2\pi} \dfrac{\partial^2}{\partial x_1 \partial x_2} [lnr + K_0(\frac{r}{l})] , \end{cases} \tag{6.9}$$

where $K_0(\frac{r}{l})$ is the modified Bessel function of third kind, i.e. the Macdonald function. The stresses are calculated by means of formulae (6.7).

b) Assume that at the origin there acts the distortion $\gamma_{22}^0 = \gamma^0 \delta(x_1)\delta(x_2)$; then $A_1 = \gamma^0 \partial_1^2 \delta(x_1)\delta(x_2)$, $A_2 = \gamma^0 \partial_1 \partial_2 \delta(x_1)\delta(x_2)$.

Solving Eqs (6.8) we obtain the following particular integrals:.

$$\begin{cases} F = \dfrac{\mu(\lambda+\mu)\gamma^0}{2\pi(\lambda+2\mu)} \dfrac{\partial^2}{\partial x_1^2} [r^2(lnr + C)] , \\ \Psi = \dfrac{(\gamma+\epsilon)\gamma^0}{2\pi} \dfrac{\partial^2}{\partial x_1 \partial x_2} (lnr + K_0(\frac{r}{l})) . \end{cases} \tag{6.10}$$

Observe that when $\gamma_{ji}^0 = \gamma^0 \delta(x_1)\delta(x_2)\delta_{ji}$, adding (6.9) and (6.10) we have

$$F = \frac{\mu(\lambda+\mu)\gamma^0}{2\pi(\lambda+2\mu)} \nabla_1^2 [r^2(lnr + C)] = \frac{2\mu(\lambda+\mu)}{\pi(\lambda+2\mu)} (lnr + C) , \quad \Psi = 0 . \tag{6.11}$$

This case can be interpreted as the action of a temperature nucleus at the origin. Observe that in this case the couple stresses μ_{13}, μ_{23}, μ_{31}, μ_{32} vanish.

2. Consider now the case of the distortion $\varkappa^0_{13} = -\varkappa^0_{23} = \varkappa^0 \delta(x_1)\,\delta(x_2)$. Thus, we have

$$A_1 = 0\,,\ A_2 = 0\quad A_3 = (\partial_1 + \partial_2)\,\varkappa^0\,\delta(x_1)\,\delta(x_2)\ .$$

Now the representation of stress in terms of the Airy-Mindlin function is unsuitable. Eliminating from Eqs (6.5) and (6.6) in turn the stresses and taking into account the equilibrium equations, we arrive at the system of equations

$$\begin{aligned}
&\nabla_1^2 \mu_{13} = -(\gamma+\epsilon)\,\partial_2 A_3\,, \quad \nabla_1^2 \mu_{23} = (\gamma+\epsilon)\,\partial_1 A_3\,,\\
&\nabla_1^2 \nabla_1^2 \sigma_{11} = p\,\partial_2^2 A_3\,, \qquad \nabla_1^2 \nabla_1^2 \sigma_{22} = p\,\partial_1^2 A_3\,, \\
&\nabla_1^2 \nabla_1^2 \sigma_{12} = -p\,\partial_1 \partial_2 A_3\,, \quad \sigma_{21} = \sigma_{12}\,, \quad p = \frac{4\mu(\lambda+\mu)}{\lambda+2\mu}\ .
\end{aligned} \tag{6.12}$$

Solving them we have

$$\begin{aligned}
&\mu_{13} = \frac{1}{2\pi}(\gamma+\epsilon)\,\varkappa^0\,\partial_2(\partial_1+\partial_2)\,I_1\,, \quad \mu_{23} = -\frac{1}{2\pi}(\gamma+\epsilon)\,\varkappa^0\,\partial_1(\partial_1+\partial_2)\,I_1\,,\\
&\sigma_{11} = \frac{p\varkappa^0}{2\pi}\,\partial_2^2(\partial_1+\partial_2)\,I_2\,, \quad \sigma_{22} = \frac{p\varkappa^0}{2\pi}\,\partial_1^2(\partial_1+\partial_2)\,I_2\,, \\
&\sigma_{12} = -\frac{p\varkappa^0}{2\pi}\,\partial_1\partial_2(\partial_1+\partial_2)\,I_2\,, \quad \sigma_{21} = \sigma_{12}\,,
\end{aligned} \tag{6.13}$$

where

$$I_1 = -(\ln r + C)\,, \quad I_2 = r^2(\ln r + C)\ .$$

Let us now examine the second system of compatibility equations related to the so-called second problem of plane state of strain, when

$$\underline{u} = (0, 0, u_3)\,, \quad \underline{\varphi} = (\varphi_1, \varphi_2, 0)\ . \tag{6.14}$$

This system has the form

$$\begin{cases} \partial_2 \gamma'_{31} + \varkappa'_{22} = \alpha_{11}, \quad -\partial_1 \gamma'_{32} + \varkappa'_{11} = \alpha_{22}, \\ \partial_2 \gamma'_{32} - \varkappa'_{21} = \alpha_{12}, \quad -\partial_1 \gamma'_{31} - \varkappa'_{12} = \alpha_{21}, \\ \partial_1 \gamma'_{23} - \partial_2 \gamma'_{13} + \varkappa'_{11} + \varkappa'_{22} = \alpha_{33}, \end{cases} \tag{6.15}$$

where

$$\begin{cases} \alpha_{11} = -\partial_2 \overset{0}{\gamma}_{31} - \overset{0}{\varkappa}_{22}, \quad \alpha_{22} = \partial_1 \overset{0}{\gamma}_{32} - \overset{0}{\varkappa}_{11}, \\ \alpha_{12} = -\partial_2 \overset{0}{\gamma}_{32} + \overset{0}{\varkappa}_{21}, \quad \alpha_{21} = \partial_1 \overset{0}{\gamma}_{31} + \overset{0}{\varkappa}_{12}, \\ \alpha_{33} = -\partial_1 \overset{0}{\gamma}_{23} + \partial_2 \overset{0}{\gamma}_{13} - \overset{0}{\varkappa}_{11} - \overset{0}{\varkappa}_{22} \ . \end{cases}$$

Making use of the constitutive relations (1.3) we transform Eqs (6.15) to the form in which only stresses occur:

$$\begin{cases} \partial_1^2 \mu_{22} + \partial_2^2 \mu_{11} - \dfrac{\beta}{2(\gamma+\beta)} \nabla_1^2 (\mu_{11} + \mu_{22}) - \partial_1 \partial_2 (\mu_{12} + \mu_{21}) = 2\gamma B_1 \ , \\ (\partial_2^2 - \partial_1^2)(\mu_{12} + \mu_{21}) + \dfrac{\gamma}{\epsilon} \nabla_1^2 (\mu_{12} - \mu_{21}) - 2\, \partial_1 \partial_2 (\mu_{22} - \mu_{11}) = 4\gamma B_2 \ , \\ \partial_1 (\sigma_{23} + \sigma_{32}) - \partial_2 (\sigma_{31} + \sigma_{13}) = 2\mu B_3 \ , \\ \mu_{11} + \mu_{22} + \dfrac{\gamma+\beta}{\alpha} (\partial_2 \sigma_{31} - \partial_1 \sigma_{32}) = (\gamma+\beta)\left(2B_5 - \dfrac{\mu+\alpha}{\alpha} B_3\right) \ , \\ \mu_{12} - \mu_{21} - \dfrac{\epsilon(\mu+\alpha)}{2\mu\alpha} (\partial_1 \sigma_{31} + \partial_2 \sigma_{32}) = 2\epsilon B_4 \ , \end{cases} \tag{6.16}$$

where

$$B_1 = \partial_1^2 \alpha_{11} + \partial_2^2 \alpha_{22} + \partial_1 \partial_2 (\alpha_{11} + \alpha_{21}) \ ,$$

$$B_2 = \partial_1 \partial_2 (\alpha_{22} - \alpha_{11}) + \partial_2^2 \alpha_{21} - \partial_1^2 \alpha_{12} \ ,$$

$$B_3 = \alpha_{33} - (\alpha_{11} + \alpha_{22}) \ , \quad B_4 = \alpha_{21} - \alpha_{12}, \quad B_5 = \alpha_{33} \ .$$

The system of five equations (6.16) contains eight unknown stresses. Completing the compatibility equations (6.16) by the three equilibrium equations

$$\partial_1 \mu_{11} + \partial_2 \mu_{12} + \sigma_{23} - \sigma_{32} = 0 \ , \quad \partial_1 \mu_{12} + \partial_2 \mu_{22} + \sigma_{31} - \sigma_{13} = 0 \ ,$$

$$\partial_1 \sigma_{13} + \partial_2 \sigma_{23} = 0 \ , \tag{6.17}$$

we obtain a system of equations, the number of which is equal to the number of the unknowns.

In the particular case of the distortion $\gamma_{23}^{0}=\gamma^{0}\delta(x_1)\,\delta(x_2)$, the remaining distortions being equal to zero, we have

$$B_1=B_2=B_4=0\ ,\quad B_3=B_5=-\gamma^0\partial_1\delta(x_1)\,\delta(x_2)\ .$$

Eliminating in turn the stresses from Eqs (6.16) and (6.17) we obtain the following equations for the stresses:

$$\left\{\begin{aligned} &D\nabla_1^2\sigma_{13}=-\partial_2 NA_5\ ,\quad D\nabla_1^2\sigma_{23}=\partial_1 NA_5\ ,\\ &D\nabla_1^2\sigma_{31}=-\partial_2 MA_5\ ,\quad D\nabla_1^2\sigma_{32}=\partial_1 MA_5\ .\end{aligned}\right. \tag{6.18}$$

Here

$$N=(\alpha+\mu)\,D+\alpha\ ,\quad M=(\mu-\alpha)\,D-\alpha\ ,\quad D=l^2\nabla_1^2-1\ ,\quad A_5=-\gamma^0\partial_1\delta(x_1)\,\delta(x_2)\ .$$

Solving the above equations we obtain

$$\left\{\begin{aligned} &\sigma_{13}=\frac{\gamma^0}{2\pi}\,\partial_1\partial_2(\mu I_1+\alpha\overset{\nu}{I}_2)\ ,\qquad \sigma_{31}=\frac{\gamma^0}{2\pi}\,\partial_1\partial_2(\mu I_1-\alpha\overset{\nu}{I}_2)\ ,\\ &\sigma_{23}=-\frac{\gamma^0}{2\pi}\,\partial_1^2(\mu I_1-\alpha\overset{\nu}{I}_2)\ ,\qquad \sigma_{32}=-\frac{\gamma^0}{2\pi}\,\partial_1^2(\mu I_1+\alpha\overset{\nu}{I}_2)\ ,\end{aligned}\right. \tag{6.19}$$

where

$$I_1=-(\ln r+C)\ ,\quad \overset{\nu}{I}_2=K_0\left(\frac{r}{\nu}\right)\ ,\quad \nu=\left(\frac{2\gamma+\beta}{4\alpha}\right)^{1/2}\ .$$

Eliminating the stresses from Eqs (6.16) and (6.17) and making use of Eqs (6.18) we arrive at the following system of differential equations for the couple stresses:

$$\begin{aligned} &D\nabla_1^2\mu_{22}=-\left(\gamma\partial_2^2+\frac{\beta}{2}\nabla_1^2\right)A_5\ ,\\ &D\nabla_1^2\mu_{11}=-\left(\gamma\partial_1^2+\frac{\beta}{2}\nabla_1^2\right)A_5\ ,\\ &D\nabla_1^2\mu_{12}=-2\gamma\partial_1\partial_2 A_5\ ,\quad \mu_{21}=\mu_{12}\ ,\quad A_5=-\gamma^0\partial_1\delta(x_1)\,\delta(x_2)\ .\end{aligned} \tag{6.20}$$

Solving them we obtain the formulae

$$\left\{\begin{aligned} \mu_{11} &= -\frac{\gamma^{0}}{2\pi}\,\partial_1\,[\frac{\beta}{\nu^2}\overset{\nu}{I}_2 - 2\gamma\partial_1^2(I_1-\overset{\nu}{I}_2)]\ , \\ \mu_{22} &= -\frac{\gamma^{0}}{2\pi}\,\partial_1\,[\frac{\beta}{\nu^2}\overset{\nu}{I}_2 - 2\gamma\partial_2^2(I_1-\overset{\nu}{I}_2)]\ , \\ \mu_{12} &= \mu_{21} = -\frac{\gamma\gamma^{0}}{\pi}\,\partial_1^2\partial_2(I_1-\overset{\nu}{I}_2)\ . \end{aligned}\right. \tag{6.21}$$

A determination of the singular solutions (the Green functions) for the distortion components makes it possible to calculate by integration the state of stress due to distortions distributed over an arbitrary bounded region Γ.

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