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Propagation of Magnetoelastic Disturbances in Viscoelastic Bodies

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1. Introduction

In the present paper, we formulate the equations of magnetoviscoelasticity and appropriate boundary conditions for a linearly viscoelastic medium with finite conductivity. In particular, we shall consider the boundary conditions for an ideal medium, that is perfectly elastic and perfectly conductive.

In magnetohydrodynamics, the boundary conditions on the surface whose normal is parallel to the vector of the initial magnetic field, assume different forms depending on the manner in which the transition to vanishing mechanical and magnetic viscosities is effected [*1*, *2*].

The same problem arises in magnetoviscoelasticity for the various models of viscoelastic bodies. In particular, the boundary conditions on the contact surface of a liquid and solid body must be discussed.

If, in a boundary problem which concerns the contact of a liquid and a solid body, we start with the equations of the ideal liquid and the elastic solid, and assume perfect conductivity of both media, we obtain a unique form of the boundary conditions. Surface waves on the contact surface of a liquid and a solid were discussed in this manner in [*3*].

If, on the other hand, we start with the equations for viscous media with finite conductivity and then pass to the limits corresponding to the elastic body and the ideal liquid of perfect conductivity, we obtain different forms of the boundary conditions, which depend on the manner in which the transition to vanishing mechanical and magnetic viscosities has been effected.

In Section 2 of this paper we present general and simplified equations for conductors with finite electric conductivity. Section 3 is devoted to general boundary conditions, while Section 4 deals with the transition to the ideal elastic conductor. In Section 5, we present the forms of the limit passage for different models and for contact between a liquid and a solid when the magnetic field vector is normal to the contact surface.

Finally, in Section 6, we consider a simple example of the transmission of an elastic wave through the contact surface of the two media under various types of boundary conditions.

2. General Equations

Let us consider a linear viscoelastic medium with finite electric conductivity and suppose that an initial magnetic field exists in this medium. The action of body forces and external loads produces not only a deformation field but also a coupled electromagnetic field. We assume that the medium is isotropic and homogeneous and disregard the coupled thermal effects as well as the effects of the relaxation of the magnetic and electric induction. These effects have been considered in [4, 5].

The points of departure of our consideration are the equations of magneto-viscoelasticity which after being linearized [5] assume the following form:

$$\operatorname{curl} \boldsymbol{E} = -\frac{\mu_0}{c}\frac{\partial \boldsymbol{h}}{\partial t},$$

$$\operatorname{curl} \boldsymbol{h} = \frac{4\pi}{c}\boldsymbol{j} + \frac{\varepsilon}{c}\frac{\partial \boldsymbol{E}}{\partial t} + \frac{\varepsilon\mu_0 - 1}{c^2}\frac{\partial}{\partial t}\left(\frac{\partial \boldsymbol{u}}{\partial t}\times \boldsymbol{H}\right),$$

$$\boldsymbol{j} = \eta\left[\boldsymbol{E} + \frac{\mu_0}{c}\left(\frac{\partial \boldsymbol{u}}{\partial t}\times \boldsymbol{H}\right)\right], \tag{2.1}$$

$$\operatorname{div} \boldsymbol{h} = 0, \qquad \operatorname{div} \boldsymbol{D} = 0,$$

$$\boldsymbol{D} = \varepsilon\left[\boldsymbol{E} + \frac{1}{c}\frac{\mu_0\varepsilon - 1}{\varepsilon}\left(\frac{\partial \boldsymbol{u}}{\partial t}\times \boldsymbol{H}\right)\right].$$

$$\int_0^t \left\{a(t-\tau)\frac{\partial}{\partial \tau}\nabla^2\boldsymbol{u} + [a(t-\tau) + b(t-\tau)]\frac{\partial}{\partial \tau}\operatorname{grad}\operatorname{div}\boldsymbol{u}\right\}d\tau + \frac{\mu_0}{c}(\boldsymbol{j}\times\boldsymbol{H}) + \boldsymbol{X} = \varrho\frac{\partial^2\boldsymbol{u}}{\partial t^2}. \tag{2.2}$$

The system comprises the equations of electrodynamics of slowly moving media. The vectors $\boldsymbol{h}$ and $\boldsymbol{E}$ indicate the magnetic and electric fields, respectively, $\boldsymbol{j}$ is the vector of current density, $\boldsymbol{D}$ the vector of electric induction, $\boldsymbol{H}$ the vector of the initial constant magnetic field, c the velocity of light in vacuum; μ_0 and ε are the magnetic and electric permeabilities, and η is the electric conductivity.

Eqs. (2.2) are the equations of motion of a viscoelastic medium. Here, $\boldsymbol{u}$ denotes the displacement vector, $\boldsymbol{X}$ the body force per unit volume, while $a(t)$ and $b(t)$ are relaxation functions of the viscoelastic medium, and ϱ is its density. For $\boldsymbol{H} = 0$, Eqs. (2.2) reduce to the well-known equations of viscoelasticity [6].

Restricting our considerations to good conductors we disregard the displacement currents. Thus, for $\varepsilon = \mu_0 = 1$, neglecting the term $(\mu_0/c)\,\partial \boldsymbol{E}/\partial t$ and eliminating the magnitudes of $\boldsymbol{E}$ and $\boldsymbol{j}$, we obtain the following set of equations

$$\frac{\partial \boldsymbol{h}}{\partial t} - \operatorname{curl}\left(\frac{\partial \boldsymbol{u}}{\partial t} \times \boldsymbol{H}\right) - \lambda_m \nabla^2 \boldsymbol{h} = 0, \qquad \lambda_m = \frac{c^2}{4\pi\mu_0\eta}$$

$$\int_0^t \left\{ a(t-\tau)\frac{\partial}{\partial t}\nabla^2 \boldsymbol{u} + [a(t-\tau) + b(t-\tau)\frac{\partial}{\partial t}\operatorname{grad}\operatorname{div}\boldsymbol{u}\right\} d\tau$$

$$+ \boldsymbol{X} + \frac{\mu_0}{c}(\boldsymbol{j} \times \boldsymbol{H}) = \varrho \frac{\partial^2 \boldsymbol{u}}{\partial t^2}, \tag{2.3}$$

where

$$\boldsymbol{E} = \frac{\lambda_m}{c}\operatorname{curl}\boldsymbol{h} - \frac{\mu_0}{c}\left(\frac{\partial \boldsymbol{u}}{\partial t} \times \boldsymbol{H}\right), \qquad \boldsymbol{j} = \frac{c}{4\pi}\operatorname{curl}\boldsymbol{h},$$

λ_m is the magnetic viscosity.

As in problems of viscoelasticity, an elastic-viscoelastic analogy can be formulated here. Suppose that external loads and body forces have been applied at the moment $t = 0+$, and that all load and body forces are proportional to the same function of time. After performing the one-sided Laplace transform, we may write Eqs. (2.3) in the form

$$p[\overline{\boldsymbol{h}} - \operatorname{curl}(\overline{\boldsymbol{u}} \times \boldsymbol{H})] - \lambda_m \nabla^2 \overline{\boldsymbol{h}} = 0,$$

$$\overline{\mu}\nabla^2 \overline{\boldsymbol{u}} + (\overline{\mu} + \overline{\lambda})\operatorname{grad}\operatorname{div}\overline{\boldsymbol{u}} + \frac{\mu_0}{c}(\overline{\boldsymbol{j}} \times \boldsymbol{H}) + \overline{\boldsymbol{X}} = \varrho p^2 \overline{\boldsymbol{u}}, \tag{2.4}$$

where

$$\overline{\mu}(p) = p\overline{a}(p), \qquad \overline{\lambda}(p) = p\overline{b}(p),$$

$\overline{a}$, $\overline{b}$ being the Laplace transformation of the relaxation functions $a(t)$ and $b(t)$.

Let us compare Eqs. (2.4) with the corresponding equations of magnetoelasticity:

$$p[\overline{\boldsymbol{h}}_0 - \operatorname{curl}(\overline{\boldsymbol{u}}_0 \times \boldsymbol{H})] - \lambda_m \nabla^2 \overline{\boldsymbol{h}}_0 = 0,$$

$$\mu\nabla^2 \overline{\boldsymbol{u}}_0 + (\mu + \lambda)\operatorname{grad}\operatorname{div}\overline{\boldsymbol{u}}_0 + \frac{\mu_0}{c}(\overline{\boldsymbol{j}}_0 \times \boldsymbol{H}) + \overline{\boldsymbol{X}} = \varrho p^2 \overline{\boldsymbol{u}}_0, \tag{2.5}$$

where u_0, h_0, j_0, etc., refer to perfectly elastic bodies of finite electric conductivity. The comparison shows that instead of solving a magneto-viscoelastic problem we may solve the "associated" magnetoelastic problem. When its solution has been obtained, the LAMÉ constants μ, λ are replaced by the quantities $\overline{\mu}$, $\overline{\lambda}$, which are the functions of the parameter p; the inverse LAPLACE transform of u_0 then furnishes the solution of the magnetoviscoelastic problem. This analogy, however, is only valid when the load and body forces as well as the displacements in the boundary conditions of the problem are proportional to the same function of time.

3. Boundary Conditions

Let us now discuss the boundary conditions for two viscoelastic media of finite electric conductivity with a plane contact surface. Suppose that the vector of the initial magnetic field is arbitrarily oriented with respect to the plane of the contact. Taking into account the homogeneous initial conditions assumed in Section 2, we establish the boundary conditions for the Laplace transforms of the mechanical and electromagnetic quantities.

Two types of boundary conditions will be considered:

(a) boundary conditions for two viscoelastic media in contact, and

(b) boundary conditions for a viscoelastic medium in contact with a vacuum.

In case (a) we obtain a set of twelve boundary conditions (see [7]). The indices (1) and (2) denote the two media, and $\boldsymbol{n}$ the unit vector normal to the plane of contact. We have

$$\overline{\boldsymbol{u}}^{(1)} = \overline{\boldsymbol{u}}^{(2)}, \qquad \overline{\sigma}_{ni}^{(1)} + \overline{T}_{ni}^{(1)} = \overline{\sigma}_{ni}^{(2)} + \overline{T}_{ni}^{(2)},$$

$$\overline{D}_n^{(1)} = \overline{D}_n^{(2)}, \qquad \overline{b}_n^{(1)} = \overline{b}_n^{(2)},$$

$$[\boldsymbol{n} \times (\overline{\boldsymbol{E}}^{(2)} - \overline{\boldsymbol{E}}^{(1)})] = \frac{p\overline{u}_n}{c} (\mu_0^{(2)} - \mu_0^{(1)}) \boldsymbol{H}_t,$$

$$[\boldsymbol{n} \times (\overline{\boldsymbol{h}}^{(2)} - \overline{\boldsymbol{h}}^{(1)})] = 0, \tag{3.1}$$

where

$$\overline{\sigma}_{ik} = \overline{\mu}(\overline{u}_{i,k} + \overline{u}_{k,i}) + \delta_{ik}\overline{u}_{k,k},$$

$$\overline{T}_{ik} = \frac{\mu_0}{4\pi} [H_i \overline{h}_k + H_k \overline{h}_i - \delta_{ik} \overline{\boldsymbol{h}} \cdot \boldsymbol{H}],$$

$$\boldsymbol{B} = \boldsymbol{B}_0 + \boldsymbol{b} = \mu_0(\boldsymbol{H} + \boldsymbol{h}). \tag{3.2}$$

The system (3.1) contains the conditions of continuity of the transforms of the displacements, the conditions of the equality of the transforms of the sums of the mechanical stresses and the components of MAXWELL's tension tensor, and the conditions of continuity of the transforms of the normal inductions and tangential fields for the moving boundary surface. The last two conditions are given in linearized form using an approximation for the terms in v/c and without introducing two-sided values of the tangent field $\boldsymbol{H}_t$. If the quantities $\mu^{(\alpha)}$, $\alpha = 1.2$ of the media are approximately equal, then the term connected with the motion of the contact surface may be disregarded in (3.1). If, in the boundary conditions of the type (b), the medium characterized by subscript 1 is the vacuum, we must set

$$\mu_0^{(1)} = 1, \quad \sigma_{ik}^{(1)} = 0, \quad b_n^{(1)} = h_n^{(1)}, \quad D_n^{(1)} = E_n^{(1)} \quad \text{in} \quad (3.1).$$

4. Limit Passage to the Ideal Conductor

Lettering $\lambda_m \to 0$ in (2.3) and replacing the relaxation functions $a(t)$, $b(t)$ by the LAMÉ constants μ, λ, we pass from a viscoelastic body with finite electric conductivity to the perfectly elastic and perfectly conducting body. Eliminating the quantities $\boldsymbol{E}$ and $\boldsymbol{h}$ from (2.3), we then obtain the following equations:

$$\mu\nabla^2\boldsymbol{u} + (\lambda + \mu)\,\text{grad div}\,\boldsymbol{u} + \frac{\mu_0}{4\pi}\,[\text{curl curl}\,(\boldsymbol{u} \times \boldsymbol{H})] \times \boldsymbol{H} + \boldsymbol{X} = \varrho\,\frac{\partial^2\boldsymbol{u}}{\partial t^2}, \tag{4.1}$$

where

$$\boldsymbol{E} = -\frac{\mu_0}{c}\left(\frac{\partial\boldsymbol{u}}{\partial t} \times \boldsymbol{H}\right), \qquad \boldsymbol{h} = \text{curl}\,(\boldsymbol{u} \times \boldsymbol{H}). \tag{4.2}$$

For a perfectly conducting viscoelastic body, it is sufficient to assume that $\lambda_m \to 0$. We then obtain the equation of motion

$$\int_0^t \left\{a(t-\tau)\,\frac{\partial}{\partial\tau}\,\nabla^2\boldsymbol{u} + [a(t-\tau) + b(t-\tau)]\,\frac{\partial}{\partial\tau}\,\text{grad div}\,\boldsymbol{u}\right\} d\tau + \frac{\mu_0}{4\pi}\,[\text{curl curl}\,(\boldsymbol{u} \times \boldsymbol{H})] \times \boldsymbol{H} + \boldsymbol{X} = \varrho\,\frac{\partial^2\boldsymbol{u}}{\partial t^2}, \tag{4.3}$$

the boundary conditions (4.2) remaining valid.

Note that with $\mu = 0$ and $\partial\boldsymbol{u}/\partial t \not\equiv 0$ Eq. (4.1) reduces to the acoustic equation of an ideal fluid. Similarly, taking coefficients $a(t)$, $b(t)$ in (4.3) that correspond to VOIGT's model with $\bar{a} = 0$, we obtain the acoustic equation of a viscous liquid.

Passing from a body with mechanical and magnetic viscosities to a body that is perfectly conductive and elastic, we must also make the appropriate transitions to the limit in the boundary conditions. Here, the manner in which the ratio of the magnetic and mechanical viscosities in the liquid and in the solid tends to zero is essential. This concerns the coefficients of viscosity when the initial field $\boldsymbol{H}$ is perpendicular to the contact surface (see Section 5).

In the present section, we give the boundary conditions for the case when the vector of the initial magnetic field is parallel to the contact plane. We consider two basic types of contact

(a) The displacements of the media are continuous across the contact surface. We then have the following boundary conditions:

$$\overline{\boldsymbol{u}}^{(1)} = \overline{\boldsymbol{u}}^{(2)}, \qquad \overline{\sigma}_{ni}^{(1)} + \overline{T}_{ni}^{(1)} = \overline{\sigma}_{ni}^{(2)} + \overline{T}_{ni}^{(2)}, \tag{4.4}$$

where the quantities $\overline{T}_{ni}$ are given by (3.2) and the quantities $\overline{h}_i$ by (4.2). The quantities $\overline{T}_{ik}$ can be then expressed explicitly in terms of the displacements $\overline{u}_i$.

(b) In the contact plane there appears the ideal tangential slip. The boundary conditions now take the form

$$\begin{aligned} \overline{\sigma}_{33}^{(1)} + \overline{T}_{33}^{(1)} = \overline{\sigma}_{33}^{(2)} + \overline{T}_{33}^{(2)}, \qquad \overline{u}_n^{(1)} = \overline{u}_n^{(2)}, \\ \overline{\sigma}_{ni}^{(1)} = 0, \qquad \overline{\sigma}_{ni}^{(2)} = 0, \qquad i = 1,2. \end{aligned} \tag{4.5}$$

The second condition (4.5) can be replaced by the relation between the tangential components of the vectors h^{α} $(\alpha = 1, 2)$ and the surface currents [7].

5. Passage to the Limit in Boundary Conditions for H=H$_3$

If the initial magnetic field is perpendicular to the contact surface then the boundary conditions assume one of two forms depending on how the mechanical and magnetic viscosities tend to zero. These boundary conditions will be discussed for a particular one-dimensional problem with shear strains since these influence the alternative forms of the boundary conditions.

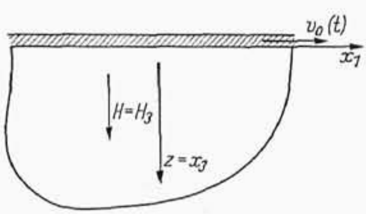

Fig. 1.

Let us consider a rigid plate on the surface of a viscoelastic semi-space with finite electric conductivity. This plate is set in motion in a tangential direction, the initial conditions being assumed homogeneous (Fig. 1). The equations of motion (2.3), then contain the function $a(t)$ and the dependent variables $u_1(z, t) = u$ and $h_1(z, t) = h$. The remaining quantities either are equal to zero or do not enter the problem. Thus, the system (2.3) takes the form

$$\begin{gathered}\lambda_m \frac{\partial^2 h}{\partial z^2} - \frac{\partial h}{\partial t} + H \frac{\partial^2 u}{\partial t \partial z} = 0, \\ \int_0^t a(t-\tau) \frac{\partial^3 u}{\partial z^2 \partial \tau} d\tau + \frac{H}{4\pi} \frac{\partial h}{\partial z} = \varrho \frac{\partial^2 u}{\partial t^2}, \qquad H_3 = H.\end{gathered} \tag{5.1}$$

Neglecting the inertia term in the equations of motion, we treat the problem as quasi-static. This does not influence the physical conditions on the boundary. After performing the Laplace transform on Eqs. (5.1), the initial conditions being assumed homogeneous, and after solving these equations we obtain the following relations:

$$\begin{gathered}\overline{u} - \overline{u}_\infty = C_1 e^{-\overline{\varkappa} z}, \\ \overline{h} - \overline{h}_\infty = \frac{H \overline{\varkappa}}{a_0^2 \overline{\nu}^2} C_1 e^{-\overline{\varkappa} z},\end{gathered} \tag{5.2}$$

where

$$\overline{\varkappa}^2 = \frac{p}{\lambda_m}(1 + a_0^2 \overline{\nu}^2), \qquad a_0^2 = \frac{H^2}{4\pi\varrho}, \qquad \overline{\nu}^2 = \frac{\varrho}{\overline{\mu}},$$

$$\overline{\mu} = \overline{a} p, \qquad \overline{u}_\infty = \overline{u}(z, p)|_{z=\infty}, \qquad \overline{h}_\infty = \overline{h}(z, p)|_{z=\infty}.$$

Making use of the boundary condition for $z = 0$,

$$\overline{v}(0, p) = \overline{v}_0, \tag{5.3}$$

we obtain

$$\overline{v} - \overline{v}_\infty = (\overline{v}_0 - \overline{v}_\infty) e^{-\overline{\varkappa} z}, \tag{5.4}$$

$$\overline{h} - \overline{h}_\infty = \frac{H \overline{\varkappa}}{a_0^2 \overline{\nu}^2} (\overline{v}_0 - \overline{v}_\infty) e^{-\overline{\varkappa} z}. \tag{5.5}$$

Here, $\overline{v} = p \overline{u}$ is the transform of the velocity.

It follows from relations (5.4) and (5.5) that

$$\overline{v}_0 - \overline{v}_\infty = \frac{a_0^2 \overline{\nu}^2}{H} \left[\frac{\lambda_m p}{1 + a_0^2 \overline{\nu}^2}\right]^{\frac{1}{2}} (\overline{h}_0 - \overline{h}_\infty). \tag{5.6}$$

Eq. (5.6) is essential for the discussion of possible combinations of boundary conditions when the mechanical and magnetic viscosities tend to zero. For a solid body with $\overline{\mu} \neq 0$, when the magnetic viscosity λ_m as well as the parameters characterizing the mechanical viscosity tend simultaneously to zero, condition (5.6) always yields

$$\overline{v}_0 = \overline{v}_\infty, \tag{5.7}$$

that is the continuity of tangential velocities. For the Voigt model with $\overline{\mu} = \mu(1 + p\tau)$, $\tau = \beta/\mu$, where β is the coefficient of viscosity, Eq. (5.6) takes the form

$$\overline{v}_0 - \overline{v}_\infty = \frac{a_0^2 \nu_0^2}{H} \sqrt{\frac{p\lambda_m}{(1 + p\tau)(1 + p\tau + a_0^2\nu_0^2)}} (\overline{h}_0 - \overline{h}_\infty), \qquad \nu_0^2 = \frac{\varrho}{\mu}. \quad (5.8)$$

It is apparent from (5.8) that in the limit we obtain relation (5.7) when λ_m and τ tend to zero in such a way that their ratio remains constant. For the MAXWELL model, we have $\overline{\mu} = \frac{p\tau}{1 + p\tau}$, where $\tau = \beta/\mu$. Eq. (5.6) therefore furnishes

$$\overline{v}_0 - \overline{v}_\infty = \frac{a_0^2 \nu_0^2}{H} \sqrt{\frac{\lambda_m (1 + p\tau)^2}{\tau[p\tau(1 + a_0^2\nu_0^2) + a_0^2\nu_0^2]}} (\overline{h}_0 - \overline{h}_\infty). \quad (5.9)$$

If the process is assumed, in the limit, to be stationary, (5.9) becomes

$$v_0 - v_\infty = \sqrt{\frac{\lambda_m}{4\pi\beta}} (h_0 - h_\infty). \quad (5.10)$$

It is apparent that when $\lambda_m \to 0$ while $\beta \neq 0$, in the limit we obtain the boundary condition (5.7). If $\beta \to 0$, then according to the three possibilities

$$\frac{\lambda_m}{\beta} \to 0, \qquad \frac{\lambda_m}{\beta} \to \infty, \qquad \frac{\lambda_m}{\beta} \to s, \quad (5.11)$$

where s is a constant, we obtain the following boundary conditions:

$$\overline{v}_0 = \overline{v}_\infty, \qquad \overline{h}_0 = \overline{h}_\infty, \qquad \overline{v}_0 - \overline{v}_\infty = \sqrt{\frac{s}{4\pi}} (\overline{h}_0 - \overline{h}_\infty) \quad (5.12)$$

respectively. The variant $\beta \to 0$ is physically artificial, because it implies a vanishing time of relaxation, and consequently excludes elastic stresses. For the stationary laminar flow of a viscous liquid, (5.8) yields

$$v_0 - v_\infty = \sqrt{\frac{\lambda_m}{4\pi\beta'}} (h_0 - h_\infty). \quad (5.13)$$

Here, β' denotes the coefficient of viscosity of the liquid. For $\beta' \to 0$ the liquid becomes ideal. Depending on the manner in which λ_m and β' tend to zero [see (5.11)], we obtain three variants of the boundary conditions corresponding to the physically significant cases.

When two media, for instance a viscous liquid and a viscoelastic solid are in contact, we obtain the following equations for the MAXWELL model (here the process may also be stationary)

$$v(z) - v_\infty = \sqrt{\frac{\lambda_m}{4\pi\beta}}\,(h(z) - h_\infty),$$
$$v(z) - v_\infty = \sqrt{\frac{\lambda_m}{4\pi\beta'}}\,(h(z) - h_\infty). \tag{5.14}$$

The first of these equations concerns the solid with the viscosity β, and the second the viscous liquid with the viscosity β'. It follows from these equations that on the contact surface $z = 0$ of these media the variants (5.12) of the boundary conditions may occur.

We conclude that a condition of the type $h^{(1)} = h^{(2)}$ may obtain, provided the surface currents vanish in the transition from the visco-elastic to the perfect conductor. The disappearance of the surface currents is identical with the disappearance of the tangential component of the stress tensor. This case occurs for a liquid, for $\beta'/\lambda_m \to 0$. It also appears for the Maxwell model for $\beta/\lambda_m \to 0$; this case, however, should be regarded as unrealistic.

6. Example

The simple example that will now be discussed, illustrates the extreme alternatives of the boundary conditions. Consider two perfectly conductive media: an ideal liquid and a perfectly elastic solid, and suppose the initial magnetic field to be perpendicular to their plane of contact. Assume moreover that a tangential force, varying harmonically with time, acts in this plane. We consider the one-dimensional problem in the x, z plane.

The liquid in the negative half-space $z < 0$ is denoted by the index (2) and the elastic solid in the half-space $z \geq 0$ by the index (1). The magnetic permeability of both media is assumed to be 1.

The equations of motion of the media take the form

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{a^2}\frac{\partial^2 u}{\partial t^2} = 0, \tag{6.1}$$

$$h = H\frac{\partial u}{\partial z}. \tag{6.2}$$

In these equations, the symbols

$$u = u^{(2)}, \qquad h = h^{(2)}, \qquad a^2 = {}_2a_0^2 = H^2/(4\pi\varrho_2)$$

should be used for the liquid, and the symbols

$$u = u^{(1)}, \qquad h = h^{(1)}, \qquad a^2 = {}_1a_0^2 + c^2, \qquad {}_1a_0^2 = \frac{H^2}{4\pi\varrho_1}, \qquad c^2 = \frac{\mu}{\varrho_1}$$

for the elastic solid. According to the discussion in the preceding section we consider two extreme variants of the boundary conditions corresponding to the transition to the limit in the liquid:

$$1.\ \frac{\lambda_m}{\beta'} = 0; \qquad 2.\ \frac{\lambda_m}{\beta'} = \infty.$$

For the first boundary condition, we have the following relations for $z = 0$:

$$\begin{aligned} u^{(1)} &= u^{(2)}, \qquad \sigma_{31}^{(1)} + T_{31}^{(1)} = T_{31}^{(2)} + Pe^{i\omega t} \\ \text{or} \qquad & \varrho_1 a^2 \frac{\partial u^{(1)}}{\partial z} - \varrho_{22} a_0^2 \frac{\partial u^{(2)}}{\partial z} = Pe^{i\omega t}. \end{aligned} \tag{6.3}$$

For the second boundary condition we find for $z = 0$

$$\begin{aligned} h^{(1)} &= h^{(2)} \quad \text{or} \quad \frac{\partial u^{(1)}}{\partial z} = \frac{\partial u^{(2)}}{\partial z}, \\ \sigma_{31}^{(1)} &= Pe^{i\omega t} \quad \text{or} \quad \varrho_1 c^2 \frac{\partial u^{(1)}}{\partial z} = Pe^{i\omega t}. \end{aligned} \tag{6.4}$$

Setting

$$u(z, t) = u^*(z) e^{i\omega t}, \tag{6.5}$$

we obtain

$$u^{*(1)} = A_1 e^{-i\alpha_1 z}, \qquad u^{*(2)} = A_2 e^{i\alpha_2 z}, \tag{6.6}$$

where

$$\alpha_1^2 = \frac{\omega^2}{a^2}, \qquad \alpha_2^2 = \frac{\omega^2}{{}_2a_0^2}.$$

Taking into account the boundary conditions (6.3), we obtain

$$\begin{aligned} u^{(1)} &= -\frac{P}{\Delta} e^{i(\omega t - \alpha_1 z)} \qquad u^{(2)} = -\frac{P}{\Delta} e^{i(\omega t + \alpha_2 z)}, \\ h^{(1)} &= \frac{HP\alpha_1}{\Delta} e^{i(\omega t - \alpha_1 z)} \qquad h^{(2)} = -\frac{HP\alpha_2}{\Delta} e^{i(\omega t + \alpha_2 z)}, \end{aligned} \tag{6.7}$$

where

$$\Delta = \varrho_1 a^2 \alpha_1 + \varrho_{22} a_0^2 \alpha_2.$$

For the boundary conditions (6.4) we find

$$\begin{aligned} u^{(1)} &= -\frac{P}{\mu\alpha_1} e^{i(\omega t - \alpha_1 z)} \qquad u^{(2)} = \frac{P}{\mu\alpha_2} e^{i(\omega t + \alpha_2 z)}, \\ h^{(1)} &= \frac{PH}{\mu} e^{i(\omega t - \alpha_1 z)} \qquad h^{(2)} = \frac{PH}{\mu} e^{i(\omega t + \alpha_2 z)}. \end{aligned} \tag{6.8}$$

As is readily seen from (6.7) and (6.8), we have radiation of waves of identical displacement or velocity amplitudes in the first case and identical amplitudes of the field h or the derivative of the displacement in the second case. Thus, depending on the manner in which the mechanical and magnetic viscosities in the liquid tend to zero, we have obtained different solutions to the contact problem between a liquid and a solid.

The manner in which the viscosities tend to zero in the solid has no influence on the solution for all physically realistic cases.

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