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On the stress functions for thermodiffusion in solids

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RIASSUNTO - *Il lavoro contiene una rappresentazione dello spostamento* **u**, *della temperatura* ϑ_1 *e del potenziale chimico* ϑ_2 *per mezzo: degli sforzi, di una funzione vettoriale* χ *e di due funzioni scalari* ψ_1, ψ_2. *Questa rappresentazione, che costituisce una generalizzazione di quella di Galerkin nell'Elastostatica, implica che ciascuna delle funzioni* χ, ψ_1, ψ_2 *verifichi la semplice equazione delle onde. Viene provata la completezza delle soluzioni delle equazioni differenziali della termodiffusione fornite dalla suddetta rappresentazione.*

1. Introduction.

Consider an elastic body which undergoes a deformation under the action of external factors of mechanical, thermal or diffusional character. The external factors are the following sources: body forces $\mathbf{X}$, heat W_1 and the mass being diffused into the body W_2. Furthermore, the external factors include those entering the boundary conditions, namely the prescribed surface traction, displacements on the boundary, heating (or cooling) of the boundary of the body and finally the given on the boundary concentration or the chemical potential. The initial sources are also included into the external factors.

The results are the displacement $\mathbf{u}$, temperature θ_1, and the chemical potential θ_2 inside of the body. We assume that all resulting quantities depend on the position x and time t.

The system of the partial differential equations describing the process of thermodiffusion in an elastic solid is the following [1], [2], [3]:

$$(1.1) \qquad \Box_2 \mathbf{u} + (\lambda + \mu) \operatorname{grad} \operatorname{div} \mathbf{u} + \mathbf{X} = \operatorname{grad} (\gamma_1 \theta_1 + \gamma_2 \theta_2)$$

$$(1.2) \qquad D_1 \theta_1 = \gamma_1 \operatorname{div} \dot{\mathbf{u}} + d\dot{\theta}_2 - W_1 ,$$

$$(1.3) \qquad D_2 \theta_2 = \gamma_2 \operatorname{div} \dot{\mathbf{u}} + d\dot{\theta}_1 - W_2 ,$$

where we have introduced the differential operators

$$\Box_2 = \mu \nabla^2 - \varrho \partial_t^2, \; D_1 = k_1 \nabla^2 - c_1 \partial_t , \; D_2 = k_2 \nabla^2 - c_2 \partial_t ,$$

$$\nabla^2 = \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_i} , \; \partial_t = \frac{\partial}{\partial t}, \; \dot{\mathbf{u}} = \frac{\partial \mathbf{u}}{\partial t} .$$

Eq. (1.1) is the vectorial displacement equation of the theory of elasticity completed by the thermal and diffusional terms. Eq. (1.2) constitutes the generalized heat conduction equation and (1.3) the generalized diffusional equation. The above equations are coupled.

By μ, λ we denoted the material Lamé constants, ρ is the density, $\gamma_1 = 3 K \alpha_1$, $\gamma_2 = 3 K \alpha_2$, where $K = \lambda + \frac{2}{3} \mu$ is the compression modulus and α_1, α_2 the coefficients of thermal and diffusional expansion. k_1 is the coefficient of heat conduction, k_2 the diffusion coefficient, c_1 the specific heat at constant deformation and chemical potential. The quantities $c_2 > 0$, d are material constants appearing in the constitutive equation for the concentration.

The system of equations (1.1) - (1.3) is very complicated and difficult to solve. Consequently, we shall attempt to reduce it to a simpler wave equations by means of an introduction of special representation for $\mathbf{u}, \theta_1, \theta_2$. We present here a representation constituting a generalization of M. Iacovache's representation [4], who applied it to problems of dynamic elasticity. Furthermore, we shall prove the completeness of the solutions obtained by means of this representation. Finally, we shall present relations between our representation and the Lamé representation [3].

2. The representation of functions $\mathbf{u}$, θ_1, θ_2, by functions χ, ψ_1, ψ_2.

To separate a system of partial differential equations we usually apply G. C. Moisil's method [5]. However, the method of associated matrices is hardly suitable for our system of five equations. Consequently, we shall apply a different method, taking into account the structure of Eqs. (1.1) - (1.3).

First apply to Eq. (1.1) the divergence operation and then operation $\Box_1$ making us of the relation (2.1). Thus, we arrive at the equations

$$(2.1)\qquad \Box_1 \operatorname{div} \mathbf{u} + \operatorname{div} \mathbf{X} = \nabla^2(\gamma_1 \theta_1 + \gamma_2 \theta_2),\ \Box_1 = (\lambda + 2\mu)\nabla^2 - \varrho\partial^2_t.$$

$$(2.2)\qquad \Box_1 \Box_2 \mathbf{u} = -\Box_1 \mathbf{X} + (\lambda + \mathbf{X}) \operatorname{grad} \operatorname{div} \mathbf{X} + \operatorname{grad} \Box_2 (\gamma_1 \theta_1 + \gamma_2 \theta_2).$$

Eliminating from Eqs. (1.2) and (1.3) the quantity div u and bearing in mind (2.1) we obtain the system of equations

$$(2.3)\qquad H_1 \theta_1 = \Gamma\theta_2 - \gamma_1 \partial_t \operatorname{div} \mathbf{X} - \Box_1 W_1,$$

$$(2.4)\qquad H_2 \theta_2 = \Gamma\theta_1 - \gamma_2 \partial_t \operatorname{div} \mathbf{X} - \Box_1 W_2,$$

where

$$H_1 = D_1 \Box_1 - \gamma_1^2 \partial_t \nabla^2,\ H_2 = D_2 \Box_1 - \gamma_2^2 \partial_t \nabla^2,\ \Gamma = (d\Box_1 + \gamma_1 \gamma_2 \nabla^2)\partial_t.$$

A further elimination in Eqs. (2.3) and (2.4) leads to the equations

$$(2.5)\qquad \Omega\theta_1 = -P_1 \partial_t \operatorname{div} \mathbf{X} - \Box_1 H_2 W_1 - \Box_1 \Gamma W_2,$$

$$(2.6)\qquad \Omega\theta_2 = -P_2 \partial_t \operatorname{div} \mathbf{X} - \Box_1 H_1 W_2 - \Box_1 \Gamma W_1,$$

where

$$\Omega = H_1 H_2 - \Gamma^2,\ P_1 = \gamma_1 H_2 + \gamma_2 \Gamma,\ P_2 = \gamma_1 \Gamma + \gamma_2 H_1.$$

Observe that the right-hand sides of Eqs. (2.5) and (2.6) contain only the sources $\mathbf{X}$, W_1, W_2.

Eliminating from Eqs. (2.2), (2.5) and (2.6) the functions θ_1, θ_2 we arrive at the equations

$$(2.7)\qquad \Omega\Box_2 \mathbf{u} = -\Omega\mathbf{X} + \operatorname{grad} \Box_1 M\mathbf{X} - \operatorname{grad} \Box_2 P_1 W_1 - \operatorname{grad} \Box_2 P_2 W_2.$$

Here

$$M = (\lambda + \mu) H - \partial_t(\gamma_1 M_1 + \gamma_2 M_2),\ H = D_1 D_2 - d^2 \partial^2_t,$$

$$M_1 = \gamma_1 D_2 + \gamma_2 d\partial_t,\quad M_2 = \gamma_2 D_1 + \gamma_1 d\partial_t.$$

The right-hand side of the above equation contain the sources $\mathbf{X}$, W_1, W_2, Eqs. (2.5), (2.6) and (2.7) are already separated.

Introduce now the vectorial functions χ and two scalar functions ψ_1, ψ_2 and let us connect them with the functions $\mathbf{u}, \theta_1, \theta_2$ as follows:

$$\mathbf{u} = -\Omega\chi + \operatorname{grad} \square_1 M \chi - P_1 \operatorname{grad} \psi_1 - P_2 \operatorname{grad} \psi_2, \tag{2.8}$$

$$\theta_1 = - \square_1 P_1 \partial_t \operatorname{div} \chi - \square_1 H_2 \psi_1 - \square_1 \Gamma \psi_2, \tag{2.9}$$

$$\theta_2 = - \square_2 P_2 \partial_t \operatorname{div} \chi - \square_1 \Gamma \psi_1 - \square_1 H_1 \psi_2. \tag{2.10}$$

Introducing the above representations of the functions $\mathbf{u}$, θ_1, θ_2 into Eqs. (2.7), (2.5) and (2.6) we obtain the differential equations

$$\square_2 \Omega\chi = \mathbf{X}, \ \Omega\psi_1 = W_1, \ \Omega\psi_2 = W_2. \tag{2.11}$$

Thus, we have arrived at the simplest wave equations. The solution of them yields the functions χ, ψ_1, ψ_2. Introducing the latter into the representation (2.8) - (2.10) completes the solution of our problem.

Our method of separating the system of equations (1.1) - (1.3) contains a number of particular cases.

a) When there is no diffusion, Eqs. (1.1) - (1.3) are reduced to the system

$$\square_2 \mathbf{u} + (\lambda+\mu) \operatorname{grad} \operatorname{div} \mathbf{u} + \mathbf{X} = \gamma_1 \operatorname{grad} \theta_1, \tag{2.12}$$

$$D_1 \theta - \gamma_1 \partial_t \operatorname{div} \mathbf{u} + W_1 = 0, \tag{2.13}$$

i. e. to the equations of thermoelasticity [6].

Introducing the representation

$$\mathbf{u} = - H_1 \chi + \operatorname{grad} \operatorname{div} [(\lambda+\mu) D_1 - \gamma_1^2 \partial_t] \chi - \gamma_1 \operatorname{grad} \psi_1 \tag{2.14}$$

$$\theta_1 = -\gamma_1 \square_2 \partial_t \operatorname{div} \chi - \square_1 \psi_1 \tag{2.15}$$

into the Eqs. (2.12) (2.13), we obtain the differential equations

$$\square_2 H_1 \chi = \mathbf{X}, \ H_1 \theta_1 = W_1 \tag{2.16}$$

b) If we are faced with a diffusion of the matter into the body at the isothermal state ($\theta_1 = 0$), then the process of the diffusion is described by the system of equations

(2.17) $$\square_2 \mathbf{u} + (\lambda+\mu) \operatorname{grad} \operatorname{div} \mathbf{u} + \mathbf{X} = \gamma_2 \operatorname{grad} \theta_2 ,$$

(2.18) $$\square_2 \theta_2 = \gamma_2 \partial_t \operatorname{div} \mathbf{u} - W_2 .$$

Now we make use of the representation

(2.19) $$\mathbf{u} = -H_2 \chi + \operatorname{grad} \operatorname{div} [(\lambda+\mu) D_2 - \gamma_2^2 \partial_t] \chi - \gamma_2 \operatorname{grad} \psi_2 ,$$

(2.20) $$\theta_2 = -\gamma_2 \square_2 \partial_t \operatorname{div} \chi - \square_1 \psi_2 .$$

which, when substituted into Eqs. (2.17), (2.18) leads to the equations

(2.21) $$\square_2 H_2 \chi = \mathbf{X}, \quad H_2 \theta_2 = W_2$$

c) In the case of absence of a diffusional process and heat sources, assuming the adiabatic process, Eqs. (1.2), (1.3) do not appear and (1.1) is reduced to the displacement equations of the classical dynamic elasticity

(2.22) $$\square_2 \mathbf{u} + (\lambda' + \mu) \operatorname{grad} \operatorname{div} \mathbf{u} + \mathbf{X} = 0.$$

This system of equations is reduced to the form

(2.23) $$\square_1 \square_2 \mathbf{u} = -\square_1 \mathbf{X} + \operatorname{grad} \operatorname{div} \mathbf{X},$$

and then, introducing into (2.23) the representation of the displacements

(2.24) $$\mathbf{u} = -\square_1 \mathbf{X} + \operatorname{grad} \operatorname{div} \boldsymbol{\chi},$$

we arrive at the repeated wave equation

(2.25) $$\square_1 \square_2 \boldsymbol{\chi} = \mathbf{X},$$

where $\boldsymbol{\chi}$ is IACOVACHE's vector [4].

3. Elastic potentials. Connection between the potentials Φ, Ψ and the functions χ, ψ_1, ψ_2.

In the paper [3] we made use of the decomposition of the displacements

(3.1) $$\mathbf{u} = \operatorname{grad} \Phi + \operatorname{rot} \mathbf{\Psi}, \qquad \operatorname{div} \mathbf{\Psi} = 0.$$

Assuming that

$$\mathbf{X} = \operatorname{grad} \vartheta + \operatorname{rot} \boldsymbol{\eta}, \qquad \operatorname{div} \boldsymbol{\eta} = 0 \tag{3.2}$$

and substituting (3.1) and (3.2) into Eqs. (1.1) and (1.3) we obtain the system of equations

$$\square_1 \Phi = \gamma_1 \theta_1 + \gamma_2 \theta_2 - \vartheta, \tag{3.3}$$

$$\square_2 \mathbf{\Psi} = -\boldsymbol{\eta}, \tag{3.4}$$

$$D_1 \theta_1 = \gamma_1 \nabla^2 \dot{\Phi} + d\,\dot{\theta} - W_1, \tag{3.5}$$

$$D_2 \theta_2 = \gamma_2 \nabla^2 \dot{\Phi} + d\dot{\theta}_1 - W_2. \tag{3.6}$$

Evidently, Eqs. (3.3), (3.5) and (3.6) are coupled. Eliminating from then the functions θ_1, θ_2 we obtain the equation of longitudinal wave

$$[\square_1 H - (\gamma_1 M_1 + \gamma_2 M_2) \nabla^2 \partial_t]\, \Phi = -(H\vartheta + M_1 W_1 + M_2 W_2) \tag{3.7}$$

Eqs. (3.4) represent the transversal wave. The wave is not perturbed either by the temperature field or by the field of the chemical potential.

We are interested in the relations between the functions $\Phi, \mathbf{\Psi}$ and the functions χ, ψ_1, ψ_2.

Consider therefore the homogeneous equations (2.11)

$$\square_2 \Omega \boldsymbol{\chi} = 0, \qquad \Omega \psi_1 = 0, \qquad \Omega \psi_2 = 0. \tag{3.8}$$

We assume the solution of $(3.8)_1$ in the form

$$\boldsymbol{\chi} = \boldsymbol{\chi}' + \boldsymbol{\chi}''. \tag{3.9}$$

In accordance with Boggio's theorem [7], the functions $\boldsymbol{\chi}', \boldsymbol{\chi}''$ satisfy the equations

$$\Omega \boldsymbol{\chi}' = 0, \quad \square_2 \boldsymbol{\chi}'' = 0. \tag{3.10}$$

Introducing (3.9) and (3.10) into the representation (2.8) we obtain

$$\mathbf{u} = -\Omega \boldsymbol{\chi}'' + \operatorname{grad} \square_1 M(\boldsymbol{\chi}' + \boldsymbol{\chi}'') - P_1 \operatorname{grad} \psi_1 - P_2 \operatorname{grad} \psi_2. \tag{3.11}$$

Connecting the terms containing the vector $\boldsymbol{\chi}''$, performing the transformation

$$\Omega = H_1 H_2 - \Gamma^2 = \square_1 (\square_1 H - \partial_t (\gamma_1 M_1 + \gamma_2 M_2) \nabla^2)$$

and using known relation of vector analysis

$$\nabla^2 \chi'' = \text{grad div}\, \chi'' - \text{rot rot}\, \chi''$$

we arrive at the relation

$$\mathbf{u} = \text{grad}\,(\text{div}\, \Box_1 M \chi' - P_1 \psi_1 - P_2 \psi_2) - \text{rot}[\text{rot}\, \Box_1 M \chi''].\tag{3.12}$$

Comparing the representation (3.1) and (3.12) we obtain the required relation between the elastic potentials Φ, $\mathbf{\Psi}$ and the fonctions χ, ψ_1, ψ_2:

$$\Phi = -P_1 \psi_1 - P_2 \psi_2 + \Box_1 M \,\text{div}\, \chi',\tag{3.13}$$

$$\mathbf{\Psi} = -\text{rot}\, \Box_1 M \chi''.\tag{3.14}$$

Substituting the functions Φ and ψ into the homogeneous equations (3.4) and (3.7) we find that they are identically satisfied.

4. Theorem on the completeness of solutions.

As proved by F. DUHEM [9] the representation (3.1) yields a solution of the dynamic elasticity equations. E. STERNBERG [9] proved that the theorem on completeness can be generalized to the problems of coupled thermoelasticity. As in [9] it may proved that the representation (3.1) leads to a complete solution for the system of thermodiffusion equations.

Now, on the basis of the representation

$$\mathbf{u} = \text{grad}\, \Phi + \text{rot}\, \mathbf{\Psi}, \qquad \text{div}\, \mathbf{\Psi} = 0,\tag{4.1}$$

we shall prove that also the representation (2.8) - (2.10) yields a complete solution.

Introducing (4.1) into Eqs. (1.1) - (1.3) we obtain the equations

$$\text{grad}\, \Box_1 \Phi + \text{rot}\, \Box_2 \mathbf{\Psi} + \mathbf{X} = \text{grad}\,(\gamma_1 \theta_1 + \gamma_2 \theta_2)\tag{4.2}$$

$$D_1 \theta_1 = \gamma_1 \nabla^2 \dot{\Phi} + d\, \dot{\theta}_2 - W_1,\tag{4.3}$$

$$D_2 \theta_2 = \gamma_2 \nabla^2 \dot{\Phi} + d\, \dot{\theta}_1 - W_2 .\tag{4.4}$$

Eliminating from (4.2) the function θ_1, θ_2, taking into account (2.5), (2.6) and (2.11) we obtain

$$\begin{aligned}\text{grad}\,[\Omega \Box_1 (\Phi + P_1 \psi_1 + P_2 \psi_2) + P \Box_2 \Omega\, \partial_t\, \text{div}\, \chi] + \\ + \text{rot}\,(\Box_2 \Omega \mathbf{\Psi}) + \Omega \mathbf{X} = \mathbf{0}\end{aligned}\tag{4.5}$$

where

$$P = P_1 \gamma_1 + P_2 \gamma_2.$$

Let us now decompose the Galerkin vector χ into its potential and rotational parts

$$\chi = \operatorname{grad} \zeta + \operatorname{rot} \sigma. \tag{4.6}$$

and substitute into $(2.11)_1$: then

$$\operatorname{grad}(\Omega \Box_2 \zeta) + \operatorname{rot}(\Box_2 \Omega \sigma) = \mathbf{X}. \tag{4.7}$$

Comparing Eq. (4.5) with Eq. (4.7) multiplied by Ω we arrive at the relations

$$\Box_2 \Omega \zeta = -\Box_1 (\Phi + P_1 \psi_1 + P_2 \psi_2) - P \Box_2 \partial_t \operatorname{div} \chi, \tag{4.8}$$

$$\Omega \sigma = -\mathbf{\Psi}. \tag{4.9}$$

In view of (4.6) and (4.9) we find that

$$\Omega \chi = \operatorname{grad} \Omega \zeta - \operatorname{rot} \mathbf{\Psi}. \tag{4.10}$$

Eliminating from (4.1) and (4.10) the quantity $\operatorname{rot} \mathbf{\Psi}$ and bearing in mind (4.8), we have

$$\Box_1 \mathbf{u} = -\Omega \Box_1 \chi - \operatorname{grad}[\Box_1 (P_1 \psi_1 + P_2 \psi_2) + \\ + P \Box_2 \partial_t \operatorname{div} \chi + (\Box_2 - \Box_1) \Omega \zeta]. \tag{4.11}$$

However, since

$$\Box_1 \zeta = \Box_1 + (\lambda + \mu) \nabla^2 \zeta, \quad \nabla^2 \zeta = \operatorname{div} \chi,$$

substituting into (4.11), after simple transformations, we arrive at the representation

$$\mathbf{u} = -\Omega \chi + \operatorname{grad} \operatorname{div} \Box_1 M \chi - P_1 \operatorname{grad} \psi_1 - P_2 \operatorname{grad} \psi_2. \tag{4.12}$$

Thus, we have deduced the representation (2.8).

Consider now the equation

$$\Box_1 \nabla^2 \Phi + \operatorname{div} \mathbf{X} = \nabla^2 (\gamma_1 \theta_1 + \gamma_2 \theta_2)$$

derived from (4.2) by an application of the divergence operation. Eliminating Φ from Eqs. (4.3), (4.4) and (4.13) we obtain Eqs. (2.5) and (2.6). Making use in Eqs. (2.5) and (2.6) of Eqs. (2.11) we obtain the representations (2.9) and (2.10).

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