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DYNAMIC PROBLEMS OF THERMODIFFUSION IN ELASTIC SOLIDS

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1. Introduction

An earlier attempt to describe the phenomenon of thermodiffusion in an elastic body was made in papers [1, 2] in which a governing system of the differential equations was derived. In the present paper, we give another way of deriving these equations and prove a number of general theorems relating "external forces" and resulting "states" of a transient process of thermodiffusion in elastic solids.

We deal with a two-component model in which a mobile component is to coexist together with an immobile one. In particular, a gas diffusing into a solid body may be described by such a model. As a reference system for the diffusion flow, a crystal lattice of the immobile component may be assumed.

The constitutive equations for such a body reads (cf. [1, 2]):

$$\begin{aligned} \sigma_{ij} &= 2G\varepsilon_{ij}+(\lambda_0\varepsilon_{kk}-\beta_\theta\theta-\beta_c C)\delta_{ij}, \\ S &= \beta_\theta\varepsilon_{kk}+c^{\varepsilon c}\theta+aC, \\ \mu &= -\beta_c\varepsilon_{kk}-a\theta+bC. \end{aligned} \tag{1.1}$$

Here, σ_{ij} and ε_{ij} denote the stress tensor and the strain tensor, respectively, $\theta = T-T_0$, where T is the absolute temperature of the solid and T_0 is the reference temperature of a natural state with zero strains and zero stresses, C stands for a concentration field. Moreover, G, λ_0 denote the Lamé constants, $\beta_\theta = 3K\alpha_t$, $\beta_c = 3K\alpha_c$, where $K = \lambda_0+(2/3)G$ is the bulk modulus and α_t is the coefficient of linear thermal expansion, while α_c is the coefficient of linear diffusive expansion. The coefficients $c^{\varepsilon,c}$, a and b occurring in $(1.1)_2$, $(1.1)_3$ denote the specific heat at constant strain and concentration, the coefficient of thermodiffusion and the coefficient of diffusion, respectively. The functions S and μ are to be identified with the entropy and the chemical potential of the solid.

For a dynamic process, the functions ε_{ij}, σ_{ij}, θ_{ij}, C, S, μ are functions of position $\mathbf{x}$ and time t. If ε_{ij}, θ, μ are treated as independent variables, the Eqs. (1.1) can be rewritten in the alternate form:

$$\begin{aligned} \sigma_{ij} &= 2G\varepsilon_{ij}+(\lambda\varepsilon_{kk}-\gamma_\theta\theta-\gamma_\mu\mu)\delta_{ij}, \\ S &= \gamma_\theta\varepsilon_{kk}+c\theta+d\mu, \\ C &= \gamma_\mu\varepsilon_{kk}+d\theta+n\mu, \end{aligned} \tag{1.2}$$

where we adopted the following notations

$$\lambda = \lambda_0 - \frac{\beta_c^2}{b}, \quad \gamma_0 = \beta_0 + \frac{a}{b}\beta_c, \quad \gamma_\mu = \frac{\beta_c}{b},$$

$$c = \frac{c^{\varepsilon,c}}{T_0} + \frac{a^2}{b}, \quad d = \frac{a}{b}, \quad n = \frac{1}{b}.$$

For our two-component model obeying the laws of irreversible thermodynamics, the entropy source function is given by the formula (cf. [1]):

$$\hat{\sigma} = \frac{1}{T}(\mathbf{q}\cdot\mathbf{X}^{(q)} + \boldsymbol{\eta}\cdot\mathbf{X}^{(\eta)}), \tag{1.3}$$

where $\mathbf{q}$ and $\boldsymbol{\eta}$ denote the flux of heat and the flux of mass, respectively. These two fluxes are related linearly to the thermodynamic stimulus $(\mathbf{X}^{(q)}, \mathbf{X}^{(\eta)})$ through

$$\begin{aligned} \mathbf{q} &= L_{qq}\mathbf{X}^{(q)} + L_{q\eta}\mathbf{X}^{(\eta)}, \\ \boldsymbol{\eta} &= L_{\eta q}\mathbf{X}^{(q)} + L_{\eta\eta}\mathbf{X}^{(\eta)}, \end{aligned} \tag{1.4}$$

where, according to the Onsager postulate, $L_{\eta q} = L_{q\eta}$. Since the entropy source $\hat{\sigma}$ is to be a positive function, $L_{qq}L_{\eta\eta} - L_{\eta q}L_{q\eta} > 0$, and we have

$$\begin{aligned} \mathbf{q} &= -\frac{1}{T}L_{qq}\operatorname{grad}T - L_{q\eta}T\operatorname{grad}\left(\frac{\mu}{T}\right), \\ \boldsymbol{\eta} &= -\frac{1}{T}L_{\eta q}\operatorname{grad}T - L_{\eta\eta}T\operatorname{grad}\left(\frac{\mu}{T}\right). \end{aligned} \tag{1.5}$$

Eliminating T grad (μ/T) from the Eqs. $(1.5)_1$ and $(1.5)_2$, we obtain

$$\mathbf{q} = -k\operatorname{grad}T + \alpha\boldsymbol{\eta}, \quad k = \frac{1}{T}\frac{L_{qq}L_{\eta\eta} - L_{q\eta}^2}{L_{\eta\eta}} > 0, \quad \alpha = \frac{L_{\eta q}}{L_{\eta\eta}}. \tag{1.6}$$

An alternate form of $(1.5)_2$ reads:

$$\boldsymbol{\eta} = -\frac{1}{T}(L_{\eta q} - \mu L_{\eta\eta})\operatorname{grad}T - L_{\eta\eta}\operatorname{grad}\mu. \tag{1.7}$$

Neglecting non-linear terms of (1.6) and (1.7), we get

$$\mathbf{q} = -k\operatorname{grad}T, \quad \boldsymbol{\eta} = -D\operatorname{grad}\mu, \quad D = L_{\eta\eta}. \tag{1.8}$$

Now using the entropy equation

$$T\dot{S} = -\operatorname{div}\mathbf{q} + \mu\operatorname{div}\boldsymbol{\eta} \tag{1.9}$$

and again neglecting non-linear terms, we obtain:

$$T\dot{S} \approx k\nabla^2 T. \tag{1.10}$$

To derive the heat conduction equation involving θ, ε_{kk} and μ, we assume that $|\theta/T_0| \ll 1$ and substitute $(1.2)_2$ into (1.10). As a result we arrive at:

$$\frac{k}{T_0}\nabla^2\theta = \gamma_0\dot{\varepsilon}_{kk} + c\dot{\theta} + d\dot{\mu}. \tag{1.11}$$

A generalized diffusion equation can also be derived if we use the mass conservation law in the form

(1.12) $$C = -\operatorname{div}\boldsymbol{\eta},$$

Substituting $(1.2)_3$ into LHS of (1.12) and using $(1.8)_2$, we get

(1.13) $$D\nabla^2\mu = \gamma_\nu \dot{\varepsilon}_{kk} + d\dot{\theta} + n\ddot{\mu}.$$

The last equation generalizes the well known diffusion equation to thermal and dilatational effects.

To obtain a complete set of the governing equations of thermodiffusive elasticity, consider the equation of motion in the form:

(1.14) $$\sigma_{ji,j} + X_i = \varrho \ddot{u}_i,$$

where u_i stands for the displacement vector and X_i denotes the body force vector.

Substituting σ_{ij} from $(1.2)_1$ into (1.14) and using the strain-displacement relation

(1.15) $$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}),$$

we get

(1.16) $$G\nabla^2 u_i + (\lambda + G)u_{j,ji} + X_i = \varrho\ddot{u}_i + \gamma_\theta \theta_{,i} + \gamma_\mu \mu_{,i}.$$

The Eqs. (1.16), (1.13) (1.11) constitute a fundamental system of field equations describing the phenomenon of thermodiffusion in a two-component solid body.

If w and σ denote the amount of heat and mass produced in the body in unit of time and volume, and if these functions are prescribed, then the Eqs. (1.10) and (1.12) are to be replaced by

(1.17) $$T\dot{S} \approx k\nabla^2 T + w, \quad \dot{C} = -\operatorname{div}\boldsymbol{\eta} + \sigma,$$

while the Eqs. (1.16), (1.11) and (1.13) take the forms:

(1.18) $$G\nabla^2\mathbf{u} + (G+\lambda)\operatorname{grad}\operatorname{div}\mathbf{u} + \mathbf{X} = \varrho\ddot{\mathbf{u}} + \gamma_\theta \operatorname{grad}\theta + \gamma_\mu \operatorname{grad}\mu,$$

(1.19) $$\frac{k}{T_0}\nabla^2\theta = \gamma_\theta \dot{\varepsilon}_{kk} + c\dot{\theta} + d\dot{\mu} - W,$$

(1.20) $$D\nabla^2\mu = \gamma_\mu \dot{\varepsilon}_{kk} + d\dot{\theta} + n\mu - \sigma, \quad W = w/T_0.$$

This is a coupled system of equations in which a „state", described by the functions $\mathbf{u}$, θ and μ, is produced by the external „forces" $\mathbf{X}$, σ, W and the boundary-initial data.

If an elastic body V is bounded by a regular surface A, the boundary conditions associated with the Eqs. (1.18)–(1.20) can be assumed in the form:

(1.21) $$\sigma_{ji}n_j = p_i(\mathbf{x}, t), \quad t > 0, \quad \mathbf{x} \in A.$$

(1.22) $$\theta(\mathbf{x}, t) = k(\mathbf{x}, t), \quad \mu(\mathbf{x}, t) = h(\mathbf{x}, t), \quad t > 0, \quad \mathbf{x} \in A,$$

while the initial conditions read:

(1.23) $$u_i(\mathbf{x}, 0) = f_i(\mathbf{x}), \quad \dot{u}_i(\mathbf{x}, 0) = g_i(\mathbf{x}), \quad t = 0, \quad \mathbf{x} \in V,$$

(1.24) $$\theta(\mathbf{x}, 0) = m(\mathbf{x}), \quad \mu(\mathbf{x}, 0) = n(\mathbf{x}).$$

Here, p_i, k, h, f_i, g_i, m and n are prescribed functions. If the displacement vector $\mathbf{u}$ is given on A, and the flux of heat and the flux of mass are prescribed on A, then (1.21), (1.22) are to be replaced by

$$u_i = U_i(\mathbf{x}, t), \quad t > 0, \quad \mathbf{x} \in A,$$

(1.25)
$$\frac{\partial \theta}{\partial n} = s(\mathbf{x}, t), \qquad \frac{\partial \mu}{\partial n} = r(\mathbf{x}, t), \qquad \mathbf{x} \in A.$$

where U_i, s, r are given functions.

2. Wave-Like Equations. Potentials of Thermodiffusive Elasticity

To discuss a solution to the system of Eqs. (1.18)–(1.20), let us decompose the vectors $\mathbf{u}$ and $\mathbf{X}$ into potential and solenoidal parts according to the formulae:

(2.1)
$$\mathbf{u} = \operatorname{grad}\phi + \operatorname{rot}\mathbf{\Psi}, \qquad \mathbf{X} = \varrho(\operatorname{grad}\vartheta + \operatorname{rot}\boldsymbol{\chi}),$$

subject to the conditions

(2.2)
$$\operatorname{div}\mathbf{\Psi} = 0, \quad \operatorname{div}\boldsymbol{\chi} = 0.$$

Substituting (2.1) into (1.18)–(1.20), we find that (1.18)–(1.20) is satisfied if

(2.3)
$$\Box_1 \Phi = m_0\theta + m_\mu \mu - \frac{1}{c_1^2}\vartheta,$$

(2.4)
$$\Box_2 \mathbf{\Psi} = -\frac{1}{c_2^2}\boldsymbol{\chi},$$

(2.5)
$$D_1\theta = d\dot{\mu} + \gamma_0 \nabla^2 \dot{\Phi} - W,$$

(2.6)
$$D_2\mu = d\dot{\theta} + \gamma_\mu \nabla^2 \dot{\Phi} - \sigma,$$

where the following notations are introduced:

$$\Box_1 = \nabla^2 - \frac{1}{c_1^2}\partial_t^2, \quad \Box_2 = \nabla^2 - \frac{1}{c_2^2}\partial_t^2, \quad c_1 = \left(\frac{\lambda + 2\mu}{\varrho}\right)^{1/2}, \quad c_2^2 = \left(\frac{\mu}{\varrho}\right)^{1/2},$$

$$D_1 = \frac{k}{T_0}\nabla^2 - c\partial_t, \quad D_2 = D\nabla^2 - n\partial_t, \quad m_0 = \frac{\gamma_0}{\varrho c_1^2}, \quad m_\mu = \frac{\gamma_\mu}{\varrho c_1^2}.$$

The Eq. (2.3) describes a longitudinal wave in thermodiffusive elastic medium, while (2.4) covers a transverse wave. Note that the Eqs. (2.3), (2.5) and (2.6) are mutually coupled. In an infinite thermodiffusive elastic solid, the Eq. (2.4) is independent of the Eqs. (2.3), (2.5) and (2.6), and in this particular case the temperature θ and the chemical potential μ have no influence on the shear wave which propagates with the constant velocity c_2, without damping and without dispersion.

Eliminating the functions θ and μ from the coupled system (2.3), (2.5), (2.6), we obtain the higher order wave-like equation for ϕ.

(2.7)
$$[\Box_1 H - (m_0 M_1 + m_\mu M_2)\nabla^2\partial_t]\phi = -\frac{1}{c_1^2\varrho}(H\vartheta + M_1 W + M_2\sigma),$$

where

$$H = D_1 D_2 - d^2\partial_t^2, \quad M_1 = \gamma_0 D_2 + \gamma_\mu d\partial_t, \quad M_2 = \gamma_\mu D_1 + \gamma_0 d\partial_t.$$

The Eq. (2.7) does not lend itself to discussion. Assuming a periodic vibrations of the solid, we can show that the longitudinal wave ϕ is to be damped and dispersed throughout the solid.

Similar separated equations for θ and μ can be derived by means of the Eqs. (2.3), (2.5), (2.6).

A solution (ϕ, θ, μ) of the system (2.3), (2.5), (2.6), which is similar to the Galerkin solution of classical elastostatics, can be obtained if (Φ, θ, μ) is expressed by a vector field Ω_i $(i = 1, 2, 3)$ by means of the determinants:

$$\phi = \begin{vmatrix} \Omega_1 & -m_0 & -m_\mu \\ \Omega_2 & D_1 & -d\partial_t \\ \Omega_3 & -d\partial_t & D_2 \end{vmatrix}, \quad \vartheta = \begin{vmatrix} \Box_1 & \Omega_1 & -m_\mu \\ -\gamma_0\nabla^2\partial_t & \Omega_2 & -d\partial_t \\ -\gamma_\mu\nabla^2\partial_t & \Omega_3 & D_2 \end{vmatrix}, \tag{2.8}$$

$$\mu = \begin{vmatrix} \Box_1 & -m_\theta & \Omega_1 \\ -\gamma_0\nabla^2\partial_t & D_1 & \Omega_2 \\ -\gamma_\mu\nabla^2\partial_t & -d\partial_t & \Omega_3 \end{vmatrix},$$

or by

$$\Phi = H\Omega_1 + \frac{1}{c_1^2\varrho}(M_1\Omega_2 + M_2\Omega_3), \tag{2.9}$$

$$\theta = M_1\nabla^2\partial_t\Omega_1 + (\Box_1 D_2 - \gamma_\mu m_\mu\nabla^2\partial_t)\Omega_2 + (\Box_1 d + \gamma_0 m_\mu\nabla^2)\partial_t\Omega_3, \tag{2.10}$$

$$\mu = M_2\nabla^2\partial_t\Omega_1 + (\Box_1 d + \gamma_\mu m_0\nabla^2)\partial_t\Omega_2 + (\Box_1 D_1 - \gamma_\theta m_\theta\nabla^2\partial_t)\Omega_3. \tag{2.11}$$

Substituting the Eqs. (2.9)–(2.11) into (2.3), (2.5), (2.6), we obtain wave-like equations which must be satisfied by Ω_i:

$$[H\Box_1 - (m_\theta M_1 + m_\mu M_2)\nabla^2\partial_t]\Omega_1 = -\frac{1}{c_1^2}\vartheta, \tag{2.12}$$

$$[H\Box_1 - (m_0 M_1 + m_\mu M_2)\nabla^2\partial_t]\Omega_2 = -W, \tag{2.13}$$

$$[H\Box_1 - (m_\theta M_1 + m_\mu M_2)\nabla^2\partial_t]\Omega_3 = -\sigma. \tag{2.14}$$

If $W = \sigma = 0$, we can put $\Omega_2 = \Omega_3 = 0$; while if $\vartheta = 0$, we can assume $\Omega_1 = 0$.

Let us discuss one more treatment of the system (2.3), (2.5), (2.6) with $\vartheta = W = \sigma = 0$. The Eqs. (2.5) and (2.6) can be reduced to the form

$$H\theta = M_1\partial_t\nabla^2\Phi, \quad H\mu = M_2\partial_t\nabla^2\Phi, \tag{2.15}$$

the RHS of which contains the dilatation $\nabla^2\phi = \operatorname{div}\mathbf{u}$.

Let $G = G(\mathbf{x}, \boldsymbol{\xi}, t)$ be the Green function satisfying the equation

$$HG = \delta(\mathbf{x} - \boldsymbol{\xi})\delta(t) \tag{2.16}$$

in an infinite space subject to the condition $G \to 0$ for $|\mathbf{x} - \boldsymbol{\xi}| \to \infty$. Assume also that G is to meet the homogeneous initial conditions.

Applying the Laplace transform to (2.15), and assuming homogeneous initial data for θ and μ, we obtain:

$$\begin{aligned} \mathscr{H}\bar{\theta} &= \mathscr{M}_1 p\nabla^2\bar{\Phi}, \\ \mathscr{H}\bar{\mu} &= \mathscr{M}_2 p\nabla^2\bar{\Phi}, \end{aligned} \tag{2.17}$$

where

$$\mathscr{H} = \left(\frac{k}{T_0}\nabla^2 - cp\right)(D\nabla^2 - np) - d^2p^2,$$

$$\mathscr{M}_1 = \gamma_\mu dp + \gamma_\theta \overline{D}_2, \quad \mathscr{M}_2 = \gamma_\theta dp + \gamma_\mu \overline{D}_1,$$

$$\overline{D}_1 = \frac{k}{T_0}\nabla^2 - cp, \quad \overline{D}_2 = D\nabla^2 - np.$$

The Eq. (2.10) in the Laplace transform domain reads:

$$\mathscr{H}\overline{G} = \delta(\mathbf{x} - \boldsymbol{\xi}). \tag{2.18}$$

Combining now $(2.17)_1$ with (2.18), we get

$$\int_V (\overline{G}\mathscr{H}\overline{\theta} - \overline{\theta}\mathscr{H}\overline{G})\, d\nabla = \int_V \overline{G}\mathscr{M}_1 p\nabla^2\overline{\phi}\, dV - \theta(\boldsymbol{\xi}, p). \tag{2.19}$$

LHS of (2.19) can be transformed by means of the divergence theorem to a surface integral. If we assume that V covers an infinite domain and (ϕ, θ, μ) are to vanish at infinity, the surface integral vanishes because $G \to 0$ as $|\mathbf{x} - \boldsymbol{\xi}| \to \infty$, and the Eq. (2.19) yields:

$$\overline{\theta}(\boldsymbol{\xi}, p) = \int_V \mathscr{M}_1 p\overline{G}\nabla^2\overline{\Phi}\, dV(\mathbf{x}). \tag{2.20}$$

Similarly, combining (2.18) with $(2.17)_2$, we get

$$\overline{\mu}(\boldsymbol{\xi}, p) = \int_V p\mathscr{M}_2\overline{G}\nabla^2\overline{\Phi}\, dV(\mathbf{x}). \tag{2.21}$$

Now applying the Laplace transform to (2.3), we obtain

$$\left(\nabla^2 - \frac{1}{c_1^2}p^2\right)\overline{\Phi} = m_\theta\overline{\theta} + m_\mu\overline{\mu}. \tag{2.22}$$

If $\overline{\theta}$ and $\overline{\mu}$ from (2.20) and (2.21) are substituted into RHS of (2.22) and the resulting equation is inverted, we find that Φ satisfies the following integro-differential equation:

$$\Box_1 \Phi(\mathbf{x}, t) = \frac{1}{c_1^2\varrho}\int_0^t d\tau \int_V G(\mathbf{x}, \mathbf{x}', t-\tau)\left(2\gamma_\mu\gamma_\theta d\frac{\partial^2}{\partial\tau^2} + \gamma_\theta^2 D_2'\frac{\partial}{\partial\tau} + \gamma_\mu^2 D_1'\frac{\partial}{\partial\tau}\right)\nabla^2\Phi(\mathbf{x}', \tau)\, dV(\mathbf{x}'), \quad D_1' = \frac{k}{T_0}\nabla^2 - c\partial\tau, \quad D_2' = D\nabla^2 - n\partial_\tau. \tag{2.23}$$

The potential $\boldsymbol{\Psi}$ describing a shear wave in the infinite elastic body satisfies the Eq. (2.4) with $\boldsymbol{\chi} = 0$:

$$\Box_2 \boldsymbol{\Psi}(\mathbf{x}, t) = 0. \tag{2.24}$$

If Φ satisfying (2.23) is known, then θ and μ can be computed from (2.20) and (2.21).

In this way, the problem of finding a solution of the Eqs. (2.3)–(2.6) in an infinite domain has been reduced to finding the solution of (2.23) and (2.24).

Now we have to show that the Green function $G(\mathbf{x}, \boldsymbol{\xi}, t)$ can be obtained in a closed form. To this end, we transform (2.16) into the Laplace transform domain. We obtain:

$$(2.25) \qquad (\nabla^2 - k_1^2 p)(\nabla^2 - k_2^2)\bar{G} = \frac{1}{k_0 D}\,\delta(\mathbf{x}-\boldsymbol{\xi}), \qquad k_0 = \frac{k}{T_0},$$

where

$$\left\{\begin{matrix} k_1^2 \\ k_2^2 \end{matrix}\right. = \frac{1}{2}\left(\alpha \pm \sqrt{\alpha^2 - 4\beta}\right), \qquad \alpha = \frac{k_0 n + cD}{k_0 D}, \qquad \beta = \frac{nc - d^2}{k_0 D} > 0, \qquad \alpha^2 > 4\beta.$$

Clearly, the only solution of this equation vanishing at infinity takes the form:

$$(2.26) \qquad \bar{G} = -\frac{1}{4\pi k_0 DR}\left(e^{-k_1 R\sqrt{p}} - e^{-k_2 R\sqrt{p}}\right).$$

Inverting now (2.26), we arrive at the following form of G:

$$G(\mathbf{x}, \boldsymbol{\xi}, t) = -\frac{1}{4\pi k_0 DR}\,\frac{1}{(\lambda_1^2 - \lambda_2^2)}\left(\psi_1(R, t) - \psi_2(R, t)\right),$$

where

$$\psi_1 = \frac{\lambda_1 R}{2\sqrt{\pi t^3}}\exp\left(\frac{-\lambda_1^2 R^2}{4t}\right), \qquad \psi_2 = \frac{\lambda_2 R}{2\sqrt{\pi t^3}}\exp\left(\frac{-\lambda_2^2 R^2}{4t}\right).$$

3. Variational theorem of thermodiffusive elasticity

A starting point of this Section is the principle of virtual work under variations of displacement. This principle is valid for an arbitrary elastic solid, and reads:

$$(3.1) \qquad \int_V (X_i - \varrho\ddot{u}_i)\,\delta u_i\, dV + \int_A p_i\,\delta u_i\, dA = \int_V \sigma_{ji}\,\delta\varepsilon_{ji}\, dV.$$

LHS of (3.1) contains three components: the first stands for the virtual work of the body forces X_i, the second denotes the virtual work of the inertia forces $\varrho\ddot{u}_i$, and the third one covers the virtual work of the surface forces p_i. RHS of (3.1) gives the virtual work of the internal forces.

Making use of the constitutive equations

$$(3.2) \qquad \sigma_{ij} = 2G\varepsilon_{ij} + (\lambda\varepsilon_{kk} - \gamma_\theta\theta - \gamma_\mu\mu)\,\delta_{ij},$$

we write the Eq. (3.1) in the form:

$$(3.3) \qquad \int_V (X_i - \varrho\ddot{u}_i)\,\delta u_i\, dV + \int_A p_i\,\delta u_i\, dA = \delta W - \int_V (\gamma_\theta\theta + \gamma_\mu\mu)\,\delta\varepsilon_{kk}\, dV.$$

Here, the following integral is introduced:

$$\mathscr{W} = \int_V \left(G\varepsilon_{ij}\varepsilon_{ij} + \frac{\lambda}{2}\,\varepsilon_{kk}\varepsilon_{nn}\right) dV.$$

If θ and μ are prescribed functions, the Eq. (3.3) yields the well known variational principle of elasticity. In a thermodiffusive elastic solid, the functions $\mathbf{u}$, θ and μ are mutually coupled and the Eq. (3.3) is to be coupled with two additional relations characterizing the phenomenon of thermal conductivity and diffusion. Taking into account the fundamental relations of thermal conductivity discussed in Sec. 1

$$T_0 \dot{S} = -\operatorname{div}\mathbf{q}, \quad \mathbf{q} = -k \operatorname{grad}\theta, \tag{3.4}$$

$$S = \gamma_0 \varepsilon_{kk} + c\theta + d\mu, \tag{3.5}$$

and introducing, after M. A. Biot [3], the vector **H** connected with the entropy S through the relation

$$S = -\operatorname{div}\mathbf{H}, \tag{3.6}$$

we obtain:

$$T_0 \dot{H}_i + k\theta_{,i} = 0, \quad -H_{i,i} = \gamma_0 \varepsilon_{kk} + c\theta + d\mu. \tag{3.7}$$

It can be shown that elimination of H_i from $(3.7)_{1,2}$ leads to the Eq. (1.11).

Multiplying the Eq. $(3.7)_1$ by δH_i and integrating the result over V, we obtain:

$$\int_V \left(\theta_{,i} + \frac{T_0}{R} \dot{H}_i\right) \delta H_i dV = 0, \tag{3.8}$$

or

$$\int_V \theta n_i \delta H_i dA - \int_V \theta \delta H_{i,i} dV + \frac{T_0}{k} \int_V \dot{H}_i \delta H_i dV = 0. \tag{3.9}$$

The Eq. (3.9), together with $(3.7)_2$, implies:

$$\gamma_0 \int_V \theta \delta \varepsilon_{kk} dV + d \int_V \theta \delta \mu dV + \int_V \theta n_i \delta H_i dA + \delta(\mathscr{P} + \mathscr{D}) = 0, \tag{3.10}$$

where the thermal potential $\mathscr{P}$ and the function of thermal dissipation $\mathscr{D}$ are defined by

$$\mathscr{P} = \frac{c}{2} \int_V \theta^2 dV, \quad \delta\mathscr{P} = c \int_V \theta \delta\theta dV, \tag{3.11}$$

$$\delta\mathscr{D} = \frac{T_0}{k} \int_V \dot{H}_i \delta H_i dV. \tag{3.12}$$

The Eq. (3.10) is the second integral relation of the variational principle of thermodiffusive elasticity. It contains the term $\int_V \theta \delta \varepsilon_{kk} dV$, which appears also in (3.3).

The last integral relation of our variational principle will be obtained by making use of the relations (cf. Sec. 1):

$$\boldsymbol{\eta} = -D \operatorname{grad}\mu, \quad \dot{C} = -\operatorname{div}\boldsymbol{\eta}, \tag{3.13}$$

$$C = \gamma_\mu \varepsilon_{kk} + d\theta + n\mu. \tag{3.14}$$

Introducing the vector F through

$$C = -\operatorname{div}\mathbf{F}, \tag{3.15}$$

we reduce (3.13) and (3.14) into the forms

$$\dot{F}_i + D\mu_{,i} = 0, \quad -F_{i,i} = \gamma_\mu \varepsilon_{kk} + d\theta + n\mu. \tag{3.16}$$

Proceeding in a manner similar to that used in obtaining (3.10), from (3.16) we find:

$$\int_V \left(\mu_{,i} + \frac{1}{D}\dot{F}_i\right)\delta F_i dV = 0, \tag{3.17}$$

$$\gamma_\mu \int_V \mu\delta\varepsilon_{kk} dV + d\int_V \mu\delta\theta dV + \delta(\mathscr{A} + \mathscr{B}) = 0, \tag{3.18}$$

where the diffusion potential $\mathscr{A}$ and the function of diffusive dissipation $\mathscr{B}$ are defined through:

$$\mathscr{A} = \frac{n}{2}\int_V \mu^2 dV, \quad \delta\mathscr{A} = n\int_V \mu\delta\mu dV, \tag{3.19}$$

$$\delta\mathscr{B} = \frac{1}{D}\int_V \dot{F}_i \delta F_i dV. \tag{3.20}$$

Combining now (3.3), (3.10) and (3.18), we arrive at a final form of the variational theorem of thermodiffusive elasticity:

$$\delta\left(\mathscr{W} + \mathscr{P} + \mathscr{A} + \mathscr{D} + \mathscr{B} + d\int_V \mu\theta dV\right) = \int_V (X_i - \varrho\ddot{u}_i)\delta u_i dV + \int_V p_i \delta u_i dA - \int_V \theta n_i \delta H_i dA - \int_V \mu n_i \delta F_i dA. \tag{3.21}$$

Note that RHS of this relation include: the body forces, the inertia forces and the surface data p_i, θ and μ on A.

4. Fundamental Energy Theorem and Uniqueness Theorem of Thermodiffusive Elasticity

Assume that the virtual displacements δu_i, the virtual temperature $\delta\theta$ and the virtual chemical potential μ satisfy the relations:

$$\delta u_i = \frac{\partial u_i}{\partial t}dt = v_i dt, \quad \delta\theta = \frac{\partial\theta}{\partial t}dt = \dot{\theta}dt, \quad \delta\mu = \frac{\partial\mu}{\partial t}dt = \dot{\mu}dt, \tag{4.1}$$

$$\delta H_i = \dot{H}_i dt = -\frac{k}{T_0}\theta_{,i}, \quad \delta F_i = \dot{F}_i dt = -D\mu_{,i}, \quad \text{etc.}$$

where (u_i, θ, μ) stands for a solution of the governing equations of thermodiffusive elasticity.

Then substituting (4.1) into RHS of (3.21), we get

$$\frac{d}{dt}\left(\mathscr{K} + \mathscr{W} + \mathscr{P} + \mathscr{A} + d\int_V \mu\theta dV\right) + \chi^\theta + \chi^\mu = \int_V X_i v_i dV + \int_A p_i v_i dA + \frac{k}{T_0}\int_V \theta\theta_{,i} n_i dA + D\int_V \mu\mu_{,i} n_i dA, \tag{4.2}$$

where χ^{θ}, χ^{μ} and $\mathscr{K}$ in this order denote the function of thermal dissipation, the function of diffusive dissipation and the kinetic energy of the body given by

(4.3) $$\chi^{\theta} = \frac{k}{T_0} \int\limits_V (\theta_{,i})^2 dV, \quad \chi^{\mu} = D \int\limits_V (\mu_{,i})^2 dV,$$

(4.4) $$\mathscr{K} = \frac{\varrho}{2} \int\limits_V v_i v_i dV.$$

Eq. (4.2) is a fundamental energy equation of thermodiffusive elasticity, RHS of this equation reveals the energy of external forces: the body force X_i, the surface force p_i, the surface temperature θ (or the flow $\partial\theta/\partial n$) as well as the surface chemical potential μ (or the flow $\partial\mu/\partial n$). Note also that the expression

(4.5) $$\mathscr{P} + \mathscr{A} + d \int\limits_V \mu\theta dV = \frac{1}{2} \int\limits_V (c^2\theta^2 + 2d\mu\theta + n\mu^2) dV,$$

occuring in LHS of (4.2) is always positive since, by laws of thermodynamics, $cn > d^2$.

Now let us prove a uniqueness theorem of thermodiffusive elasticity to the effect that there exists at most one state (u_i, θ, μ) satisfying the Eqs. (1.18)–(1.20) in V and subject to the conditions (1.21)–(1.24).

To carry out the proof, we assume that there exist two solutions (u_i', θ', μ') and (u_i'', θ'', μ'') satisfying (1.18)–(1.24). Then the difference

(4.6) $$\hat{u}_i = u_i' - u_i'', \quad \hat{\theta} = \theta' - \theta'', \quad \hat{\mu} = \mu' - \mu'',$$

satisfies the homogeneous set of equations associated with (1.18)–(1.24). We shall prove that the state $(\hat{u}, \hat{\theta}, \hat{\mu})$ vanishes throughout the domain $V \times [0, \infty)$.

To this end, we use the fundamental energy theorem (4.2) in which we set:

(4.7) $$\hat{X} = 0, \quad \mathbf{x} \in V, \quad \hat{p}_i = 0, \quad \hat{\theta} = 0, \quad \hat{\mu} = 0, \quad \mathbf{x} \in A.$$

As a result, we obtain:

(4.8) $$\frac{d}{dt} \left(\hat{\mathscr{K}} + \hat{\mathscr{W}} + \hat{\mathscr{P}} + \hat{\mathscr{A}} + d \int\limits_V \hat{\mu}\hat{\theta} dV \right) = -(\hat{\chi}^{\theta} + \hat{\chi}^{\mu}) \leqslant 0,$$

or

(4.9) $$\frac{d}{dt} \int\limits_V \left[\frac{1}{2} \varrho \hat{v}_i \hat{v}_i + \mu \hat{\varepsilon}_{ij} \hat{\varepsilon}_{ij} + \frac{\lambda}{2} \hat{\varepsilon}_{kk} \hat{\varepsilon}_{mm} + \frac{1}{2} (c\hat{\theta}^2 + n\hat{\mu}^2 + 2d\hat{\mu}\hat{\theta}) \right] dV$$
$$= -\frac{k}{T_0} \int\limits_V (\hat{\theta}_{,i})^2 dV - D \int\limits_V (\hat{\mu}_{,i})^2 dV \leqslant 0.$$

Since LHS of (4.9) vanishes for $t = 0$ due to the homogeneous initial data and it is always negative for $t > 0$, RHS of (4.9) also vanishes at $t = 0$, and the integral of RHS is a decreasing function of the time t for $t > 0$. Since this function is non-negative for $t \geqslant 0$, we obtain:

(4.10) $$\hat{\mathscr{K}} + \hat{\mathscr{W}} + \hat{\mathscr{P}} + \hat{\mathscr{A}} + d \int\limits_V \hat{\mu}\hat{\theta} dV = 0,$$

which reduces to

(4.11) $$\hat{v}_i = 0, \quad \hat{\varepsilon}_{ij} = 0, \quad \hat{\theta} = 0, \quad \hat{\mu} = 0 \quad \text{for} \quad t \geqslant 0.$$

The Eqs. (4.6), (4.11), together with the constitutive Eqs. (1.2), imply:

(4.12) $$v'_i = v''_i, \quad \varepsilon'_{ij} = \varepsilon''_{ij}, \quad \theta' = \theta'', \quad \mu' = \mu'', \quad \sigma'_{ij} = \sigma''_{ij}.$$

The Eqs. $(4.12)_3$, $(4.12)_4$ show that $\hat{\theta} = \hat{\mu} = 0$. On the other hand, $(4.12)_1$ leads to $\hat{v}_i = 0$ in $V \times [0, \infty)$, from which, in view of the homogeneous initial data, $\hat{u}_i(x, t) = \hat{u}_i(x, 0) = 0$ in $V \times [0, \infty)$. This completes the proof.

5. Reciprocal Theorem of Thermodiffusive Elasticity

We say that $G = \{u_i, \theta, \mu\}$ is a state of thermodiffusive elasticity (on V) corresponding to the homogeneous initial data and to the external forces X_i on V and (p_i, θ, μ) on A, if G satisfies the Eqs. (1.18)–(1.24) with $W = \sigma = f_i = g_i = m = n = 0$.

These external forces are denoted by

(5.1) $$I = \{X_i, p_i; \theta, \mu\},$$

and the elastic thermodiffusive state by

(5.2) $$G = \{u_i, \theta, \mu\}, \quad \mathbf{x} \in V.$$

Let G' be another elastic thermodiffusive state produced by a second system of external forces I':

(5.3) $$I' = \{X'_i, p'_i; \theta', \mu'\}, \quad G' = \{u'_i, \theta'_i, \mu'\}.$$

Applying the Laplace transform to the governing equations for G and G', and appropriately combining the resulting equations, we arrive at the identity:

(5.4) $$\bar{\sigma}_{ij}\bar{\varepsilon}'_{ij} - \bar{\sigma}'_{ij}\bar{\varepsilon}_{ij} = (\gamma_\theta \bar{\theta}' + \gamma_\mu \bar{\mu}')\bar{e} - (\gamma_\theta \bar{\theta} + \gamma_\mu \bar{\mu})\bar{e}', \quad e = \varepsilon_{kk},$$

where

$$\bar{\sigma}_{ij}(\mathbf{x}, p) = \int_0^\infty \bar{e}^{pt} \sigma_{ij}(\mathbf{x}, t)\, dt, \text{ etc.}$$

This identity can be used to obtain the following integral relation

(5.5) $$\int_V (\bar{X}_i \bar{u}'_i - \bar{X}'_i \bar{u}_i)\, dV + \int_A (\bar{p}_i \bar{u}'_i - \bar{p}'_i \bar{u}_i)\, dA + \int_V (\gamma_\theta \bar{\theta} + \gamma_\mu \bar{\mu})\bar{e}'\, dV - \int_V (\gamma_\theta \bar{\theta}' + \gamma_\mu \bar{\mu}')\bar{e}\, dV = 0,$$

which constitutes a first part of the reciprocity theorem. To get its second part, we combine the following equations

(5.6) $$\frac{k}{T_0} \nabla^2 \bar{\theta} = p(\gamma_\theta \bar{e} + c\bar{\theta} + d\bar{\mu}), \qquad \frac{k}{T_0} \nabla^2 \bar{\theta}' = p(\gamma_\theta \bar{e}' + c\bar{\theta}' + d\bar{\mu}'),$$

which are obtained from (1.11).

Combining $(5.6)_{1,2}$, integrating the result over the body volume and using the Green theorem, we arrive at

$$(5.7)\qquad \frac{k}{T_0}\int_A\left(\bar{\theta}'\frac{\partial\bar{\theta}}{\partial n}-\bar{\theta}\frac{\partial\bar{\theta}'}{\partial n}\right)dA = p\int_V(\gamma_\theta\bar{e}+d\bar{\mu})\bar{\theta}'dV-p\int_V(\gamma_\theta\bar{e}'+d\bar{\mu}')\bar{\theta}\,dV.$$

Similarly, taking advantage of the equations

$$(5.8)\qquad D\nabla^2\bar{\mu} = p(\gamma_\mu\bar{e}+d\bar{\theta}+n\bar{\mu}),\quad D\nabla^2\bar{\mu}' = p(\gamma_\theta\bar{e}'+d\bar{\theta}'+n\bar{\mu}'),$$

which are obtained from (1.13), we obtain:

$$(5.9)\qquad D\int_A\left(\bar{\mu}'\frac{\partial\bar{\mu}}{\partial n}-\bar{\mu}\frac{\partial\bar{\mu}'}{\partial n}\right)dA = p\int_V(\gamma_\mu\bar{e}+d\bar{\theta})\bar{\mu}'dV-p\int_V(\gamma_\mu\bar{e}'+d\bar{\theta}')\bar{\mu}dV.$$

The Eq. (5.9) constitutes a third part of the reciprocity theorem. If we now eliminate the common terms of the Eqs. (5.5), (5.7) and (5.9), we arrive at a final form of the reciprocity theorem in the Laplace transform domain:

$$(5.10)\qquad p\left[\int_V(\bar{X}_i\bar{u}_i'-\bar{X}_i'\bar{u}_i)dV+\int_A(\bar{p}_i\bar{u}_i'-\bar{p}_i'\bar{u}_i)dA\right]$$
$$+\frac{k}{T_0}\int_A\left(\bar{\theta}\frac{\partial\bar{\theta}'}{\partial n}-\bar{\theta}'\frac{\partial\bar{\theta}}{\partial n}\right)dA+D\int_A\left(\bar{\mu}\frac{\partial\bar{\mu}'}{\partial n}-\bar{\mu}'\frac{\partial\bar{\mu}}{\partial n}\right)dA = 0.$$

In this equation are present only the states G and G' corresponding to the external forces I and I'.

Inverting (5.10), we obtain the reciprocity theorem of thermodiffusive elasticity in the space-time domain:

$$(5.11)\qquad \int_V(X_i\odot u_i'-X_i'\odot u_i)dV+\int_A(p_i\odot u_i'-p_i'\odot u_i)dA$$
$$+\frac{k}{T_0}\int_A\left(\theta*\frac{\partial\theta'}{\partial n}-\theta'*\frac{\partial\theta}{\partial n}\right)dA+D\int_A\left(\mu*\frac{\partial\mu'}{\partial n}-\mu'*\frac{\partial\mu}{\partial n}\right)dA = 0,$$

where

$$(5.12)\qquad \begin{aligned}&\int_A X_i\odot u_i'dV = \int_0^t d\tau\int_V X(\mathbf{x},t-\tau)\frac{\partial u_i(\mathbf{x},\tau)}{\partial\tau}dV(\mathbf{x}),\ \text{etc.}\\ &\int_A\theta*\frac{\partial\theta'}{\partial n}dA = \int_0^t d\tau\int_A\theta(\mathbf{x},t-\tau)\frac{\partial\theta'(\mathbf{x},\tau)}{\partial n}dA(\mathbf{x}),\ \text{etc.}\end{aligned}$$

The Eq. (5.11) is also valid if G and G' are quasi-static states, or if one of the two states G and G' is quasi-static. For a static state of thermodiffusive elasticity, the reciprocity theorem reduces to the following relations:

$$\int_V(X_iu_i'-X_i'u_i)dV+\int_A(p_iu_i'-p_i'u_i)dA+\int_V(\gamma_\theta\theta+\gamma_\mu\mu)e'dV-\int_V(\gamma_\theta\theta'+\gamma_\mu\mu')edV = 0,$$

(5.13) $$\int_A \left(\theta \frac{\partial \theta'}{\partial n} - \theta' \frac{\partial \theta}{\partial n}\right) dA = 0, \qquad \int_A \left(\mu \frac{\partial \mu'}{\partial n} - \mu' \frac{\partial \mu}{\partial n}\right) dA = 0.$$

The Eq. (5.11) can easily be generalized to include nonhomogeneous initial data and nonvanishing heat and mass sources.

6. Relations Resulting from the Reciprocal Theorem

In Sec. 5, we proved the reciprocal theorem under the assumption $W = \sigma = 0$. If $W \neq 0$ and $\sigma \neq 0$, the fundamental system of field equations takes the form:

(6.1) $$G\nabla^2 \mathbf{u} + (\lambda + G)\,\mathrm{grad}\,\mathrm{div}\,\mathbf{u} + \mathbf{X} = \varrho \ddot{\mathbf{u}} + \gamma\;\; \mathrm{grad}\,\theta + \gamma_\mu\, \mathrm{grad}\,\mu,$$

(6.2) $$\frac{k}{T_0} \nabla^2 \theta = c\dot{\theta} + d\dot{\mu} + \gamma_\mu \mathrm{div}\,\dot{\mathbf{u}} - Q,$$

(6.3) $$D\nabla^2 \mu = n\dot{\mu} + d\dot{\theta} + \gamma\; \mathrm{div}\,\dot{\mathbf{u}} - \sigma, \qquad Q = \frac{W}{T_0},$$

while the Eq. (5.11) should be replaced by:

(6.4) $$\int_V (X_i \odot u_i' - X_i' \odot u_i)\,dV + \int_A (p_i \odot u_i' - p_i' \odot u_i)\,dA + \frac{k}{T_0} \left(\theta * \frac{\partial \theta'}{\partial n} - \theta' * \frac{\partial \theta}{\partial n}\right) dA + D \int_A \left(\mu * \frac{\partial \mu'}{\partial n} - \mu' * \frac{\partial \mu}{\partial n}\right) dA + \int_V (Q * \theta' - Q' * \theta)\,dV + \int_V (\sigma * \mu' - \sigma' * \mu)\,dV = 0,$$

where

$$X_i \odot u_i' = \int_0^t X_i(\mathbf{x}, t-\tau) \frac{\partial u_i'(\mathbf{x}, \tau)}{\partial \tau}\, d\tau,$$

$$Q * \theta' = \int_0^t Q(\mathbf{x}, t-\tau)\theta'(\mathbf{x}, \tau)\,d\tau, \text{ etc.}$$

Note that the Eq. (6.4) is required to hold under the homogeneous initial data only, and the following identities to be observed

$$p_i = \sigma_{ji} n_j, \qquad \frac{\partial \theta}{\partial n} = \frac{\partial \theta}{\partial x_i} n_i, \qquad \frac{\partial \mu}{\partial n} = \frac{\partial \mu}{\partial x_i} n_i,$$

where $\mathbf{n}$ in the outward unit normal to A.

Consider an infinite thermodiffusive elastic medium in which the source functions $\mathbf{X}$, Q and σ are defined in the bounded domain V, and assume that the state (u_i, θ, μ)

due to these sources is to vanish as $|x_1^2+x_2^2+x_3^2| \to \infty$. In this case, the surface integrals of (6.4) vanish, and we obtain:

$$\int\limits_V (X_i \odot u_i' - X_i' \odot u_i)\,dV + \int\limits_V (Q \ast \theta' - Q' \ast \theta)\,dV + \int\limits_V (\sigma * \mu' - \sigma' * \mu)\,dV = 0, \tag{6.5}$$

where the external forces

$$I = \{X_i, Q, \sigma\}, \quad I' = \{X_i', Q', \sigma'\}$$

and resulting states

$$C = \{u_i, \theta, \mu\}, \quad C' = \{u_i', \theta', \mu'\},$$

are included.

In what follows we shall define a number of singular elastic thermodiffusive states (Green functions) which will be used to give further reciprocal relations and to obtain some integral representations of a thermodiffusive elastic state. If $I = \{\delta(\mathbf{x}-\boldsymbol{\xi})\,\delta(t)\,\delta_{ij}, 0, 0\}$ is the set of external forces in the Eqs. (6.1)–(6.3), then the resulting state will be denoted by

$$G = \{U_{ij}(\mathbf{x}, \boldsymbol{\xi}, t), \Theta_j^X(\mathbf{x}, \boldsymbol{\xi}; t), \quad \mathcal{M}_j^X(\mathbf{x}, \boldsymbol{\xi}; t)\}.$$

Thus, G is a singular state of thermodiffusive elasticity produced in an infinite space by an instantaneous concentrated force which is parallel to the x_j-axis and applied at the point $\boldsymbol{\xi}$.

Similarly, we can introduce a singular state $\{U_i^Q, \Theta^Q, \mathcal{M}^Q\}$ corresponding to an instantaneous concentrated source of heat: $Q = \delta(\mathbf{x}-\boldsymbol{\xi})\,\delta(t)$, as also a state $\{U_i^\sigma, \Theta^\sigma, \mathcal{M}^\sigma\}$ due to the source of mass of the form $\sigma = \delta(\mathbf{x}-\boldsymbol{\xi})\,\delta(t)$.

These three singular states are listed in the Table

I	C		
	u_i	θ	μ
$\{X_i = \delta(\mathbf{x}-\boldsymbol{\xi})\,\delta_{ij}\,\delta(t), 0, 0\}$	U_{ij}	Θ_j^X	$\mathcal{M}_j^X$
$\{0, \delta(\mathbf{x}-\boldsymbol{\xi})\,\delta(t), 0\}$	$U_i^{'}$	$\Theta^{'}$	$\mathcal{M}^0$
$\{0, 0, \delta(\mathbf{x}-\boldsymbol{\xi})\,\delta(t)\}$	U_i^σ	Θ^σ	$\mathcal{M}^\sigma$

Consider two different sets of external forces acting on the solid

$$I = \{\delta(\mathbf{x}-\boldsymbol{\xi})\delta(t)\,\delta_{ij}, 0, 0\}, \quad I' = \{\delta(\mathbf{x}-\boldsymbol{\xi}')\,\delta(t)\delta_{ik}, 0, 0\}.$$

Corresponding states are given by

$$G = \{U_{ij}(\mathbf{x}, \boldsymbol{\xi}, t), \Theta_j^X(\mathbf{x}, \boldsymbol{\xi}, t), \mathcal{M}_j^X(\mathbf{x}, \boldsymbol{\xi}, t)\}, \; G' = \{U_{ik}(\mathbf{x}, \boldsymbol{\xi}', t), \Theta_k^X(\mathbf{x}, \boldsymbol{\xi}', t), \mathcal{M}_k^X(\mathbf{x}, \boldsymbol{\xi}', t)\}.$$

The reciprocity equation (6.5) associated with these two states reads

$$\int\limits_V dV(\mathbf{x}) \int\limits_0^t \delta(\mathbf{x}-\boldsymbol{\xi})\delta_{ij}\delta(t-\tau)\, \frac{\partial U_{ik}(\mathbf{x}, \boldsymbol{\xi}', \tau)}{\partial \tau}\, d\tau =$$

$$= \int_V dV(\mathbf{x}) \int_0^t \delta(\mathbf{x}-\boldsymbol{\xi}')\,\delta_{ik}\delta(t-\tau)\,\frac{\partial U_{ij}(\mathbf{x},\boldsymbol{\xi},\tau)}{\partial\tau}\,d\tau.$$

Thus,

$$\dot{U}_{jk}(\boldsymbol{\xi},\boldsymbol{\xi}',t) = \dot{U}_{kj}(\boldsymbol{\xi}',\boldsymbol{\xi},t);$$

and by the homogeneous initial conditions we obtain:

$$U_{jk}(\boldsymbol{\xi},\boldsymbol{\xi}',t) = U_{kj}(\boldsymbol{\xi}',\boldsymbol{\xi},t). \tag{6.6}$$

Assume now that I and I' reduce to

$$I = \{0,\,\delta(\mathbf{x}-\boldsymbol{\xi})\,\delta(t),\,0\},\qquad I' = \{0,\,\delta(\mathbf{x}-\boldsymbol{\xi}')\,\delta(t),\,0\};$$

then

$$G = \{U_i^Q(\mathbf{x},\boldsymbol{\xi},t),\,\Theta^Q(\mathbf{x},\boldsymbol{\xi},t),\,\mathscr{M}^Q(\mathbf{x},\boldsymbol{\xi},t)\},$$

$$G' = \{U_i^Q(\mathbf{x},\boldsymbol{\xi}',t),\,\Theta^Q(\mathbf{x},\boldsymbol{\xi}',t),\,\mathscr{M}^Q(\mathbf{x},\boldsymbol{\xi}',t)\},$$

and the reciprocity equation (6.5) leads to:

$$\Theta^Q(\boldsymbol{\xi}',\boldsymbol{\xi},t) = \Theta^Q(\boldsymbol{\xi},\boldsymbol{\xi}',t). \tag{6.7}$$

If I and I' take the form:

$$I = \{0,\,0,\,\delta(\mathbf{x}-\boldsymbol{\xi})\,\delta(t)\},\qquad I' = \{0,\,0,\,\delta(\mathbf{x}-\boldsymbol{\xi}')\,\delta(t)\},$$

then we obtain:

$$\mathscr{M}^Q(\boldsymbol{\xi}',\boldsymbol{\xi},t) = \mathscr{M}^Q(\boldsymbol{\xi},\boldsymbol{\xi}',t). \tag{6.8}$$

Assume next that I and I' are of the form:

$$I = \{\delta(\mathbf{x}-\boldsymbol{\xi})\,\delta(t)\,\delta_{ij},\,0,\,0\},\qquad I' = \{0,\,\delta(\mathbf{x}-\boldsymbol{\xi}')\,\delta(t),\,0\}.$$

Then the Eq. (6.5) reduces to

$$\int_V dV(\mathbf{x}) \int_0^t \delta(\mathbf{x}-\boldsymbol{\xi})\,\delta(t-\tau)\,\frac{\partial U_i^Q(\mathbf{x},\boldsymbol{\xi}',\tau)}{\partial\tau}\,d\tau$$

$$= \int_V dV(\mathbf{x}) \int_0^t \delta(\mathbf{x}-\boldsymbol{\xi}')\,\delta(t-\tau)\Theta_j^X(\mathbf{x},\boldsymbol{\xi},\tau)\,d\tau,$$

which implies

$$\dot{U}_j^Q(\boldsymbol{\xi},\boldsymbol{\xi}',t) = \Theta_j^X(\boldsymbol{\xi}',\boldsymbol{\xi},t). \tag{6.9}$$

Similarly, if we set

$$I = \{\delta(\mathbf{x}-\boldsymbol{\xi})\,\delta(t)\,\delta_{ij},\,0,\,0\},\qquad I' = \{0,\,0,\,\delta(\mathbf{x}-\boldsymbol{\xi}')\,\delta(t)\},$$

we obtain:

$$\dot{U}_j^q(\boldsymbol{\xi},\boldsymbol{\xi}',t) = \mathscr{M}_j^X(\boldsymbol{\xi}',\boldsymbol{\xi},t). \tag{6.10}$$

Finally, if I and I' are given by

$$I = \{0,\,\delta(\mathbf{x}-\boldsymbol{\xi})\,\delta(t),\,0\},\qquad I' = \{0,\,0,\,\delta(\mathbf{x}-\boldsymbol{\xi}')\,\delta(t)\},$$

by virtue of (6.5) we obtain:

$$\Theta^{\sigma}(\xi, \xi', t) = \mathcal{M}^{Q}(\xi', \xi, t). \tag{6.11}$$

The Eq. (6.6) generalizes well the known J. C. Maxwell theorem of the classical elastokinetics.

Consider now the case in which

$$I = \{0, Q, 0\}, \quad I' = \{0, \delta(\mathbf{x}-\mathbf{x}')\delta(t), 0\},$$
$$G = \{u_i, \theta, \mu\}, \quad G' = \{U^{Q}, \Theta^{Q}, \mathcal{M}^{Q}\}.$$

Then, by (6.5) we obtain:

$$\theta(\mathbf{x}', t) = \int_V dV(\mathbf{x}) \int_0^t Q(\mathbf{x}, t-\tau)\Theta^{Q}(\mathbf{x}, \mathbf{x}', \tau)\,d\tau. \tag{6.12}$$

If Q describes a concentrated source of heat moving with a constant velocity v along the x_3-axis, then substituting

$$Q(\mathbf{x}, t) = Q_0\,\delta(x_1)\,\delta(x_2)\,\delta(x_3-vt), \tag{6.13}$$

into (6.12), we obtain:

$$\theta(\mathbf{x}', t) = Q_0 \int_0^t \Theta^{Q}(0, 0, v\tau; x_1', x_2', x_3'; t-\tau)\,d\tau. \tag{6.14}$$

The last formula yields the temperature at the point x' and at time t, if Θ^{Q} is available.

If we assume that

$$I = \{0, Q, 0\}, \quad I' = \{0, 0, \delta(\mathbf{x}-x')\,\delta(t)\},$$

and if Q is given by (6.13), then by (6.5)

$$\mu(\mathbf{x}', t) = Q_0 \int_0^t \mathcal{M}^{\sigma}(0, 0, v\tau; x_1', x_2', x_3', t-\tau)\,d\tau. \tag{6.15}$$

Finally, if we set

$$I = \{0, Q, 0\}, \quad I' = \{\delta(\mathbf{x}-\mathbf{x}')\,\delta(t)\,\delta_{ij}, 0, 0\},$$

the reciprocity equation (6.5) leads to

$$\dot{u}_j(\mathbf{x}', t) = Q_0 \int_0^t \Theta_j^{X}(0, 0, v\tau; x_1', x_2', x_3', t-\tau)\,d\tau. \tag{6.16}$$

Similar integral representations of other functions describing a thermodiffusive elastic state can be achieved if we take into account a concentrated source of mass moving with a constant velocity or a concentrated movable force.

Also note that the Eqs. (6.6)–(6.11) remain valid for a bounded body V provided we let

$$u = 0, \quad \theta = 0, \quad \mu = 0 \quad \text{on} \quad A_u,$$
$$p_i = 0, \quad \frac{\partial\theta}{\partial n} = 0, \quad \frac{\partial\mu}{\partial n} = 0 \quad \text{on} \quad A_\sigma, \quad (A = A_u + A_\sigma)$$

$$U_{ij} = U_j^Q = U_j^\sigma = \Theta_j^X = \Theta^Q = \Theta^\sigma = \mathcal{M}_j^X = \mathcal{M}^Q = \mathcal{M}^\sigma = 0 \quad \text{on} \quad A_u,$$

$$p_i^X = p^Q = p_i^\sigma = \frac{\partial \Theta_j^X}{\partial n} = \frac{\partial \Theta^Q}{\partial n} = \frac{\partial \Theta^\sigma}{\partial n} = \frac{\partial \mathcal{M}^X}{\partial n} = \frac{\partial \mathcal{M}^Q}{\partial n} = \frac{\partial \mathcal{M}^\sigma}{\partial n} = 0 \quad \text{on} \quad A_\sigma,$$

where

$$p_j^X = \sigma_{jk}^X n_k, \quad p_j^Q = \sigma_{jk}^Q n_k, \quad p_j^\sigma = \sigma_{jk}^\sigma n_k.$$

7. Somigliana Formulae of Thermodiffusive Elasticity

In the classical theory of elastostatics, there are integral formulae, called the Somigliana formulae, which relate the displacement vector at an internal point of a solid to the displacement vector and the stress vector on the boundary of the solid. In this Section, we shall obtain similar formulae for a thermodiffusive elastic solid.

Assume that we are given the state G':

$$G' = \{U_{ij}^X(\mathbf{x}, \mathbf{x}'; t), \Theta_j^X(\mathbf{x}, \mathbf{x}'; t), \mathcal{M}_j^X(\mathbf{x}, \mathbf{x}'; t)\},$$

corresponding to the external forces

$$I' = \{\delta(\mathbf{x}-\mathbf{x}')\delta(t)\delta_{ij}, 0, 0\}.$$

Substituting these sets of functions into the reciprocity equation (6.4), we obtain:

$$(7.1) \quad \dot{u}_j(\mathbf{x}', t) = \int_V (X_i \odot U_{ij} + Q \ast \Theta_j^X + \sigma \ast \mathcal{M}_j^X) dV + \int_A (p_i \odot U_{ij} - p_i^X \odot u_i) dA + \frac{k}{T_0} \int_A \left(\theta \ast \frac{\partial \Theta_j^X}{\partial n} - \Theta_j^X \ast \frac{\partial \theta}{\partial n}\right) dA + D \int_A \left(\mu \ast \frac{\partial \mathcal{M}_j^X}{\partial n} - \mathcal{M}_j^X \ast \frac{\partial \mu}{\partial n}\right) dA,$$

where

$$p_i^{X_j} = \sigma_{ik}^{X_j}(\mathbf{x}, \mathbf{x}', t) n_k(\mathbf{x}), \quad \sigma_{ik}^{X_j} = 2\mu \varepsilon_{ik}^{X_j} + \lambda \delta_{ik} \varepsilon_{nn}^{X_j},$$

$$\varepsilon_{ik}^{X_j} = \frac{1}{2}\left(\frac{\partial U_{ij}}{\partial x_k} + \frac{\partial U_{kj}}{\partial x_i}\right).$$

If I' and G' are chosen such that

$$I' = \{0, \delta(\mathbf{x}-\mathbf{x}')\delta(t), 0\}, \quad G' = \{U_i^Q, \Theta^Q, \mathcal{M}^Q\},$$

then by (6.4) we obtain:

$$(7.2) \quad \theta(\mathbf{x}, t) = \int_V (X_i \odot U_i^Q + Q \ast \Theta^Q + \sigma \ast \mathcal{M}^Q) dV + \int_A \left((p_i \odot U_i^Q - p_i^Q \odot u_i) dA + \frac{k}{T_0} \int_A \left(\theta \ast \frac{\partial \Theta^Q}{\partial n} - \Theta^Q \ast \frac{\partial \theta}{\partial n}\right) dA + D \int_A \left(\mu \ast \frac{\partial \mathcal{M}^Q}{\partial n} - \mathcal{M}^Q \ast \frac{\partial \mu}{\partial n}\right) dA,\right.$$

where

$$p_i^{\theta} = \sigma_{ik}^{\theta} n_k, \quad \sigma_{ki} = 2\mu\varepsilon_{ik}^{\theta} + \lambda\delta_{ij}\varepsilon_{nn}^{\theta},$$

$$\varepsilon_{ki} = \frac{1}{2}\left(\frac{\partial U_i^{\theta}}{\partial x_k} + \frac{\partial U_k^{\theta}}{\partial x_i}\right).$$

Finally, choosing $I' = \{0, 0, \delta(\mathbf{x}-\boldsymbol{\xi})\delta(t)\}$ and using again (6.4), we obtain:

$$\mu(\mathbf{x}', t) = \int_A (X_i \odot U_i^{\sigma} + Q \ast \Theta^{\sigma} + \sigma \ast \mathcal{M}^{\sigma}) dV \tag{7.3}$$

$$= \int_A (p_i \odot U_i^{\sigma} - p_i^{\sigma} \odot u)\, dA + \frac{k}{T_0} \int_A \left(\theta \ast \frac{\partial \Theta^{\sigma}}{\partial n} - \Theta^{\sigma} \ast \frac{\partial \theta}{\partial n}\right) dA$$

$$+ D \int_A \left(\mu \ast \frac{\partial \mathcal{M}^{\sigma}}{\partial n} - \mathcal{M}^{\sigma} \ast \frac{\partial \mu}{\partial n}\right) dA,$$

where

$$p_i^{\sigma} = \sigma_{ik}^{\sigma} n_j, \quad \sigma_{ki}^{\sigma} = 2\mu\varepsilon_{ki}^{\sigma} + \lambda\delta_{ik}\varepsilon_{nn}^{\sigma},$$

$$\varepsilon_{ik}^{\sigma} = \frac{1}{2}\left(\frac{\partial U_i^{\sigma}}{\partial x_k} + \frac{\partial U_k^{\sigma}}{\partial x_i}\right).$$

The formulae (7.1)–(7.3) constitute a generalization of the Somigliana formulae of the classical elasticity to dynamic thermodiffusive elasticity. The surface integrals of (7.1)–(7.3) involve the six functions: $u_i, p_i, \theta, \frac{\partial \theta}{\partial n}, \mu, \frac{\partial \mu}{\partial n}$. Since only three of these functions are prescribed in a boundary-initial value problem of thermodiffusive elasticity, the formulae (7.1)–(7.3) are of a somewhat theoretical character. They may be modified to a useful form if the Green functions involved in (7.1)–(7.3) concern a bounded solid and if they satisfy proper boundary conditions.

First assume that $U_{ij}^X, \Theta^X, \ldots, \mathcal{M}^{\sigma}$ vanish on A. Then (7.1)–(7.3) reduce to

$$u(\mathbf{x}', t) = -\int_A p_i^{Xj} \odot u_i dA + \frac{k}{T_0} \int_A \theta \ast \frac{\partial \Theta_j^X}{\partial n} dA + D \int_A \mu \ast \frac{\partial \mathcal{M}_j^X}{\partial n} dA, \tag{7.4}$$

$$\theta(\mathbf{x}', t) = -\int_A p_i^{Q} \odot u_i dA + \frac{k}{T_0} \int_A \theta \ast \frac{\partial \Theta^{Q}}{\partial n} dA + D \int_A \mu \ast \frac{\partial \mathcal{M}^{Q}}{\partial n} dA, \tag{7.5}$$

$$\mu(\mathbf{x}', t) = -\int_A p_i^{\sigma} \odot u_i dA + \frac{k}{T_0} \int_A \theta \ast \frac{\partial \Theta^{\sigma}}{\partial n} dA + D \int_A \mu \ast \frac{\partial \mathcal{M}^{\sigma}}{\partial n} dA, \tag{7.6}$$

where we have assumed: $X_i = Q = \sigma = 0$.

The formulae (7.4)–(7.6) determine the state (u_i, θ, μ) subject to the boundary conditions

$$u_i = f_i, \quad \theta = g, \quad \mu = h, \quad \mathbf{x} \in A, \quad t > 0, \tag{7.7}$$

where f_i, g and h are prescribed functions.

If we choose the Green functions occurring in (7.1)–(7.3) in such a way that

$$p_i^{Xj} = 0, \qquad p_i^Q = 0, \qquad p_i^\sigma = 0,$$

$$\frac{\partial \Theta_j^X}{\partial n} = 0, \qquad \frac{\partial \Theta^Q}{\partial n} = 0, \qquad \frac{\partial \Theta^\sigma}{\partial n} = 0,$$

$$\frac{\partial \mathcal{M}_j^X}{\partial n} = 0, \qquad \frac{\partial \mathcal{M}^Q}{\partial n} = 0, \qquad \frac{\partial \mathcal{M}^\sigma}{\partial n} = 0 \quad \text{on} \quad A,$$

then the Eqs. (7.1)–(7.3) imply:

$$\dot{u}_j(\mathbf{x}', t) = \int_A p_i \odot U_{ij} dA - \frac{k}{T_0} \int_A \Theta_j^X * \frac{\partial \theta}{\partial n} dA - D \int_A \mathcal{M}_j^X * \frac{\partial \mu}{\partial n} dA, \tag{7.8}$$

$$\theta(\mathbf{x}', t) = \int_A p_i \odot U_i^Q dA - \frac{k}{T_0} \int_A \Theta^Q * \frac{\partial \theta}{\partial n} dA - D \int_A \mathcal{M}^Q * \frac{\partial \mu}{\partial n} dA, \tag{7.9}$$

$$\mu(\mathbf{x}', t) = \int_A p_i \odot U_i^\sigma dA - \frac{k}{T_0} \int_A \Theta^\sigma * \frac{\partial \theta}{\partial n} dA - D \int_A \mathcal{M}^\sigma * \frac{\partial \mu}{\partial n} dA. \tag{7.10}$$

The last formulae determine a thermodiffusive elastic state subject to the boundary conditions

$$p_i = h_i, \qquad \frac{\partial \theta}{\partial n} = r, \qquad \frac{\partial \mu}{\partial n} = s, \qquad \mathbf{x} \in A, \quad t > 0, \tag{7.11}$$

where h_i, r, s are given functions.

8. Generalized V. M. Maysel Theorem

Assume that the source functions X_i, Q and σ do not vanish in V, and the state (u_i, θ, μ) is required to satisfy the following boundary conditions:

$$\begin{aligned} u_i = f_i, \quad \theta = g_i, \quad \mu = h, \quad \mathbf{x} \in A_u, \quad t > 0, \\ p_i = g_i, \quad \frac{\partial \theta}{\partial n} = r, \quad \frac{\partial \mu}{\partial n} = s, \quad \mathbf{x} \in A_\sigma, \quad t > 0. \end{aligned} \tag{8.1}$$

Assume also that it is possible to find the Green functions $U_{ij}, \ldots, \mathcal{M}^\sigma$ that satisfy the boundary conditions:

$$\begin{aligned} U_{ij}^X = \ldots = \mathcal{M}^\sigma = 0 \quad \text{on} \quad A_u, \\ p_i^X = 0, \quad p_i^Q = 0, \quad p_i^\sigma = 0, \\ \frac{\partial \Theta_j^X}{\partial n} = 0, \quad \frac{\partial \Theta^Q}{\partial n} = 0, \quad \frac{\partial \Theta^\sigma}{\partial n} = 0, \\ \frac{\partial \mathcal{M}_j^X}{\partial n} = 0, \quad \frac{\partial \mathcal{M}^Q}{\partial n} = 0, \quad \frac{\partial \mathcal{M}}{\partial n} = 0 \quad \text{on} \quad A_\sigma. \end{aligned} \tag{8.2}$$

Then using (7.1)–(7.3) we arrive at the generalized Maysel formulae

$$(8.3) \qquad \dot{u}_j(\mathbf{x}', t) = \int_V (X_i \odot U_{ij} + Q * \Theta_j^X + \sigma * \mathcal{M}_j^X) dV + \int_{A_\sigma} \left(p_i \odot U_{ij} - \frac{k}{T_0} \Theta_j^X \frac{\partial \theta}{\partial n} - D \mathcal{M}_j^X * \frac{\partial \mu}{\partial n} \right) dA + \int_{A_u} \left(-p_i^X \odot u_i + \frac{k}{T_0} \theta * \frac{\partial \Theta_j^X}{\partial n} + D\mu * \frac{\partial \mathcal{M}_j^X}{\partial n} \right) dA,$$

$$(8.4) \qquad \theta(\mathbf{x}', t) = \int_V (X_i \odot U_i^Q + Q * \Theta_j^Q + \sigma * \mathcal{M}^Q) dV + \int_{A_\sigma} \left(p_i \odot U_i^Q - \frac{k}{T_0} \Theta^Q * \frac{\partial \theta}{\partial n} - D \mathcal{M}^Q * \frac{\partial \mu}{\partial n} \right) dA + \int_{A_u} \left(-p_i^Q \odot u_i + \frac{k}{T_0} \theta * \frac{\partial \Theta^Q}{\partial n} + D\mu * \frac{\partial \mathcal{M}^Q}{\partial n} \right) dA,$$

$$(8.5) \qquad \mu(\mathbf{x}', t) = \int_V (X_i \odot U_i^\sigma + Q * \Theta^\sigma + \sigma * \mathcal{M}^\sigma) dV + \int_{A_\sigma} \left(p_i \odot U_i^\sigma - \frac{k}{T_0} \Theta^\sigma * \frac{\partial \theta}{\partial n} - D \mathcal{M}^\sigma * \frac{\partial \mu}{\partial n} \right) dA + \int_{A_u} \left(-p_i^\sigma \odot u_i + \frac{k}{T_0} \theta * \frac{\partial \Theta^\sigma}{\partial n} + D\mu * \frac{\partial \mathcal{M}^\sigma}{\partial n} \right) dA.$$

Note that these formulae refer to a mixed boundary-value problem of thermodiffusive elasticity. They reduce to the Eqs. (7.4)–(7.6) for $X_i = Q = \sigma = 0$ and $A_\sigma = 0$, and they imply the Eqs. (7.8)–(7.10) if $X_i = Q = \sigma = 0$ and $A_u = 0$.

9. Simplification of the Governing Equations of Thermodiffusive Elasticity

One of the assumptions of the classical elastokinetics is that of a slow heat exchange between two adjacent parts of the body. Under this assumption, every part of the solid can be treated as thermally insulated and a thermodynamic process taking place in the solid is classified as adiabatic. In addition, it must be assumed that the heat sources vanish throughout the solid and its boundary is thermally insulated.

Since in an adiabatic process the time derivative of the entropy vanishes, from the constitutive equation

$$(9.1) \qquad \dot{S} = \gamma \operatorname{div}\dot{\mathbf{u}} + c\dot{\theta} + d\dot{\mu},$$

we obtain:

$$(9.2) \qquad \theta = -\frac{1}{c}(\gamma \operatorname{div}\mathbf{u} + d\mu).$$

The Eq. (9.2) implies that the temperature θ is proportional to the elastic dilatation and to the chemical potential. Substituting (9.2) into (1.18), (2.10), we arrive at

$$(9.3) \qquad G\nabla^2\mathbf{u} + (\lambda' + G)\operatorname{grad}\operatorname{div}\mathbf{u} + \mathbf{X} = \varrho\ddot{\mathbf{u}} + \eta \operatorname{grad}\mu,$$

$$(9.4) \qquad D\nabla^2\mu = n'\dot{\mu} + \eta \operatorname{div}\dot{\mathbf{u}} - \sigma,$$

where

$$\lambda' = \lambda + \frac{\gamma_\theta^2}{c}, \quad n' = \frac{nc-d^2}{c} > 0, \quad \eta = \gamma_\mu\left(1-\frac{d\gamma_\theta}{c\gamma_\mu}\right) > 0.$$

Thus the governing equations of thermodiffusive elasticity (1.18)–(1.20) have been reduced to two coupled equations (9.3) and (9.4). If a solution (u_i, μ) of (9.3)–(9.4) is known, θ may be computed from (9.2).

Observe that the resulting system of equations (9.3)–(9.4) is similar to that of the coupled thermoelasticity [3]. Solution of it can be obtained using the decomposition formulae (2.1). The resulting system of wave-like equations now reads:

(9.5) $$\Box_1 \Phi = m'_\mu \mu - \frac{1}{c_1^2},$$

(9.6) $$\Box_2 \boldsymbol{\Psi} = -\frac{1}{c_2^2}\boldsymbol{\chi},$$

(9.7) $$D'_2 \mu - \eta \nabla^2 \dot{\Phi} = -\sigma,$$

where

$$m'_\mu = \frac{\eta}{c_1^2 \varrho}, \quad c_1 = \left(\frac{\lambda'+2G}{\varrho}\right)^{1/2}, \quad D'_2 = D\nabla^2 - n'\partial_t.$$

Eliminating first μ and then Φ from the Eqs. (9.5), (9.7), we obtain:

(9.8) $$(\Box_1 D'_2 - \eta m'_\mu \nabla^2 \partial_t)\Phi = -\frac{1}{c_1^2} D'_2 \vartheta - m'_\mu \sigma,$$

(9.9) $$\Box_2 \boldsymbol{\Psi} = -\frac{1}{c_2^2}\boldsymbol{\chi},$$

(9.10) $$(\Box_1 D'_2 - \eta m'_\mu \nabla^2 \partial_t)\mu = -\frac{1}{c_1^2}\eta \nabla^2 \partial_t \vartheta - \Box_1 \sigma.$$

It can be seen that the longitudinal wave Φ and the chemical potential μ are subject to dispersion and damping. The Eqs. (9.8) and (9.10) can be used to find singular solutions of the diffusive elasticity in an infinite space.

To obtain the reciprocal theorem of diffusive elasticity, we make use of (6.4) and (9.2). We obtain:

(9.11) $$\int_V (X_i \odot u'_i - X'_i \odot u_i)\,dV + \int_A (p_i \odot u'_i - p'_i \odot u_i)\,dA + D\int_N \left(\mu * \frac{\partial \mu'}{\partial n} - \mu' * \frac{\partial \mu}{\partial n}\right) dA + \int_V (\sigma * \mu' - \sigma' * \mu)\,dA,$$

which for an infinite domain reduces to

(9.12) $$\int_V (X_i \odot u'_i + \sigma * \mu')\,dV = \int_V (X'_i \odot u_i + \sigma' * \mu)\,dV.$$

The Green functions associated with the system (9.3)–(9.4) can be listed in the following Table:

I	u	μ
$\{X_i = \delta(\mathbf{x}-\boldsymbol{\xi})\,\delta(t)\,\delta_{ij}, \quad \sigma = 0\}$	U_{ij}	$\mathcal{M}_j^X$
$\{X_i = 0, \quad \sigma = \delta(\mathbf{x}-\boldsymbol{\xi})\,\delta(t)\}$	U_i^σ	$\mathcal{M}^\sigma$

It can be shown that, by virtue of (9.12), these Green functions satisfy the following reciprocal relations:

$$(9.13) \qquad U_{jk}(\boldsymbol{\xi}, \boldsymbol{\xi}', t) = U_{kj}(\boldsymbol{\xi}', \boldsymbol{\xi}, t), \quad \mathcal{M}^\sigma(\boldsymbol{\xi}, \boldsymbol{\xi}', t) = \mathcal{M}^\sigma(\boldsymbol{\xi}', \boldsymbol{\xi}, t),$$
$$U_j^\sigma(\boldsymbol{\xi}, \boldsymbol{\xi}', t) = \mathcal{M}_j^X(\boldsymbol{\xi}', \boldsymbol{\xi}, t).$$

It can also be shown that the Somigliana formulae of the diffusive elasticity take the form:

$$(9.14) \quad u_j'(\mathbf{x}', t) = \int_V (X_i \odot U_{ij} + \sigma \ast \mathcal{M}_j^X)\,dV + \int_A (p_i \odot U_{ij} - p_i^{X_j} \odot u_i)\,dA + D\int_A \left(\mu \ast \frac{\partial M_j^X}{\partial n} - \mathcal{M}_j^X \ast \frac{\partial \mu}{\partial n}\right) dA,$$

$$(9.15) \quad \mu(\mathbf{x}', t) = \int_V (X_i \odot U^\sigma + \sigma \ast \mathcal{M}^\sigma)\,dV + \int_A (p_i \odot U_i^\sigma - p_i^\sigma \odot u_i)\,dA + D\int_A \left(\mu \ast \frac{\partial \mathcal{M}^\sigma}{\partial n} - \mathcal{M}^\sigma \ast \frac{\partial u}{\partial n}\right) dA.$$

In a similar way to that of Sec. 8, we can obtain the Maysel formulae of the diffusive elasticity.

10. Neglecting Elastic Dilatation in Like Heat and Diffusive Equations

The governing equations of thermodiffusive elasticity are considerably simplified if div $\dot{\mathbf{u}}$ is neglected in the Eqs. (1.19) and (1.20). Under this assumption the temperature and the chemical potential satisfy the equations which are independent of the elastic deformation and the Eqs. (1.18)–(1.20) reduce to the system:

$$(10.1) \qquad G\nabla^2\mathbf{u} + (\lambda + G)\,\mathrm{grad\,div}\,\mathbf{u} + \mathbf{X} = \varrho\ddot{\mathbf{u}} + \gamma_\theta\,\mathrm{grad}\,\theta + \gamma_\mu\,\mathrm{grad}\,\mu,$$

$$(10.2) \qquad \frac{k}{T_0}\nabla^2\dot{\theta} = c\dot{\theta} + d\dot{\mu} - W,$$

$$(10.3) \qquad D\nabla^2\mu = n\dot{\mu} + d\dot{\theta} - \sigma.$$

If a solution (Θ, μ) of the Eqs. (10.2) and (10.3) is found, RHS of (10.1) is known, and to solve our simplified problem we have to integrate the Eq. (10.1) to obtain $\mathbf{u}$.

Assuming that

$$(10.4) \qquad \begin{aligned} \mathbf{u} &= \mathrm{grad}\,\Phi + \mathrm{rot}\,\boldsymbol{\Psi}, & \mathrm{div}\,\boldsymbol{\Psi} &= 0, \\ \mathbf{X} &= \varrho(\mathrm{grad}\,\vartheta + \mathrm{rot}\,\boldsymbol{\chi}), & \mathrm{div}\,\boldsymbol{\chi} &= 0, \end{aligned}$$

and eliminating θ and μ from (10.1) by means of (10.2), (10.3), we obtain the wave-like equations

(10.5) $$H_1\Phi = -\frac{1}{c_1^2\varrho}(H\vartheta+M_1W+M_2\sigma), \qquad \Box_2\Psi = -\frac{1}{c_2^2}\chi.$$

If $W = \sigma = 0$, the functions Φ and Ψ can be found from the equations

(10.6) $$\Box_1\Phi = -\frac{1}{c_1^2\varrho}\vartheta, \qquad \Box_2\Psi = -\frac{1}{c_2^2}\chi$$

and it is seen that both the longitudinal wave Φ and the shear wave Ψ are undamped and undispersed.

If $\vartheta = W = \chi = 0$, we have:

(10.7) $$H\Box_1\Phi = -\frac{1}{c_1^2\varrho}M_2\sigma, \qquad \Psi = 0,$$

and the longitudinal wave consists of an elastic part and a diffusive part.

In our simplified theory, a reciprocal theorem can be also obtained. Setting $\bar{e} = \operatorname{div}\bar{\mathbf{u}} = 0$, $\bar{e}' = \operatorname{div}\bar{\mathbf{u}}' = 0$ in the Eqs. (5.7) and (5.9), from (5.5), (5.7), (5.9) we get:

(10.8) $$\int_V (\bar{X}_i\bar{u}_i' - \bar{X}_i'\bar{u}_i)dV + \int_A (\bar{p}_i\bar{u}_i' - \bar{p}_i'\bar{u}_i)dA + \int_V (\gamma_0\bar{\theta}+\gamma_\mu\bar{\mu})\bar{e}'dV - \int_V (\gamma_0\bar{\theta}'+\gamma_\mu\bar{\mu}')\bar{e}dV = 0,$$

(10.9) $$\frac{k}{T_0}\int_A \left(\bar{\theta}'\frac{\partial\bar{\theta}}{\partial n} - \bar{\theta}\frac{\partial\bar{\theta}'}{\partial n}\right)dA = pd_*\int_V (\bar{\mu}\bar{\theta}' - \bar{\mu}'\bar{\theta})dV,$$

(10.10) $$D\int_A \left(\bar{\mu}'\frac{\partial\bar{\mu}}{\partial n} - \bar{\mu}\frac{\partial\bar{\mu}'}{\partial n}\right)dA = pd\int_V (\bar{\theta}\bar{\mu}' - \bar{\theta}'\bar{\mu})dV.$$

If we add the Eqs. (10.9) and (10.10), we arrive at

(10.11) $$D\int_A \left(\bar{\mu}'\frac{\partial\bar{\mu}}{\partial n} - \bar{\mu}\frac{\partial\bar{\mu}'}{\partial n}\right)dA + \frac{k}{T_0}\int_A \left(\bar{\theta}'\frac{\partial\bar{\theta}}{\partial n} - \bar{\theta}\frac{\partial\bar{\theta}'}{\partial n}\right)dA = 0.$$

Inverting now (10.8) and (10.11), we obtain the following reciprocal relations:

(10.12) $$\int_V (X_i * u_i' - X_i' * u_i)dV + \int_A (p_i * u_i' - p_i' * u_i)dA + \gamma_J\int_V (\theta * e' - \theta' * e)dV + \gamma_\mu\int_V (\mu * e' - \mu' * e)dV = 0;$$

(10.13) $$D\int_A \left(\mu' * \frac{\partial\mu}{\partial n} - \mu * \frac{\partial\mu'}{\partial n}\right)dA + \frac{k}{T_0}\int_A \left(\theta' * \frac{\partial\theta}{\partial n} - \theta * \frac{\partial\theta'}{\partial n}\right)dA = 0.$$

In a similar way to that of Sec. 9 and by means of the Eqs. (10.12), (10.13), we can arrive at simplified formulae of the Somigliana and of the Maysel type.

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Streszczenie

DYNAMICZNY PROBLEM TERMODYFUZJI W CIELE STAŁYM

W pracy wyprowadzono odmienną niż w [1] postać równań termodyfuzji, przyjmując jako funkcje niezależne przemieszczenie **u**, temperaturę θ oraz potencjał chemiczny μ. Układ podstawowych równań różniczkowych termodyfuzji daje się sprowadzić do układu równań falowych. Wykazuje się, że fale podłużne, propagujące się w nieskończonej przestrzeni sprężystej, są tłumione i podlegają dyspersji, podczas gdy fale poprzeczne są nietłumione i nie ulegają dyspersji.

Przedstawiono szereg twierdzeń podstawowych termodyfuzji, jak twierdzenie wariacyjne, podstawowe twierdzenie energetyczne, twierdzenie o jednoznaczności rozwiązań oraz twierdzenie o wzajemności prac.

Omówiono wnioski wynikające z twierdzenia o wzajemności prac, podając rozszerzone na termodyfuzję twierdzenie Somigliana i Majziela.

Wresznie omówiono dwa przybliżone modele termodyfuzji. W pierwszym zakłada się adiabatyczność procesu termodynamicznego, w drugim pomija się wpływ dylatacji na pole temperatury i potencjału chemicznego.

Резюме

ДИНАМИЧЕСКАЯ ЗАДАЧА ТЕРМОДИФФУЗИИ В ТВЕРДОМ ТЕЛЕ

В работе выведен другой чем в [1] вид уравнений термодиффузии, принимая как независимые функции перемещение **u**, температуру θ и химический потенциал μ. Систему основных дифференциальных уравнений термодиффузии удается свести к системе волновых уравнений. Показывается, что продольные волны, распространяющиеся в бесконечном упругом пространстве, затухают и подлежат дисперсии, тогда как поперечные волны не затухают и не подлежат дисперсии.

Представлен ряд основных теорем термодиффузии таких как: вариационная теорема, основная энергетическая теорема, теорема единственности решений, а также теорема взаимности работ.

Обсуждены следствия вытекающие из теоремы взаимности работ, приводя расширенную на термодиффузию теорему Сомигляна и Майзеля.

Наконец обсуждены две приближенные модели термодиффузии. В первой предполагается адиабативность термодинамического процесса, а во второй пренебрегается влиянием дилятации на поля температуры и химического потенциала.

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