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ON SOME DYNAMICAL PROBLEMS OF THERMOELASTICITY

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§ 1. General relations and equations

Beginning in 1956, as the result of papers by M. A. Biot [1], M. Lessen [2], and P. Chadwick and I. N. Sneddon [3], a new topic of investigation in dynamical thermoelasticity was initiated which concerned the coupling between the temperature and strain fields. In 1958, P. Chadwick and I. N. Sneddon [3] made a detailed examination of the influence of thermal volume changes in a body on the form of plane harmonic waves. E. J. Lockett [4] considered the influence of temperature and strain fields on the velocity of propagation of Rayleigh surface waves. I. N. Sneddon [5] investigated the propagation of thermal stresses in thin metallic rods, produced by periodic forces and an impulse, or by heat sources situated at the end of the rod. In two papers [6, 7], H. Zorski examined the propagation of stresses in an infinite space, produced by the action of a thermal impulse.

In this paper, we shall be concerned with the propagation of elastic spherical, cylindrical and plane waves, due to the action of heat sources or centers of pressure which vary harmonically in time. We deal also with the generation of longitudinal waves in an infinite space with a spherical or cylindrical cavity, and with the propagation of thermal stresses produced by heating the plane boundary of a half-space.

The equations for a thermoelastic medium have the form

$$\mu \nabla^2 \mathbf{u} + (\lambda + \mu) \operatorname{grad} \operatorname{div} \mathbf{u} - \rho \ddot{\mathbf{u}} = \gamma_0 \operatorname{grad} \theta, \tag{1.1}$$

$$\nabla^2 \theta - \frac{1}{\kappa} \dot{\theta} + \eta \operatorname{div} \dot{\mathbf{u}} = -\frac{Q}{\kappa}, \tag{1.2}$$

where $\mathbf{u}$ is the displacement vector, θ is the temperature (under the assumption that $T + \theta$ is the absolute temperature and the state $\theta = 0$ is free of stresses and displacements), λ, μ are the Lamé constants, ρ is the density of the material and κ is the thermal conductivity. In addition, $Q = W/(\rho c)$, $\eta = \gamma_0 T/(\rho c)$, $\gamma_0 = \alpha_t(3\lambda + 2\mu)$, where W is the amount of heat generated in a unit volume of the body per unit time, c is the specific heat capacity and α_t is the coefficient of linear thermal expansion.

Introducing into the equations (1.1) and (1.2) the displacement function

$$\mathbf{u} = \operatorname{grad} \phi + \operatorname{rot} \boldsymbol{\psi}, \tag{1.3}$$

we reduce the system of equations (1.1)—(1.2) to the system of three equations

$$\nabla^2\phi - \frac{1}{c_1^2}\ddot{\phi} = \vartheta_0\theta, \tag{1.4}$$

$$\nabla^2\psi_i - \frac{1}{c_2^2}\ddot{\psi}_i = 0, \qquad (i = 1, 2, 3) \tag{1.5}$$

$$\nabla^2\theta - \frac{1}{\kappa}\dot{\theta} - \eta\nabla^2\dot{\phi} = -\frac{Q}{\kappa}. \tag{1.6}$$

In the above formulae $c_1 = \sqrt{\lambda + 2\mu}/\sqrt{\rho}$ is the velocity of propagation of the elastic longitudinal wave and $c_2 = \sqrt{\mu/\rho}$ is the velocity of the transverse wave. Moreover

$$\vartheta_0 = \frac{3\lambda + 2\mu}{\lambda + 2\mu}\alpha_t.$$

In the quasi-steady treatment of the problem the inertia forces are neglected; this assumption is admissible, if the temperature varies slowly in time. In this case we have the system of equations

$$\nabla^2\phi = \vartheta_0\theta, \tag{1.7}$$

$$\nabla^2\theta - \frac{1}{\kappa}\dot{\theta} - \eta\nabla^2\dot{\phi} = -\frac{Q}{\kappa}. \tag{1.8}$$

We can eliminate the function ϕ from the second equation. Then the conduction equation in the infinite space takes the form

$$\nabla^2\theta - \frac{1}{\kappa'}\dot{\theta} = -\frac{Q}{\kappa}, \quad \frac{1}{\kappa'} = \frac{1}{\kappa} + \eta\vartheta_0. \tag{1.9}$$

For a stationary field temperature the coupling of the fields vanishes, and in the infinite space we have the system

$$\nabla^2\theta = -\frac{Q}{\kappa}, \quad \nabla^2\phi = \vartheta_0\theta. \tag{1.10}$$

§ 2. Stresses due to the action of heat sources in an infinite body

We proceed to the investigation of thermoelastic waves in an infinite space. We shall successively consider the effects of point, linear and plane heat sources. We assume that the heat sources vary harmonically in time, i.e., $Q(P, t) = Q_0(P)e^{i\omega t}$, the frequency ω of the vibrations of the source being real and positive. Since the effect of heat sources results only in generation of longitudinal waves, we consider the equations (1.4) and (1.6). Setting

$$\phi(P, t) = e^{i\omega t}\phi^*(P),\ \theta(P, t) = e^{i\omega t}\theta^*(P),$$

we arrive at the system of equations

$$(\nabla^2 + \sigma^2)\phi^* = \vartheta_0\theta^*, \tag{2.1}$$

$$(\nabla^2 - q)\theta^* - q\eta'\nabla^2\phi^* = -\frac{Q_0}{\kappa}, \tag{2.2}$$

where

$$\sigma^2 = \frac{\omega^2}{c_1{}^2},\ q = i\frac{\omega}{\kappa},\ \eta' = \eta\kappa.$$

Eliminating from the equations (2.1) and (2.2) first the function ϕ^*, and then the function θ^*, we obtain

$$(\nabla^2 - q)(\nabla^2 + \sigma^2)\theta^* - q\varepsilon\nabla^2\theta^* = -\frac{1}{\kappa}(\nabla^2 + \sigma^2)Q_0(P), \tag{2.3}$$

$$(\nabla^2 - q)(\nabla^2 + \sigma^2)\phi^* - q\varepsilon\nabla^2\phi^* = -\frac{\vartheta_0}{\kappa}Q_0(P), \tag{2.4}$$

where

$$\varepsilon = \eta\kappa\vartheta_0.$$

Let a concentrated heat source be situated at the origin of the coordinate system. The solution of the equation (2.4) in cylindrical coordinates can be represented in the integral form

$$\phi^* = -\frac{Q_0\vartheta_0}{2\pi^2\kappa}\int_0^\infty\int_0^\infty \frac{\alpha J_0(\alpha r)\cos\gamma r}{F(\alpha, \gamma)}\,d\alpha\,d\gamma, \tag{2.5}$$

where

$$F(\alpha, \gamma) = (\alpha^2 + \gamma^2 + k_1{}^2)(\alpha^2 + \gamma^2 + k_2{}^2),$$

$$k_1{}^2 + k_2{}^2 = q(1 + \varepsilon) - \sigma^2,\ k_1{}^2k_2{}^2 = -q\sigma^2.$$

After the indicated integration has been carried out the function ϕ^* takes the closed form

$$\phi^* = -\frac{Q_0\vartheta_0}{4\pi\kappa R(k_1^2 - k_2^2)}(e^{-k_1R} - e^{-k_2R}), \tag{2.6}$$

where

$$R = (x_1^2 + x_2^2 + x_3^2)^{\frac{1}{2}}, \; k_{1,2} = a_{1,2} + ib_{1,2}, \; a_{1,2} > 0.$$

Solving the equation (2.3) we have

$$\theta^* = \frac{Q_0}{4\pi\kappa R(k_1^2 - k_2^2)}[(\sigma^2 + k_1^2)e^{-k_1R} - (\sigma^2 + k_2^2)e^{-k_2R}]. \tag{2.7}$$

Let us now move the heat source to the point (ξ_1, ξ_2, ξ_3) and set $Q_0 = 1$. Then $\phi^*(x_r, \xi_r)$ and $\theta^*(x_r, \xi_r)$ are the Green's functions for the infinite space.

The stresses corresponding to the temperature field θ are given by the relations

$$\sigma_{ij}(x_r, \xi_r, t) = 2\mu[\phi_{,ij} - \delta_{ij}\phi_{,kk}] + \rho\ddot{\phi}\delta_{ij}. \tag{2.8}$$

Neglecting the coupling between the temperature and strain fields (i.e., for $\varepsilon = 0$, $k_1 = \sqrt{q}$, $k_2 = i\sigma$) we obtain the familiar result [9]

$$\begin{aligned} \phi^* &= \frac{Q_0\vartheta_0}{4\pi\kappa R(\sigma^2 + q)}(e^{-R\sqrt{q}} - e^{-Ri\sigma}), \\ \theta^* &= \frac{Q_0}{4\pi\kappa R}e^{-R\sqrt{q}}. \end{aligned} \tag{2.9}$$

Let us now consider the axi-symmetric problem. Suppose that on the surface of a cylinder of radius ρ the heat sources $Q(r, z) = e^{i\omega t}Q_0\delta(r - \rho)$ are uniformly distributed. Solving the equation (2.4) we arrive at the integral

$$\phi^* = -\frac{Q_0\vartheta_0\rho}{\kappa}\int_0^\infty \frac{\alpha J_0(\alpha\rho)J_0(\alpha r)}{F(\alpha)}d\alpha, \tag{2.10}$$

where

$$F(\alpha) = (\alpha^2 + k_1^2)(\alpha^2 + k_2^2), \; r = (x_1^2 + x_2^2)^{\frac{1}{2}}.$$

The function ϕ^* can be represented in the form

$$\phi^* = \frac{Q_0\vartheta_0\rho}{\kappa(k_1^2 - k_2^2)}\begin{cases} I_0(k_1r)K_0(k_1\rho) - I_0(k_2r)K_0(k_2\rho) & \text{for } 0 < r < \rho, \\ I_0(k_1\rho)K_0(k_1r) - I_0(k_2\rho)K_0(k_2r) & \text{for } \rho < r < \infty. \end{cases} \tag{2.11}$$

In the particular case of a linear heat source, assuming that

$$\lim_{\rho\to 0} 2\pi\rho Q_0 = q_0,$$

we obtain from (2.10)

$$\phi^* = -\frac{q_0\vartheta_0}{2\pi\kappa}\int_0^\infty \frac{\alpha J_0(\alpha r)}{F(\alpha)}\,d\alpha, \tag{2.12}$$

or

$$\phi^* = \frac{q_0\vartheta_0}{2\pi\kappa(k_1^2-k_2^2)}\,[K_0(k_1r) - K_0(k_2r)]. \tag{2.13}$$

It can readily be proved that

$$\theta^* = \frac{q_0}{2\pi\kappa(k_1^2-k_2^2)}\,[(\sigma^2+k_1^2)K_0(k_1r) - (\sigma^2+k_2^2)K_0(k_2r)]. \tag{2.14}$$

In the above formulae $I_0(z)$ denotes the modified Bessel function of the first kind and $K_0(z)$ that of the third kind.

If the coupling between the temperature and strain fields is neglected, the formulae (2.13) and (2.14) yield the familiar results [9]

$$\phi^* = \frac{q_0\vartheta_0}{2\pi\kappa(\sigma^2+q)}\,[K_0(r\sqrt{q}) - K_0(i\sigma r)], \tag{2.15}$$

$$\theta^* = \frac{q_0}{2\pi\kappa}\,K_0(r\sqrt{q}). \tag{2.16}$$

The stresses corresponding to the axi-symmetric function ϕ^* are given by the relations

$$\begin{aligned} \sigma_{rr} &= -\,e^{i\omega t}(2\mu r^{-1}\phi^*_{,r} + \rho\omega^2\phi^*),\\ \sigma_{\varphi\varphi} &= -\,e^{i\omega t}(2\mu\phi^*_{,rr} + \rho\omega^2\phi^*). \end{aligned} \tag{2.17}$$

Assume now that there acts in the plane $x_1 = 0$ a plane heat source $Q(P, t) = e^{i\omega t}Q_0\delta(x_1)$. Solving the equations (2.3) and (2.4) we obtain

$$\phi^* = \frac{Q_0\vartheta_0}{2\kappa(k_1^2-k_2^2)}\left[\frac{e^{-k_1x_1}}{k_1} - \frac{e^{-k_2x_1}}{k_2}\right], \tag{2.18}$$

$$\theta^* = \frac{Q_0}{2\kappa(k_1^2-k_2^2)}\left[(\sigma^2+k_1^2)\frac{e^{-k_1x_1}}{k_1} - (\sigma^2+k_2^2)\frac{e^{-k_2x_1}}{k_2}\right]. \tag{2.19}$$

The stresses produced by a plane heat source are

$$\sigma_{11} = -e^{i\omega t}\rho\omega^2\phi^*, \tag{2.20}$$
$$\sigma_{22} = \sigma_{33} = -e^{i\omega t}(2\mu\phi^*_{,11} + \rho\omega^2\phi^*).$$

§ 3. Effect of a center of pressure in an infinite space

The coupled equations of thermoelasticity enable us to determine the temperature field associated with the effect of forces varying in time. Consider the simplest three-dimensional problem in which only longitudinal waves occur.

Let there be given a center of pressure, i.e., a system of three force doublets acting in the directions x_1, x_2, x_3 respectively, and assume that it acts at the origin of the coordinate system. The displacement equations in this case have the form

$$\mu\nabla^2 u_i + (\lambda + \mu)u_{k,ki} + [\delta(x_r)]_{,i} = \rho\ddot{u}_i + \gamma_0\theta_{,i}, \qquad (i = 1, 2, 3) \tag{3.1}$$

where $\delta(x_r) = \delta(x_1)\delta(x_2)\delta(x_3)$. In the case of a center of pressure varying harmonically in time we take

$$u_i = e^{i\omega t}\phi^*_{,i}. \tag{3.2}$$

Thus the system of equations (3.1) is reduced to the equation

$$(\nabla^2 + \sigma^2)\phi^* = \vartheta_0\theta^* - \frac{1}{\lambda + 2\mu}\,\delta(x_r). \tag{3.3}$$

Assuming that there are no heat sources inside the body we supplement (3.3) by the equation

$$(\nabla^2 - q)\theta^* - q\eta'\nabla^2\phi^* = 0. \tag{3.4}$$

The solution of the system (3.3)—(3.4) yields for a point center of pressure

$$\phi^* = \frac{1}{4\pi\beta R(k_1{}^2 - k_2{}^2)}\,[(k_1{}^2 - q)e^{-k_1R} - (k_2{}^2 - q)e^{-k_2R}], \tag{3.5}$$

$$\theta^* = \frac{\eta' q}{4\pi\beta R(k_1{}^2 - k_2{}^2)}\,[k_1{}^2e^{-k_1R} - k_2{}^2e^{-k_2R}]. \tag{3.6}$$

Knowing the function ϕ^* we can determine the stresses by means of the formulas (2.8). Since the temperature field has a singularity at the origin, the point center of pressure plays a role analogous to that of the heat source in the preceding considerations.

If the coupling between the temperature and strain fields be neglected ($\eta' = 0$), we obtain the known result [10]

$$\phi = \frac{1}{4\pi\beta R} e^{i(\omega t - R\sigma)}, \qquad \theta = 0. \tag{3.7}$$

In the case of the effect of a line center of pressure (along the x_3-axis) we have

$$\phi^* = \frac{1}{2\pi\beta(k_1^2 - k_2^2)} [(k_1^2 - q)K_0(k_1 r) - (k_2^2 - q)K_0(k_2 r)], \tag{3.8}$$

$$\theta^* = \frac{\eta' q}{2\pi\beta(k_1^2 - k_2^2)} [k_1^2 K_0(k_1 r) - k_2^2 K_0(k_2 r)]. \tag{3.9}$$

Taking $\eta' = 0$ (no coupling) we arrive at the familiar result

$$\phi = \frac{1}{2\pi\beta} K_0(i\sigma r) e^{i\omega t}, \ \theta = 0. \tag{3.10}$$

Finally, for a plane center of pressure in the plane $x_1 = 0$, we have

$$\phi^* = \frac{1}{2\beta(k_1^2 - k_2^2)} \left(\frac{k_1^2 - q}{k_1} e^{-k_1 x_1} - \frac{k_2^2 - q}{k_2} e^{-k_2 x_1} \right), \tag{3.11}$$

$$\theta^* = \frac{\eta' q}{2\beta(k_1^2 - k_2^2)} (k_1^2 e^{-k_1 x_1} - k_2^2 e^{-k_2 x_1}), \tag{3.12}$$

$$x_1 > 0.$$

If $\eta' = 0$, then

$$\phi = \frac{1}{2\beta\sqrt{i\sigma}} \exp [i(\omega t - \sigma x_1)], \qquad \theta = 0. \tag{3.13}$$

§ 4. State of stress in an infinite space with a spherical or cylindrical cavity

Consider first a spherical cavity of radius a in an infinite space. The solution of the homogeneous equations (2.3)—(2.4) can be represented in the form

$$\phi^* = \frac{1}{R} (A e^{-k_1 R} + B e^{-k_2 R}), \tag{4.1}$$

$$\theta^* = \frac{1}{R\vartheta_0} [A(k_1^2 + \sigma^2) e^{-k_1 R} + B(k_2^2 + \sigma^2) e^{-k_2 R}]. \tag{4.2}$$

The constants A and B are to be determined from the boundary conditions for $R = a$. If we assume that

$$\theta^*(a) = \theta_0,$$

$$\sigma^*_{RR}(a) = -\left[\frac{4\mu}{R}\phi^*_{,R} + \rho\omega^2\phi^*\right]_{R=a} = 0, \tag{4.3}$$

then

$$A = \theta_0 a\vartheta_0 \frac{m_2 e^{k_1 a}}{\Delta}, \quad B = -\theta_0 a\vartheta_0 \frac{m_1 e^{k_2 a}}{\Delta}, \tag{4.4}$$

where

$$\Delta = (k_1{}^2 + \sigma^2)m_2 - (k_2{}^2 + \sigma^2)m_1,$$

$$m_1 = 4\mu(1 + ak_1) - a^2\omega\rho, \ m_2 = 4\mu(1 + ak_2) - a^2\omega\rho.$$

Setting now

$$\theta^*(a) = 0, \ \sigma^*_{RR} = -p_0, \tag{4.5}$$

we find

$$A = \frac{p_0 a^3 e^{k_1 a}}{\Delta}, \quad B = \frac{-p_0 a^3 e^{k_1 a}}{\Delta} \cdot \frac{k_1{}^2 + \sigma^2}{k_2{}^2 + \sigma^2}. \tag{4.6}$$

Knowledge of the function ϕ^* makes it possible to determine the stresses in accordance with the formulae

$$\begin{aligned} \sigma_{RR} &= -\frac{4\mu}{R}\phi_{,R} + \rho\ddot{\phi}, \\ \sigma_{\varphi\varphi} = \sigma_{\theta\theta} &= -2\mu(\phi_{,RR} + R^{-1}\phi_{,R}) + \rho\ddot{\phi}. \end{aligned} \tag{4.7}$$

If the coupling between the temperature and strain fields is neglected ($\varepsilon = 0$), we obtain from the formulae (4.1) and (4.2), taking into account the boundary conditions (4.5), the solutions

$$\phi^* = \frac{p_0 a^3}{\bar{m}_2 R} e^{-(R-a)i\sigma}, \qquad \theta^* = 0, \tag{4.8}$$

where

$$\bar{m}_2 = 4\mu(1 + a\sqrt{q}) - a^2\omega^2\rho.$$

In the case of a cylindrical cavity the solution of the homogeneous equations (2.3) and (2.4) has the form

$$\phi^* = AK_0(k_1 r) + BK_0(k_2 r), \tag{4.9}$$

$$\theta^* = \frac{1}{\vartheta_0}\left[A(k_1{}^2 + \sigma^2)K_0(k_1 r) + B(k_2{}^2 + \sigma^2)K_0(k_2 r)\right]. \tag{4.10}$$

Assuming that

$$\theta^*(a) = \theta_0,$$

$$\sigma_{rr}{}^*(a) = -\left[\frac{2\mu}{r}\phi^*_{,r} + \rho\omega^2\phi^*\right]_{r=a} = 0, \tag{4.11}$$

we obtain

$$A = \frac{Q_0\vartheta_0 n_2}{\varDelta_1}, \quad B = -\frac{Q_0\vartheta_0 n_1}{\varDelta_1}, \tag{4.12}$$

where

$$n_1 = 2\mu k_1 K_1(k_1 a) - \rho\omega^2 a K_0(k_1 a),$$

$$n_2 = 2\mu k_2 K_1(k_2 a) - \rho\omega^2 a K_0(k_2 a),$$

$$\varDelta_1 = n_2(\sigma^2 + k_1{}^2)K_0(k_1 a) - n_1(\sigma^2 + k_2{}^2)K_0(k_2 a).$$

In an analogous manner we determine the constants A and B for other boundary conditions. The stresses in the elastic body are given by the relations

$$\sigma_{rr} = -\frac{2\mu}{r}\phi_{,r} + \rho\ddot{\phi}, \qquad \sigma_{\varphi\varphi} = -2\mu\phi_{,rr} + \rho\ddot{\phi}. \tag{4.13}$$

We have now to investigate the roots of the equations

$$(k^2 + k_1{}^2)(k^2 + k_2{}^2) = 0, \tag{4.14}$$

$$k_1{}^2 + k_2{}^2 = q(1 + \varepsilon) - \sigma^2,$$

$$k_1{}^2 k_2{}^2 = -q\sigma^2,$$

which enter into all the results of §§ 2–4. These roots can be represented in the form

$$k_{1,2}^2 = \frac{a^2}{2\kappa^2}\left[-\eta^2 + i\eta(1 + \varepsilon) \pm \varDelta\right],$$

$$\varDelta = \sqrt{\eta^2[\eta^2 - (1 + \varepsilon)^2] + 2i\eta^3(1 - \varepsilon)},$$

where $\eta = \omega/\omega^*$ is a dimensionless quantity and $\omega^* = c_1{}^2/\kappa$ is a characteristic quantity of the thermoelastic medium which was introduced by P. Chadwick and I. N. Sneddon [3].

The root k_1 corresponds to the modified thermal wave, and k_2 to the modiefied elastic longitudinal wave. In fact, $k_1 = \sqrt{q}$, $k_2 = i\sigma$ for $\varepsilon = 0$. In the problems under consideration we are interested in roots the real parts of which are positive, since they describe the thermal and elastic waves diverging from the center of excitation into infinity.

P. Chadwick and I. N. Sneddon [3] investigated in detail the behavior of the roots k_1 and k_2 in terms of the parameter η, and determined their approximate values for $\eta \ll 1$ and $\eta \gg 1$. They proved that for $\eta \ll 1$

$$
\begin{aligned}
&\frac{\kappa}{c_1} k_1 = \sqrt{\frac{1+\varepsilon}{2}\eta}\left[(1+i) - \frac{\eta\varepsilon}{2(1+\varepsilon)^2}(1-i)\right],\\
(4.15)\qquad &\frac{\kappa}{c_1} k_2 = \frac{\eta^2\varepsilon}{2(1+\varepsilon)^{3/2}} + i\frac{\eta}{(1+\varepsilon)^{1/2}},
\end{aligned}
$$

and for $\eta \gg 1$

$$
\begin{aligned}
&\frac{\kappa}{c_1} k_1 = \sqrt{\frac{\eta}{2}}\left[\left(1 - \frac{\varepsilon}{2\eta}\right) + i\left(1 + \frac{\varepsilon}{2\eta}\right)\right],\\
(4.16)\qquad &\frac{\kappa}{c_1} k_2 = \tfrac{1}{2}\varepsilon + i\eta.
\end{aligned}
$$

In the same paper the authors analysed in detail the influence of the parameter ε on the velocity of longitudinal waves and on the dispersion coefficient. The results of Chadwick and Sneddon concern the influence of the temperature field on plane harmonic waves. It is clear, however, that these results are also true for spherical and cylindrical waves, since in this case the roots k_1 and k_2 are the same. The velocity of propagation of the modified elastic wave for $\eta \ll 1$ is given by the relation

$$c_1' = (1+\varepsilon)c_1,$$

where c_1 denotes the velocity of propagation of the elastic wave in the problem in which no coupling occurs, i.e., for $\varepsilon = 0$. The dispersion coefficient has the form

$$\chi = \frac{\varepsilon}{2}(2 - 5\varepsilon)\eta^2.$$

Since the quantity ε is small (for aluminum $\varepsilon = 3.56 \times 10^{-2}$, for steel $\varepsilon = 2.97 \times 10^{-4}$ and for lead $\varepsilon = 7.33 \times 10^{-2}$) it is evident that the influence of the coupling between the temperature and strain fields on the velocity of propagation of elastic plane, cylindrical and spherical waves due to the action of heat sources or centers of pressure is insignificant. Similarly, the differences in the magnitudes of the stresses are very small. Nevertheless, it is important to observe that by taking account of the coupling, it is possible to determine the temperature field generated by the effect of forces varying harmonically in time.

Finally, let us observe that the results deduced may prove useful for the determination of the stresses produced by instantaneous sources of heat or force. Denoting by $\sigma_{ij}(P, t)$ the stresses produced by the action of such sources we have

$$\sigma_{ij}(P, t) = \int_0^\infty \sigma_{ij}^*(P, \omega) e^{i\omega t} \, d\omega, \tag{4.17}$$

where $\sigma_{ij}^*(P, \omega)$ is the stress due to the sources of heat or forces which vary harmonically in time.

§ 5. Stresses due to heating of the plane boundary of an elastic half-space

Suppose that on the plane $z = 0$, bounding the elastic half-space, the temperature $Q(r, 0, t) = \theta_0^*(r) e^{i\omega t}$ is prescribed. This heating causes in the elastic semi-space an axi-symmetric temperature field and axi-symmetric state of stress. Assume that the surface $z = 0$ is free of tractions.

Let us construct the solution in two parts: the solution of the system of equations (2.1)—(2.2) for the infinite space the boundary condition $\theta(r, 0, t) = \theta_0^*(r) e^{i\omega t}$ being satisfied on the plane $z = 0$, and the additional solution for the half-space ensuring the fulfilment of the remaining boundary conditions. To deduce the first solution consider the equations

$$(\nabla^2 - q)\theta^* - q\eta' \nabla^2 \phi^* = 0, \tag{5.1}$$

$$(\nabla^2 + \sigma^2)\phi^* = \vartheta_0 \theta^*. \tag{5.2}$$

Applying the Fourier-Hankel integral transform we arrive at the system of algebraic equations

$$\begin{aligned} -(\alpha^2 + \gamma^2 - q)\tilde{\theta}^* + q\eta'(\alpha^2 + \gamma^2)\tilde{\phi}^* + \gamma\tilde{\theta}_0^* &= 0, \\ -(\alpha^2 + \gamma^2 - \sigma^2)\tilde{\phi}^* &= \vartheta_0 \tilde{\theta}^*, \end{aligned} \tag{5.3}$$

This problem was solved by a different method by Eason and Sneddon [11], and Paria [12].

where

$$\tilde{\theta}^*(\alpha, \gamma) = \sqrt{\frac{2}{\pi}} \int_0^\infty \int_0^\infty \theta^*(r, z) r J_0(\alpha r) \sin \gamma z \, dr \, dz,$$

(5.4)
$$\tilde{\phi}^*(\alpha, \gamma) = \sqrt{\frac{2}{\pi}} \int_0^\infty \int_0^\infty \phi^*(r, z) r J_0(\alpha r) \sin \gamma z \, dr \, dz,$$

$$\tilde{\theta}_0^*(\alpha) = \int_0^\infty r J_0(\alpha r) \theta_0^*(r) \, dr.$$

Solving the system (5.3) we obtain

(5.5)
$$\tilde{\phi}^* = -\frac{\gamma \vartheta_0 \tilde{\theta}_0^*}{F(\alpha, \gamma)}, \quad \tilde{\theta}^* = \frac{\gamma \tilde{\theta}^*}{F(\alpha, \gamma)} (\alpha^2 + \gamma^2 - \sigma^2),$$

where

(5.6)
$$F(\alpha, \gamma) = (\alpha^2 + \gamma^2 + k_1^2)(\alpha^2 + \gamma^2 + k_2^2),$$
$$k_1^2 + k_2^2 = q(1 + \varepsilon) - \sigma^2,$$
$$k_1^2 k_2^2 = -q\sigma^2.$$

Inverting the transform we have

(5.7)
$$\phi^*(r, z) = -\vartheta_0 \sqrt{\frac{2}{\pi}} \int_0^\infty \int_0^\infty \frac{\tilde{\theta}_0^*(\alpha) \gamma \alpha J_0(\alpha r) \sin \gamma z}{F(\alpha, \gamma)} \, d\alpha \, d\gamma,$$

or

(5.8)
$$\phi^*(r, z) = -\sqrt{\frac{2}{\pi}} \frac{\vartheta_0}{k_1^2 - k_2^2} \int_0^\infty \tilde{\theta}_0^*(\alpha) \alpha (e^{-k_1 z} - e^{-k_2 z}) J_0(\alpha r) \, d\alpha.$$

Knowing the function ϕ^* we calculate the stresses

(5.9)
$$\sigma_{rr}^{(1)} = -e^{i\omega t} \left[2\mu \left(\frac{\phi^*_{,r}}{r} + \phi^*_{,zz} \right) + \rho \omega^2 \phi^* \right],$$
$$\sigma_{\varphi\varphi}^{(1)} = -e^{i\omega t} [2\mu (\phi^*_{,rr} + \phi^*_{,zz}) + \rho \omega^2 \phi^*],$$
$$\sigma_{zz}^{(1)} = -e^{i\omega t} \left[2\mu \left(r^{-1} \frac{\phi^*_{,r}}{r} + \phi^*_{,rr} \right) + \rho \omega^2 \phi^* \right],$$
$$\sigma_{rz}^{(1)} = e^{i\omega t} \cdot 2\mu \phi^*_{,rz}.$$

In the plane $z = 0$ we have $\sigma_{zz}^{(1)} = 0$ and $\sigma_{rz}^{(1)} \neq 0$,

$$\sigma_{rz}^{(1)}(r, 0, t) = \sqrt{\frac{2}{\pi}} \frac{\vartheta_0 e^{i\omega t}}{k_1 + k_2} \int_0^\infty \alpha^3 \tilde{\theta}_0^*(\alpha) J_0(\alpha r) d\alpha. \tag{5.10}$$

The second stage of the solution consists in completing the state $\sigma_{ij}^{(1)}$ by a state $\sigma_{ij}^{(2)}$ such that in the plane $z = 0$ the following boundary conditions are satisfied:

$$\begin{gathered} \sigma_{zz}^{(2)}(r, 0, t) = 0, \\ \sigma_{rz}^{(1)}(r, 0, t) + \sigma_{rz}^{(2)}(r, 0, t) = 0, \\ \theta^{(2)}(r, 0, t) = 0. \end{gathered} \tag{5.11}$$

In the elastic half-space occur both longitudinal and transverse waves. Hence, in the second stage, we have to consider the system of equations

$$(\nabla^2 - q)(\nabla^2 + \sigma^2)\phi^{*(2)} - q\varepsilon\nabla^2\phi^{*(2)} = 0, \tag{5.12}$$

$$(\nabla^2 - \tau^2)\psi^* = 0. \tag{5.13}$$

The displacements are related to the functions $\phi^{(2)}$ and ψ by the formulae

$$\begin{aligned} u_r^{(2)} &= \phi_{,r}^{(2)} + \psi_{,rz}, \\ u_z^{(2)} &= \phi_{,z}^{(2)} - \psi_{,rr} - \frac{\psi_{,r}}{r} = \phi_{,z}^{(2)} + \psi_{,zz} - \frac{\psi}{c_2^2}. \end{aligned} \tag{5.14}$$

Hence

$$\begin{aligned} u_r^{*(2)} &= \phi_{,r}^{*(2)} + \psi_{,rz}^*, \\ u_z^{*(2)} &= \phi_{,z}^{*(2)} + \psi_{,zz}^* + \tau^2\psi^*. \end{aligned} \tag{5.15}$$

Expressing the stresses by the displacements we obtain

$$\begin{aligned} \sigma_{zz}^{*(2)} &= 2\mu(\partial_z^2 - \nabla^2)\phi^{*(2)} - \rho\omega^2\phi^{*(2)} + 2\mu(\partial_z^2 + \tau^2)_{,z}\psi^*, \\ \sigma_{rz}^{*(2)} &= 2\mu\phi_{,rz}^{*(2)} + \mu(2\partial_z^2 + \tau^2)_{,r}\psi^*. \end{aligned} \tag{5.16}$$

The solution of the equation (5.12) can be represented by the Hankel integral

$$\phi^{*(2)} = \int_0^\infty (Ae^{-\lambda_1 z} + Be^{-\lambda_2 z}) J_0(\alpha z)\, d\alpha, \tag{5.17}$$

where λ_1, λ_2 are the roots of the equation

$$\lambda^4 + [\sigma^2 - q(1+\varepsilon) - 2\alpha^2]\lambda^2 + \alpha^4 - \alpha^2[\sigma^2 - q(1+\varepsilon)] - q\sigma^2 = 0 \tag{5.18}$$

the real parts of which are positive.

The solution of the equation (5.13) is assumed to have the form

$$\psi^* = \int_0^\infty c(\alpha) e^{-\nu z} J_0(\alpha z) d\alpha, \qquad \nu = \sqrt{\alpha^2 - \tau^2}. \tag{5.19}$$

The amplitude of the temperature is given by the relation

$$\theta^{*(2)} = \frac{1}{\vartheta_0} (\nabla^2 + \sigma^2) \phi^{*(2)}. \tag{5.20}$$

Substituting (5.17) into (5.20) we have

$$\theta^{*(2)} = \frac{1}{\vartheta_0} \int_0^\infty [A(\lambda_1^2 + \sigma^2 - \alpha^2) e^{-\lambda_1 z} + B(\lambda_2^2 + \sigma^2 - \alpha^2) e^{-\lambda_2 z}] J_0(\alpha z)\, d\alpha. \tag{5.21}$$

The quantities A, B and C are to be determined in accordance with the boundary conditions (5.11). Thus we have

$$\begin{gathered} A = -\sqrt{2\pi} \frac{\vartheta_0 \mu \alpha^3 \tilde{\theta}_0^*(\alpha)}{k_1 + k_2} \cdot \frac{\nu n_2}{\Delta_1}, \\ B = -A \frac{n_1}{n_2}, \quad C = \frac{2\mu\alpha^2 - \rho\omega^2}{2\mu\nu\alpha^2} \cdot \frac{\lambda_2^2 - \lambda_1^2}{n_2} A, \\ \Delta_1 = (\lambda_1 n_2 - \lambda_2 n_1) 4\mu\nu\alpha^2 - (2\mu\alpha^2 - \rho\omega^2)(\nu^2 + \alpha^2)(\lambda_2^2 - \lambda_1^2), \\ n_{1,2} = \lambda_{1,2}^2 + \sigma^2 - \alpha^2. \end{gathered} \tag{5.22}$$

Knowledge of the functions $\phi^{*(2)}$ and $\psi^{(*)}$ enables us to determine the displacements $u_r^{*(2)}$, $u_z^{*(2)}$, and then also the stresses $\sigma_{ij}^{*(2)}$. Adding the stresses $\sigma_{ij}^{(1)}$ and $\sigma_{ij}^{(2)}$ we obtain the final stresses σ_{ij}.

The second stage of the solution is identical with the Lamb-problem for the elastic half-space.

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