Príspevky k teórii stavebných konštrukcií

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Green Functions for Micropolar Elasticity

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1. Introduction

In this paper we shall be concerned with an infinite body which is assumed to be elastic, isotropic, homogeneous and centrosymmetric. The action of body forces and moments induces in such a body the formation of displacement $u(\mathbf{x}, t)$ and rotation $\omega(\mathbf{x}, t)$ fields, varying with the position of the point $\mathbf{x}$ and time t.

The deformation of the body will be characterized by two asymmetric tensors, namely the strain tensor γ_{ji} and the curvature twist tensor $\varkappa_{ji}$ connected with the displacement and rotation fields by the relations, [1–4]

$$\gamma_{ji} = \mu_{i,j} - \varepsilon_{kji}\omega_k, \qquad \varkappa_{ji} = \mu_{i,j}. \tag{1.1}$$

The stress state is defined by the stress tensor σ_{ji} and the couple-stress tensor μ_{ji}. The dependence between the state of stress and that of strain is described by the following formulae

$$\sigma_{ji} = (\mu + \alpha)\,\gamma_{ji} + (\mu - \alpha)\,\gamma_{ij} + \lambda\gamma_{kk}\delta_{ij}, \tag{1.2}$$

$$\mu_{ji} = (\gamma + \varepsilon)\,\varkappa_{ji} + (\gamma - \varepsilon)\,\varkappa_{ij} + \beta\varkappa_{kk}\delta_{ij}. \tag{1.3}$$

The quantities μ, λ, α, β, γ, ε are material constants. Substituting Eqs. (1.2) and (1.3) successively, into equations of motion

$$\sigma_{ji,j} + X_i - \varrho\ddot{u}_i = 0, \tag{1.4}$$

$$\varepsilon_{ijk}\sigma_{jk} + \mu_{ji,j} + Y_i - I\ddot{\omega}_i = 0, \tag{1.5}$$

and expressing the quantities γ_{ji} and $\varkappa_{ji}$ by the displacements u_i and rotations ω_i, respectively, – in accordance with formulae (1.1) – we arrive at a system of six differential equations. We write them in vector form:

$$(\mu + \alpha)\,\nabla^2\boldsymbol{u} + (\lambda + \mu - \alpha)\operatorname{grad}\operatorname{div}\boldsymbol{u} + 2\alpha\operatorname{rot}\boldsymbol{\omega} + \boldsymbol{X} = \varrho\ddot{\boldsymbol{u}}, \tag{1.6}$$

$$(\gamma + \varepsilon)\nabla^2\boldsymbol{\omega} + (\beta + \gamma - \varepsilon)\,\mathrm{grad}\,\mathrm{div}\,\boldsymbol{\omega} - 4\alpha\boldsymbol{\omega} + 2\alpha\,\mathrm{rot}\,\boldsymbol{u} + \boldsymbol{Y} = I\ddot{\boldsymbol{\omega}}. \tag{1.7}$$

In the above equations the symbol $\boldsymbol{X}$ denotes the vector of body forces, $\boldsymbol{Y}$ stands for the body-couple vector, ϱ denotes the density and I the rotational inertia of the body. The time derivatives of the functions u_i and ω_i are marked with a dot.

The aim of this paper is to give basic solutions of the system of Eqs. (1.6) and (1.7) under the assumption that the body forces and couples vary harmonically with time.

We seek for the determination of the displacements $\boldsymbol{u} = \boldsymbol{U}^{(k)}(\boldsymbol{x}, \boldsymbol{\xi}, t)$ and rotations $\boldsymbol{\omega} = \boldsymbol{\Omega}^{(k)}(\boldsymbol{x}, \boldsymbol{\xi}, t)$ induced by the action of a concentrated force applied at the point $\boldsymbol{\xi}$ and directed parallelly to the x_k-axis as well as for the determination of the displacements $\boldsymbol{u} = \boldsymbol{V}^{(k)}(\boldsymbol{x}, \boldsymbol{\xi}, t)$ and rotations $\boldsymbol{\omega} = \boldsymbol{W}^{(k)}(\boldsymbol{x}, \boldsymbol{\xi}, t)$ induced by the action of a concentrated moment applied at the point $\boldsymbol{\xi}$ and directed parallelly to the x_k-axis. In this way we obtain two pairs of tensors $(U_j^{(k)}, \Omega_j^{(k)}$ and $(V_j^{(k)}, W_j^{(k)})$; they will be called, generally, Green functions for a medium with micropolar elasticity. To determine the Green functions of the system of Eqs. (1.6) and (1.7) it proved convenient to make use of the generalized Iacovache's functions $\boldsymbol{\varphi}$ i $\boldsymbol{\psi}$ [5] which – in the theory of asymmetric elasticity – are connected with the $\boldsymbol{u}, \boldsymbol{\omega}$ field by the following relations [6, 7].

$$\boldsymbol{u} = \Box_1\Box_4\boldsymbol{\varphi} - \mathrm{graddiv}\,\Gamma\boldsymbol{\varphi} - 2\alpha\,\mathrm{rot}\,\Box_3\boldsymbol{\psi}, \tag{1.8}$$

$$\boldsymbol{\omega} = \Box_2\Box_3\boldsymbol{\psi} - \mathrm{graddiv}\,\Theta\boldsymbol{\psi} - 2\alpha\,\mathrm{rot}\,\Box_1\boldsymbol{\varphi}. \tag{1.9}$$

Introducing (1.8) and (1.9) into the system of Eqs. (1.6) and (1.7) we get

$$\Box_1(\Box_2\Box_4 + 4\alpha^2\nabla^2)\,\boldsymbol{\varphi} + \boldsymbol{X} = 0, \tag{1.10}$$

$$\Box_3(\Box_2\Box_4 + 4\alpha^2\nabla^2)\,\boldsymbol{\psi} + \boldsymbol{Y} = 0. \tag{1.11}$$

The following notations have been introduced in the relations (1.8), (1.9) and in Eqs. (1.10), 1.11)

$$\begin{aligned} \Box_1 &= (\lambda + 2\mu)\nabla^2 - \varrho\,\partial_t^2, & \Box_2 &= (\mu + \alpha)\nabla^2 - \varrho\,\partial_t^2,\\ \Box_3 &= (\beta + 2\gamma)\nabla^2 - I\,\partial_t^2 - 4\alpha, & \Box_4 &= (\gamma + \varepsilon)\nabla^2 - I\,\partial_t^2 - 4\alpha,\\ \Gamma &= (\lambda + \mu - \alpha)\,\Box_4 - 4\alpha^2, & \Theta &= (\beta + \gamma - \varepsilon)\,\Box_2 - 4\alpha^2,\end{aligned} \tag{1.12}$$

$$\nabla^2 = \partial_i\partial_i, \qquad \partial_t^2 = \frac{\partial^2}{\partial t^2}, \qquad \partial_i = \frac{\partial}{\partial x_i}.$$

As may be easily seen, Eqs. (1.10) and (1.11) are particularly useful for the determination of Green functions. It suffices to find the particular integral of these equations and then to determine the displacements $\boldsymbol{u}$ and rotations $\boldsymbol{\omega}$ from the formulae (1.8) and (1.9). It appears from Eqs. (1.10) and (1.11) that – in the absence of body fordes – there is $\boldsymbol{\psi} \equiv 0$, while in the absence of body couples we have $\boldsymbol{\psi} \equiv 0$.

2. Effect of concentrated forces and couples

We are going now to consider the state of deformation in an infinite elastic region induced by the action of body forces and couples, varying harmonically with time.

$$\mathbf{X}(\mathbf{x}, t) = \mathbf{X}^*(\mathbf{x})\, e^{-i\omega t}, \qquad \mathbf{Y}(\mathbf{x}, t) = \mathbf{Y}^*(\mathbf{x})\, e^{-i\omega t} \tag{2.1}$$

In this way, the fields of displacements $\mathbf{u}$ and rotations $\boldsymbol{\omega}$ are formed, both varying harmonically with time, tvo. Marking with asterisks the amplitudes of the corresponding functions, we rewrite Eqs. (1.10), (1.11) in the following form

$$(\nabla^2 + k_1^2)(\nabla^2 + k_2^2)(\nabla^2 + k_3^2)\,\boldsymbol{\varphi}^* + \varkappa \mathbf{X}^* = 0, \tag{2.2}$$

$$(\nabla_1^2 + k_1^2)(\nabla_2^2 + k_2^2)(\nabla^2 + \hat{k}_3^2)\,\boldsymbol{\psi}^* + \sigma \mathbf{Y}^* = 0. \tag{2.3}$$

In the above equations the following notations were used

$$k_3 = \frac{\omega}{c_1}, \qquad \hat{k}_3 = \left(\frac{\omega^2 - \omega_0^2}{c_3^2}\right)^{\frac{1}{2}}, \qquad c_1 = \left(\frac{\lambda + 2\mu}{\varrho}\right)^{\frac{1}{2}},$$

$$c_3 = \left(\frac{\beta + 2\gamma}{I}\right)^{\frac{1}{2}}, \qquad \omega_0^2 = 4\alpha/I,$$

$$\varkappa = [(\lambda + 2\mu)(\mu + \alpha)(\gamma + \varepsilon)]^{-1}, \quad \sigma = [(\beta + 2\gamma)(\mu + \alpha)(\gamma + \varepsilon)]^{-1}.$$

The quantities k_1^2, k_2^2 are roots of the equation

$$k^4 - k^2(\sigma_2^2 + \sigma_4^2 + p(r - 2)) + \sigma_2^2(\sigma_4^2 - 2p) = 0, \tag{2.4}$$

where

$$\sigma_2 = \frac{\omega}{c_2}, \qquad \sigma_4 = \frac{\omega}{c_4}, \qquad c_2 = \left(\frac{\mu + \alpha}{\varrho}\right)^{\frac{1}{2}},$$

$$c_4 = \left(\frac{\gamma + \varepsilon}{I}\right)^{\frac{1}{2}}, \qquad p = \frac{2\alpha}{I c_4^2}, \qquad r = \frac{2\alpha}{\varrho c_2^2}.$$

It results from Eq. (2.4)

$$k_{1,2}^2 = \frac{1}{2}\left[\sigma_2^2 + \sigma_4^2 + p(r - 2) \pm \sqrt{(\sigma_4^2 - \sigma_2^2 + p(r - 2))^2 + 4pr\sigma_2^2}\right] \tag{2.5}$$

Let us remark that the determinant of Eq. (2.4) is positive.

We shall consider now the homogeneous equation (2.2). According to a known theorem of Boggio [8], the solution of this equation may be presented in the form of a sum of partial solutions, namely

$$\boldsymbol{\varphi}^* = \boldsymbol{\varphi}'^* + \boldsymbol{\varphi}''^* + \boldsymbol{\varphi}'''^*, \tag{2.6}$$

fullfilling the Helmholtz vector equations

$$(\nabla^2 + k_1^2)\,\boldsymbol{\varphi}'^* = 0, \quad (\nabla^2 + k_2^2)\,\boldsymbol{\varphi}''^* = 0, \quad (\nabla^2 + k_3^2)\,\boldsymbol{\varphi}'''^* = 0. \tag{2.7}$$

The functions $\frac{e^{\pm ik_\alpha R}}{R}$ $\alpha = 1, 2, 3$ are particular solutions of Eq. (2.7). However, only the solutions $\frac{e^{ik\alpha R}}{R}$ have a physical sense since only the expressions

$$\mathrm{Re}\left[e^{-i\omega t}\frac{e^{k_\alpha R}}{R}\right] = \frac{1}{R}\cos\omega\left(t - \frac{R}{v_\alpha}\right), \qquad v_\alpha = \frac{\omega}{k_\alpha}, \qquad \alpha = 1, 2, 3,$$

represent the divergent wave propagating from the origin of disturbance to infinity. Thus the solution of the homogeneous equation (2.2) will take the form

$$\boldsymbol{\varphi}^* = \mathbf{A}\frac{e^{ik_1R}}{R} + \mathbf{B}\frac{e^{ik_2R}}{R} + \mathbf{C}\frac{e^{i\hat{k}_3R}}{R}. \tag{2.8}$$

Similarly, the solution of Eq. (2.3) will be given by the function

$$\boldsymbol{\psi}^* = \mathbf{D}\frac{e^{ik_1R}}{R} + \mathbf{E}\frac{e^{ik_2R}}{R} + \mathbf{F}\frac{e^{ik_3R}}{R}. \tag{2.9}$$

Only real phase velocities may appear in Eqs. (2.8) and (2.9). Thus, it should be $k_1^2 > 0$, $k_2^2 > 0$, $k_3^2 > 0$, $\hat{k}_3^2 > 0$. The first and third condition are already satisfied. The remaining two will be satisfied provided $\sigma_4^2 > 2p$ or else $\omega^2 > \frac{4\alpha}{I}$. This is a consequence of the relation $k_1^2k_2^2 = \sigma_2^2(\sigma_4^2 - 2p) > 0$. Two first waves in the expression (2.8) undergo dispersion as k_1 and k_2 are the functions of the frequency ω. In the formula (2.9) all three terms represent the waves undergoing dispersion as the quantity $\hat{k}_3$ is a function of the frequency ω also.

The formulae describing the amplitudes of displacements and rotations will read as follows:

$$\begin{aligned}\mathbf{u}^* = (\lambda + 2\mu)(\gamma + \varepsilon)(\nabla^2 + k_3^2)(\nabla^2 + \sigma_4^2 - 2p)\,\boldsymbol{\varphi}^* + \\ -(\gamma + \varepsilon)(\lambda + \mu - \alpha)\,\mathrm{grad\,div}\,[(\nabla^2 + \sigma_4^2 - 2p - \eta)]\boldsymbol{\varphi}^* + \\ -2\alpha(\beta + 2\gamma)(\nabla^2 + \hat{k}_3^2)\,\mathrm{rot}\,\boldsymbol{\psi}^*,\end{aligned} \tag{2.10}$$

$$\begin{aligned}\boldsymbol{\omega}^* = (\mu + \alpha)(\beta + 2\gamma)(\nabla^2 + \sigma_2^2)(\nabla^2 + \hat{k}_3^2)\,\boldsymbol{\psi}^* + \\ -(\beta + \gamma - \varepsilon)(\gamma + \varepsilon)\,\mathrm{grad\,div}\,[\nabla^2 + \sigma_2^2 - 2r - \xi]\,\boldsymbol{\psi}^* + \\ -2\alpha(\lambda + 2\mu)(\nabla^2 + k_3^2)\,\mathrm{rot}\,\boldsymbol{\varphi}^*.\end{aligned} \tag{2.11}$$

The following notations are introduced in the above expressions

$$\xi = 4\alpha^2[(\beta + \gamma - \varepsilon)(\gamma + \varepsilon)]^{-1}, \quad \eta = 4\alpha^2[(\gamma + \varepsilon)(\lambda + \mu - \alpha)]^{-1}.$$

Let us first consider the action of body forces. Since $\mathbf{Y}^* = 0$, the function $\boldsymbol{\psi}^* = 0$, also. This, we have to consider Eq. (2.2) putting $\boldsymbol{\psi}^* = 0$ in Eqs. (2.10) and (2.11).

In order to get the solution of Eq. (2.2) we shall make use of the operator method described already in [9]. In this way we obtain

$$\varphi^* = \left(\frac{\boldsymbol{H}_1^*}{(k_1^2 - k_2^2)(k_1^2 - k_3^2)} + \frac{\boldsymbol{H}_2^*}{(k_2^2 - k_1^2)(k_2^2 - k_3^2)} + \frac{\boldsymbol{H}_3^*}{(k_3^2 - k_1^2)(k_3^2 - k_2^2)} \right). \quad (2.12)$$

The vector functions $\boldsymbol{H}_1^*$, $\boldsymbol{H}_2^*$, $\boldsymbol{H}_3^*$ have to verify the Helmholtz equations

$$(\nabla^2 + k_1^2)\,\boldsymbol{H}_1^* = -\varkappa \boldsymbol{X}^*, \quad (\nabla^2 + k_2^2)\,\boldsymbol{H}_2^* = -\varkappa \boldsymbol{X}^*, \quad (\nabla^2 + k_3^2)\,\boldsymbol{H}_3^* = -\varkappa \boldsymbol{X}^*. \quad (2.13)$$

The solution of there equations are given by the functions

$$\boldsymbol{H}_j^*(\mathbf{x}) = \frac{\varkappa}{4\pi} \iiint\limits_V \boldsymbol{X}^*(\boldsymbol{\xi})\, \frac{e^{ik_jR}}{R(\boldsymbol{\xi}, \mathbf{x})}\, dV(\boldsymbol{\xi}), \qquad i = 1, 2, 3, \quad (2.14)$$

where

$$R(\boldsymbol{\xi}, \mathbf{x}) = [(\xi_i - x_i)(\xi_i - x_i)]^{1/2}.$$

Now, we can write the solution of Eq. (2.2) in the form of the function

$$\boldsymbol{\varphi}^*(\mathbf{x}) = \frac{\varkappa}{4\pi} \iiint\limits_V \frac{\boldsymbol{X}^*(\boldsymbol{\xi})}{R(\boldsymbol{\xi}, \mathbf{x})} \times$$

$$\times \left[\frac{e^{ik_1R}}{(k_1^2 - k_2^2)(k_1^2 - k_3^2)} + \frac{e^{ik_2R}}{(k_2^2 - k_3^2)(k_2^2 - k_1^2)} + \frac{e^{ik_3R}}{(k_3^2 - k_1^2)(k_3^2 - k_2^2)} \right] dV(\boldsymbol{\xi}). \quad (2.15)$$

Let us assume that at the origin of the coordinate system the concentrated force $\boldsymbol{X}(\mathbf{x}, t) = e^{-i\omega t}(X_1^*, 0, 0)$, where $X_1^* = \delta(x_1)\,\delta(x_2)\,\delta(x_3)$ is acting.

From (2.15) we have $\boldsymbol{\varphi}^* = (\varphi_1^*, 0, 0)$, where

$$\varphi_1^*(\mathbf{x}) = \frac{\varkappa}{4\pi R_0} \sum_{j=1}^{3} \frac{e^{ik_jR_0}}{A_j}, \qquad R_0 = (x_1^2 + x_2^2 + x_3^2)^{\frac{1}{2}},$$

$$A_1 = \frac{1}{(k_1^2 - k_2^2)(k_1^2 - k_3^2)}, \qquad A_2 = \frac{1}{(k_2^2 - k_3^2)(k_2^2 - k_1^2)}, \quad (2.16)$$

$$A_3 = \frac{1}{(k_3^2 - k_1^2)(k_3^2 - k_2^2)}.$$

Putting (2.16) into (2.11) and (2.12) we obtain

$$u_j^* = U_j^{*(1)} = \frac{1}{4\pi\varrho\omega^2} \left(B_1 k_1^2 \frac{e^{ik_1R_0}}{R_0} + B_2 k_2^2 \frac{e^{ik_2R_0}}{R_0} \right) \delta_{1j} +$$

$$+ \frac{1}{4\pi\varrho\omega^2}\, \partial_1 \partial_j \left(B_1 \frac{e^{ik_1R_0}}{R_0} + B_2 \frac{e^{ik_2R_0}}{R_0} + B_3 \frac{e^{ik_3R_0}}{R_0} \right), \quad (2.17)$$

$$\omega_j^* = \Omega_j^{*(1)} = \frac{p}{4\pi\varrho c_2^2 (k_1^2 - k_2^2)}\, \varepsilon_{1jk} \frac{\partial}{\partial x_k} \left(\frac{e^{ik_1R_0} - e^{ik_2R_0}}{R_0} \right), \quad (2.18)$$

where

$$B_1 = \frac{\sigma_2^2 - k_2^2}{k_1^2 - k_2^2}, \qquad B_2 = \frac{\sigma_2^2 - k_1^2}{k_2^2 - k_1^2}, \qquad B_3 = -1.$$

Thus we obtained three components of the displacement vector $\boldsymbol{U}^{*(1)}$ and three components of the rotation vector $\boldsymbol{\Omega}^{*(1)}$. If the unit concentrated force be directed along the x_r-axis, then we have to put the index „r" instead of „1" in Eqs. (2.17) and (2.18). In this case we obtain the displacements $U_j^{*(r)}$, $\Omega_j^{*(r)}$ for $j, r = 1, 2, 3$. These functions form the Green tensors of displacements and rotations.

Putting into (2.17) and (2.18) $\alpha = 0$ we obtain the known formula of classical elastokinetics [10]

$$U_j^{*(r)} = \frac{\delta_{jr}}{4\pi\mu}\frac{e^{i\tau R_0}}{R_0} - \frac{1}{4\pi\varrho\omega^2}\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_r}\left(\frac{e^{i\sigma R_0} - e^{i\tau R_0}}{R_0}\right), \tag{2.19}$$

$$\Omega_j^{*(r)} = 0.$$

Here

$$\tau = \frac{\omega_0}{c_2^0}, \qquad c_2^0 = \left(\frac{\mu}{\varrho}\right)^{\frac{1}{2}}, \qquad \sigma = \frac{\omega}{c_1}, \qquad c_1 = \left(\frac{\lambda + 2\mu}{\varrho}\right)^{\frac{1}{2}}.$$

Let us return to the formulae (2.17) and (2.18). We assume that the concentrated force acting along the x_1-axis leads the rotation $\omega_1^* = \Omega_1^{*(1)} = 0$. This reduces to zero the components $\varkappa_{1j} = 1, 2, 3$ of the curvature – twist tensor. Ten components of the tensor γ_{ij} differ from zero. The waves represented by the quantities k_1 and k_2 undergo dispersion.

Let us assume that in an infnite body only the body couples are acting, i.e. we assume $\boldsymbol{X} = 0$ and $\boldsymbol{\varphi} = 0$. Consider now Eq. (2.3), we rewrite it in the form

$$(\nabla^2 + \mu_1)(\nabla^2 + \mu_2)(\nabla^2 + \mu_3)\boldsymbol{\psi}^* + \sigma\boldsymbol{Y}^* = 0, \tag{2.20}$$

where

$$\mu_1 = k_1, \qquad \mu_2 = k_2, \qquad \mu_3 = \hat{k}_3.$$

The function $\boldsymbol{\psi}^*$ is the solution of Eq. (2.20)

$$\boldsymbol{\psi}^* = \sum_{r=1}^{3}\frac{\boldsymbol{\Gamma}_r^*}{C_r}, \tag{2.21}$$

where

$$C_1 = \frac{1}{(\mu_1^2 - \mu_2^2)(\mu_1^2 - \mu_3^2)}, \qquad C_2 = \frac{1}{(\mu_2^2 - \mu_3^2)(\mu_2^2 - \mu_1^2)},$$

$$C_3 = \frac{1}{(\mu_3^2 - \mu_1^2)(\mu_3^2 - \mu_2^2)}.$$

The vector functions $\Gamma_1^*, \Gamma_2^*, \Gamma_3^*$ have to satisfy the Helmholtz equation

$$(\nabla^2 + \mu_1^2)\boldsymbol{\Gamma}_1^* = -\sigma\boldsymbol{Y}^*, \quad (\nabla^2 + \mu_2^2)\boldsymbol{\Gamma}_2^* = -\sigma\boldsymbol{Y}^*, \quad (\nabla^2 + \mu_3^2)\boldsymbol{\Gamma}_3^* = -\sigma\boldsymbol{Y}^*. \tag{2.22}$$

In analogy with the solution (2.15) the solution of Eq. (2.20) will read as follows

$$\boldsymbol{\psi}^*(\mathbf{x}) = \frac{\sigma}{4\pi} \iiint\limits_V \frac{\mathbf{Y}^*(\boldsymbol{\xi})}{R(\boldsymbol{\xi}, \mathbf{x})} \left(\sum_{r=1}^{3} \frac{e^{i\mu_r R}}{C_r} \right) dV(\boldsymbol{\xi}). \tag{2.23}$$

Let us consider now the action of a concentrated moment applied at the origin of the coordinate system and acting parallelly to the x_1-axis. Putting $Y_j^* = \delta(x_1)\,\delta(x_2)\,\delta(x_3)\,\delta_{1j}$ in Eq. (2.23) we obtain $\boldsymbol{\psi}^* = (\psi_1^*, 0, 0)$, where

$$\psi_1^*(\mathbf{x}) = \frac{\sigma}{4\pi R_0} \sum_{r=1}^{3} \frac{c^{i\mu_r R_0}}{C_r}. \tag{2.24}$$

Substituting $\boldsymbol{\psi}^*$ and $\boldsymbol{\varphi}^* = 0$ into the formulae (2.10) and (2.11) we get

$$u_j^* = V_j^{*(1)} = \frac{r}{4\pi I c_4^2(\mu_1^2 - \mu_2^2)} \varepsilon_{1jk} \frac{\partial}{\partial x_k} \left(\frac{e^{i\mu_1 R_0} - c^{i\mu_2 R_0}}{R_0} \right), \tag{2.25}$$

$$\omega_j^* = W_j^{*(1)} = \frac{1}{4\pi I c_4^2} \left(\mu_1^2 D_1 \frac{e^{i\mu_1 R_0}}{R_0} + \mu_2^2 D_2 \frac{e^{i\mu_2 R_0}}{R_0} \right) \delta_{1j} +$$

$$+ \frac{1}{4\pi I c_4^2} \partial_1 \partial_j \left(D_1 \frac{e^{i\mu_1 R_0}}{R_0} + D_2 \frac{e^{i\mu_2 R_0}}{R_0} + D_3 \frac{e^{i\mu_3 R_0}}{R_0} \right), \tag{2.26}$$

where

$$D_1 = \frac{\mu_1^2 - \sigma_2^2}{\mu_1^2(\mu_1^2 - \mu_2^2)}, \qquad D_2 = \frac{\mu_2^2 - \sigma_2^2}{\mu_2^2(\mu_2^2 - \mu_1^2)}, \qquad D_3 = -\frac{\sigma_2^2}{\mu_1^2 \mu_2^2}.$$

It is worth noting that the action of the moment $Y_j^* = \delta(x_1)\,\delta(x_2)\,\delta(x_3)\,\delta_{1j}$ leads to the zero-value of the displacement along the x_1-axis ($V_1^{*(1)} = 0$). Consequently $\gamma_{11} = 0$, too. The components of the tensor $\varkappa_{ji}$ differ from zero. Since μ_1, μ_2, μ_3 depend on the vibration frequency ω, all the types of waves appearing in (2.25) and (2.26) undergo dispersion. Passing to the classical theory of elasticity we obtain $V_{j)}^{*\,1)} = 0$.

Now we may easily pass from the formulae (2.25) and (2.26) to the quantities $V_j^{*(r)}$, $W_j^{*(r)}$ and in this way to the displacements and rotations induced by the action of the concentrated moment acting in the point $\boldsymbol{\xi}$ and directed parallelly to the x_r-axis.

Let us consider, moreover, a certain particular case. Assume that a concentrated force $X_j^* = \delta(\mathbf{x} - \boldsymbol{\xi})\,\delta_{jr}$ acting at the point $\boldsymbol{\xi}$ parallelly to the x_r-axis. This force will induce the field of displacements $U_j^{*(r)}(\mathbf{x}, \boldsymbol{\xi})$ and rotations $\Omega_j^{*(r)}(\mathbf{x}, \boldsymbol{\xi})$. Assume now that a concentrated moment $Y_j^* = \delta(\mathbf{x} - \boldsymbol{\eta})\,\delta_{jl}$ is acting at the point $\boldsymbol{\eta}$ parallelly to the x_l-axis. It will produce the field of displacements $V_j^{*(l)}(\mathbf{x}, \boldsymbol{\xi})$ and rotations $W_j^{*(l)}(\mathbf{x}, \boldsymbol{\eta})$. To the causes and effects formulated in this way we shall apply the theorem on the reciprocity of works [6]

$$\int_V (X_i^* u_i'^* + Y_i^* \omega_i'^*)\, dV = \int_V (X_i'^* u_i^* + Y_i'^* \omega_i^*)\, dV. \tag{2.27}$$

From this equation we obtain

$$\int_V \delta(\mathbf{x} - \boldsymbol{\xi})\, \delta_{jr} V_j^{*(l)}(\mathbf{x}, \boldsymbol{\eta})\, \mathrm{d}V(\mathbf{x}) = \int_V \delta(\mathbf{x} - \boldsymbol{\eta})\, \delta_{jl} \Omega_j^{*(r)}(\mathbf{x}, \boldsymbol{\xi})\, \mathrm{d}V(\mathbf{x}),$$

or

$$V_r^{*(l)}(\boldsymbol{\xi}, \boldsymbol{\eta}) = \Omega_l^{*(r)}(\boldsymbol{\eta}, \boldsymbol{\xi}). \tag{2.28}$$

Extending the formulae (2.18) and (2.26) and making use of them we get

$$V_r^{*(l)}(\boldsymbol{\xi}, \boldsymbol{\eta}) = \frac{r}{4\pi I c_4^2(\mu_1^2 - \mu_2^2)} \varepsilon_{lrk} \left. \frac{\partial}{\partial x_k} \left(\frac{e^{i\mu_1 R} - e^{i\mu_2 R}}{R} \right) \right|_{\mathbf{x}=\boldsymbol{\xi}},$$

$$\Omega_l^{*(r)}(\boldsymbol{\eta}, \boldsymbol{\xi}) = \frac{p}{4\pi \varrho c_2^2(k_1^2 - k_2^2)} \varepsilon_{rlk} \left. \frac{\partial}{\partial x_k} \left(\frac{e^{ik_1 R} - e^{ik_2 R}}{R} \right) \right|_{\mathbf{x}=\boldsymbol{\eta}},$$

$$\mu_1 = k_1, \qquad \mu_2 = k_2.$$

It is seen that Eq. (2.28) – being an extended form of Maxwell's theorem on reciprocity known from elastokinetics – is satisfied here.

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