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MICROPOLAR ELASTICITY

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THE MICROPOLAR THERMOELASTICITY

1. Introduction

Thermoelasticity investigates the interaction of the field of deformation with the field of temperature and combines, on the basis of the thermodynamics of the irreversible processes, two separately developing branches of science, namely the theory of elasticity and the theory of heat conduction.

At the present moment, after 20 years of the development, the thermoelasticity of the Hooke's continuum is fully formed. The fundamental assumptions have been worked out [1] - [5], the fundamental relations and different equations have been elaborated, the fundamental energy and variational theorems obtained. The entire classical thermoelasticity has been formulated in a number of monographs.

On the background of the development of the classical thermoelasticity the achievements of Cosserat's continuum thermoelasticity [6 - 9], are still modest. Though all more important theorems have been derived, the domain of the particular solutions is incomparably smaller. The fundamentals of the micropolar, Cosserats' thermoelasticity were formulated in 1966 by the author of the present study [10], [11].

We present, in a concise form, the fundamental relations and the fundamental equations of Cosserats' continuum thermoelasticity. The principle of the energy conservation and the entropy balance are our point of departure

$$\frac{d}{dt}\int_V \left[\frac{1}{2}(\rho v_i v_i + I w_i w_i) + U\right] dV = \int_V (X_i v_i + Y_i w_i)\, dV + \tag{1.1}$$
$$+ \int_A (p_i v_i + m_i w_i)\, dA - \int_A q_i n_i\, dA$$

and

$$\int_V \dot{S}\, dV = - \int_A \frac{q_i n_i}{T}\, dA + \int_V \Theta\, dV . \tag{1.2}$$

In eq. (1.1) U denotes the internal energy referred to the unit of volume, X_i, Y_i are the components of the body forces and moments acting on the surface A bounding the body, u_i, φ_i denote the components of the displacement vector and rotation vector, respectively, $\dot{u}_i = v_i$, $\dot{\varphi}_i = w_i$ are their time derivatives, $\underline{q}$ is the flux of heat vector, ρ the density, I the rotational inertia.

The term on the left-hand side of eq. (1.1) represents the time change of the internal and kinetic energies. The first term on the right hand side of the equation is the power of body forces and body moments, the second term is the power of traction and surface moments. The last term expresses the amount of heat transmitted into the volume V by the heat conduction.

The left-hand side of the balance of entropy equation (1.2) represents the increase of entropy. The first term on the right-hand side is the increase of entropy due to the exchange of the entropy with environment, the second term expresses the production of entropy generated by heat conduction. Here $\Theta = -\frac{q_i T_{,i}}{T^2} > 0$, according to the postulate of the thermodynamics of irreversible processes.

In eq. (1.2) S denotes the entropy referred to the unit of volume, T is the absolute temperature, Θ is the source of entropy.

Transforming eqs. (1.1) and (1.2) by means of the equation of motion

$$\left.\begin{aligned} &\sigma_{ji,j} + X_i = \rho \ddot{u}_i , \\ &\epsilon_{ijk}\sigma_{jk} + \mu_{ji,j} + Y_i = I\ddot{\varphi}_i , \end{aligned}\right\} \tag{1.3}$$

where σ_{ji}, μ_{ji} are the components of force stress and moment stress tensors, ϵ_{ijk} is Ricci's alternator, and taking into account the definitions of the asymmetric strain tensors

$$\gamma_{ji} = u_{i,j} - \epsilon_{kji}\varphi_k \quad , \qquad \varkappa_{ji} = \varphi_{i,j} \tag{1.4}$$

we obtain, eliminating the quantity $q_{i,i}$ and introducing the free energy $F = U - ST$ the following equation

$$\dot{F} = \sigma_{ji}\dot{\gamma}_{ji} + \mu_{ji}\dot{\varkappa}_{ji} - \dot{T}S \; . \tag{1.5}$$

Since the free energy is the function of the independent variables γ_{ji}, $\varkappa_{ji}$ and T then

$$\dot{F} = \frac{\partial F}{\partial \gamma_{ji}}\dot{\gamma}_{ji} + \frac{\partial F}{\partial \varkappa_{ji}}\dot{\varkappa}_{ji} + \frac{\partial F}{\partial T}\dot{T} \; . \tag{1.6}$$

We assume that the functions Θ, $q_i, .. \sigma_{ji}$, μ_{ji} do not depend explicitly on the time derivatives of the functions γ_{ji}, $\varkappa_{ji}$, T next, defining $S = -\frac{\partial F}{\partial \gamma_{ji}}$ we obtain the following relations

$$\sigma_{ji} = \frac{\partial F}{\partial \gamma_{ji}} \; , \; \mu_{ji} = \frac{\partial F}{\partial \varkappa_{ji}} \; , \; S = -\frac{\partial F}{\partial T} \; . \tag{1.7}$$

Let us return to the inequality

$$\Theta = -\frac{T_{,i}\, q_i}{T^2} \geq 0 \; .$$

This inequality is satisfied by the Fourier's law of heat conduction

$$-q_i = k_{ij}\, T_{,j} \; . \tag{1.8}$$

Consequently, we obtain from the entropy balance, for a homogeneous and isotropic body,

$$T\dot{S} = -q_{i,i} = k\, T_{,jj} \; , \tag{1.9}$$

here k is the coefficient of the thermal conduction.

Let us expand the free energy $F(\gamma_{ji}, \varkappa_{ji}, T)$ into the Taylor series in the vicinity of the natural state ($\varkappa_{ji} = \gamma_{ji} = 0$, $T = T_0$), disregarding the terms of higher order than the second one. For isotropic, homogeneous, and centrosymmetric bodies, we obtain the following form of the expansion

$$
\begin{aligned}
F = \frac{\mu+\alpha}{2}\gamma_{ji}\gamma_{ji} + \frac{\mu-\alpha}{2}\gamma_{ji}\gamma_{ij} + \frac{\lambda}{2}\gamma_{kk}\gamma_{nn} + \frac{\gamma+\varepsilon}{2}\varkappa_{ji}\varkappa_{ji} + \\
+ \frac{\gamma-\varepsilon}{2}\varkappa_{ji}\varkappa_{ij} + \frac{\beta}{2}\varkappa_{kk}\varkappa_{nn} - \nu\gamma_{kk}\vartheta - \frac{m}{2}\vartheta^2
\end{aligned}
\qquad (1.10)
$$

Here $\vartheta = T - T_0$ where T_0 is the temperature of the natural state, the magnitudes $\mu, \alpha, \lambda, \gamma, \varepsilon, \beta$ denote the material constants ν, m are the quantities containing the mechanical and thermal constants. On the right-hand side there occurs the independent quadratic invariants $\gamma_{ji}\gamma_{ji}$, $\gamma_{ji}\gamma_{ij}$, $\gamma_{kk}\gamma_{nn}$ and the invariant γ_{kk}. The quantities $\gamma_{ji}\varkappa_{ji}$, $\gamma_{ji}\varkappa_{ij}$, $\gamma_{kk}\varkappa_{nn}$, $\varkappa_{nn}\vartheta$ cannot enter the equation because of the assumption of the centrosymmetry.

Consequently, making use of eqs. (1.6), we obtain the following constitutive equations

$$
\left.
\begin{aligned}
\sigma_{ji} &= (\mu+\alpha)\gamma_{ji} + (\mu-\alpha)\gamma_{ij} + (\lambda\gamma_{kk} - \nu\vartheta)\delta_{ij}, \\
\mu_{ji} &= (\gamma+\varepsilon)\varkappa_{ji} + (\gamma-\varepsilon)\varkappa_{ij} + \beta\varkappa_{kk}\delta_{ij}, \\
S &= \nu\gamma_{kk} + m\vartheta = \nu\gamma_{kk} + \frac{c_\varepsilon}{T_0}\vartheta .
\end{aligned}
\right\}
\qquad (1.11)
$$

Here $\mu, \lambda, \alpha, \beta, \gamma, \varepsilon$ are the material constants in the isothermal state, $\nu = (3\lambda + 2\mu)\alpha_t$ where α_t is the coefficient of the linear thermal expansion, while c_ε is the specific heat at constant deformation. We should remark that the constitutive equation $(1.11)_3$ holds true only for the limitation $\left|\frac{\vartheta}{T_0}\right| \ll 1$. From the entropy balance

$$T\dot{S} = -q_{i,i} + W \tag{1.12}$$

where W denotes the amount of heat generated in a unit of volume and time, from the Fourier law $q_i = -kT_{,i}$ and from the equation $(1.11)_3$, we are lead to the lineat heat conduction equation

$$\nabla^2\vartheta - \frac{1}{\varkappa}\dot{\vartheta} - \eta \operatorname{div}\dot{\underline{u}} = -\frac{Q}{\varkappa}\,, \quad Q = \frac{W}{c_\varepsilon}\,, \quad \vartheta = T - T_0\,. \tag{1.13}$$

2. The Dynamical Problem of Thermoelasticity

Let us consider a regular region of space $V+A$ where A is the boundary containing an elastic, homogeneous, isotropic and centrosymmetric micropolar continuum.

Let $\sigma_{ji}(\underline{x},t)$ be the components of the non-symmetric force stress tensor, while $\mu_{ji}(\underline{x},t)$ the components of the non-symmetric moment stress tensor $\underline{u}(\underline{x},t)$ denotes the displacement vector and $\underline{\varphi}(\underline{x},t)$ is the vector of rotation. By $\vartheta = T - T_0$ we denote the change of temperature.

The dynamical problem of thermoelasticity consists in determining the functions

$$\sigma_{ji}(\underline{x},t)\,,\ \mu_{ji}(\underline{x},t)\,,\ \gamma_{ji}(\underline{x},t)\,,\ \varkappa_{ji}(\underline{x},t)\,,\ \underline{u}(\underline{x},t)\,,\ \underline{\varphi}(\underline{x},t)$$

$$\text{and}\quad \vartheta(\underline{x},t)\ \text{for}\ \underline{x}\in V+A\,.$$

These functions ought to satisfy:

a) the equation of motion,

b) the equation of thermal conduction,

c) the linearized constitutive equations,

d) the boundary conditions

$$\begin{cases} \mu_{ji} n_j = m_i(\underline{x}, t), \; \sigma_{ji} n_j = p_i(\underline{x}, t), \; \theta = \vartheta(\underline{x}, t), \; \underline{x} \in A_\sigma, \; t > 0, \\ u_i = f_i(\underline{x}, t), \; \varphi_i = g_i(\underline{x}, t), \; \theta = \vartheta(\underline{x}, t), \; \underline{x} \in A_u, \; t > 0. \end{cases} \tag{2.1}$$

e) the initial conditions

$$\left.\begin{aligned} u_i = k_i(\underline{x}), \quad \dot{u}_i = h_i(\underline{x}), \quad \theta = s(\underline{x}), \\ \varphi_i = l_i(\underline{x}), \quad \dot{\varphi}_i = s_i(\underline{x}), \quad \underline{x} \in V, \; t = 0. \end{aligned}\right\} \tag{2.2}$$

The functions $p_i, m_i, f_i, g_i, \vartheta$ in the boundary conditions and k_i, h_i s, l_i, s_i in the initial conditions are given functions.

Let us pass to the representation of the differential equations of the problem choosing as unknown functions the displacements $\underline{u}(\underline{x},t)$, the rotations $\underline{\varphi}(\underline{x},t)$ and the temperature $\theta(\underline{x},t)$. Eliminating the stresses from the equations of motion by means of constitutive equations, expressing them by the functions $\underline{u}$ and $\underline{\varphi}$, we obtain, together with the equation of heat conduction, the following set of differential equations of thermoelasticity.

$$
(2.3)\qquad \begin{cases} \Box_2 \underline{u} + (\lambda + \mu - \alpha)\,\mathrm{grad\,div}\,\underline{u} + 2\alpha\,\mathrm{curl}\,\underline{\varphi} + \underline{X} = \nu\,\mathrm{grad}\,\vartheta\,, \\ \Box_4 \underline{\varphi} + (\beta + \gamma - \varepsilon)\,\mathrm{grad\,div}\,\underline{\varphi} + 2\alpha\,\mathrm{curl}\,\underline{u} + \underline{Y} = 0\,, \\ \nabla^2\vartheta - \frac{1}{\varkappa}\dot{\vartheta} - \eta\,\mathrm{div}\,\dot{\underline{u}} = -\frac{Q}{\varkappa}\,, \end{cases}
$$

where $\Box_2 = (\mu + \alpha)\nabla^2 - \rho\partial_t^2$, $\Box_4 = (\varepsilon + \gamma)\nabla^2 - 4\alpha - J\partial_t^2$.
We have obtained a coupled system of seven equations for seven unknowns, namely three components of the displacement $\underline{u}$ three components of the rotation $\underline{\varphi}$ and the temperature ϑ . These fields can be generated by loadings, a surface heating, body forces and moments and heat sources.

In eqs.(2.3) the mutually independent functions $\underline{u}, \underline{\varphi}, \vartheta$ are coupled. The change of deformation field generates the change generates the change of temperature and vice versa.

The coupled system of equations (2.3) is complicated and inconvenient to deal with; hence, our prime objective will be to uncouple it.
The dynamic equations of thermoelasticity

$$
(2.4)\qquad \begin{cases} \Box_2 \underline{u} + (\lambda + \mu - \alpha)\,\mathrm{grad\,div}\,\underline{u} + 2\alpha\,\mathrm{curl}\,\underline{\varphi} + \underline{X} = \nu\,\mathrm{grad}\,\vartheta\,, \\ \Box_4 \underline{\varphi} + (\beta + \gamma - \varepsilon)\,\mathrm{grad\,div}\,\underline{\varphi} + 2\alpha\,\mathrm{curl}\,\underline{u} + Y = 0 \\ D\vartheta - \eta\,\mathrm{div}\,\dot{\underline{u}} = -\frac{Q}{\varkappa}\,, \quad D = \nabla^2 - \frac{1}{\varkappa}\partial_t\,, \end{cases}
$$

can be separated in two different ways. The first way, analogous to Lame's procedure applied in classical elastokinetics, consists in the decomposition of the vectors $\underline{u}$ and $\underline{\varphi}$ into potential and solenoidal parts, respectively

$$\left.\begin{aligned} \underline{u} &= \operatorname{grad}\Phi + \operatorname{curl}\underline{\Psi} \quad , \qquad \operatorname{div}\underline{\Psi} = 0 , \\ \underline{\varphi} &= \operatorname{grad}\Gamma + \operatorname{curl}\underline{H} \quad , \qquad \operatorname{div}\underline{H} = 0 . \end{aligned}\right\} \tag{2.5}$$

Decomposing in a similar way the expressions for the body forces and the body couples

$$\left.\begin{aligned} \underline{X} &= \rho(\operatorname{grad}\vartheta + \operatorname{curl}\underline{\chi}) \quad , \qquad \operatorname{div}\underline{\chi} = 0 , \\ \underline{Y} &= J(\operatorname{grad}\sigma + \operatorname{curl}\underline{\eta}) \quad , \qquad \operatorname{div}\underline{\eta} = 0 , \end{aligned}\right\} \tag{2.6}$$

and substituting (2.5) and (2.6) to the equations of thermoelasticity (2.4), we obtain the following system of equations

$$\left.\begin{aligned} &\Box_1\Phi + \rho\vartheta = \nu\theta , \\ &D\theta - \eta\partial_t\nabla^2\dot{\Phi} = -\frac{Q}{\varkappa} , \\ &\Box_3\Gamma + J\sigma = 0 , \\ &\Box_2\underline{\Psi} + 2\alpha\operatorname{curl}\underline{H} + \rho\underline{\chi} = 0 , \\ &\Box_4\underline{H} + 2\alpha\operatorname{curl}\underline{\Psi} + J\underline{\eta} = 0 , \\ &D = \nabla^2 - \frac{1}{\varkappa}\partial_t , \\ &\Box_3 = \nabla^2 - \frac{1}{c_3^2}\partial_t^2 - 4\alpha . \end{aligned}\right\} \tag{2.7}$$

The complex system of equations of thermoelasticity has been reduced to the solution of simple wave equations (2.7). Eq. $(2.7)_1$ represents a longitudinal wave, $(2.7)_2$ the heat conduction equation, eq. $(2.7)_3$ the longitudinal micro-rotational wave, eq. $(2.7)_4$, $(2.7)_5$ correspond to a transversal displacement and transversal micro-rotational wave.

The form of eqs. $(2.7)_1$ and $(2.7)_2$ is identical with the form of the longitudinal wave equation of the classical thermoelasticity; on the contrary, eq. $(2.7)_3$ is a new one, namely the Klein-Gordon differential equation. Let us notice that eqs. $(2.7)_1$, $(2.7)_2$ and eqs. $(2.7)_4$, $(2.7)_5$ are mutually coupled.

After elimination, we obtain

$$
(2.8) \qquad \left\{
\begin{aligned}
&(\Box_1 D - \nu\eta\partial_t\nabla^2)\Phi = -\frac{\nu}{\varkappa}Q - \rho D\vartheta \, , \\
&(\Box_1 D - \nu\eta\partial_t\nabla^2)\vartheta = -\frac{\Box_1^2 Q}{\varkappa} - \rho\eta\partial_t\nabla^2\vartheta \, , \\
&\Box_3\Gamma + J\sigma = 0 \, , \\
&(\Box_2\Box_4 + 4\alpha^2\nabla^2)\underline{\Psi} = 2\alpha \ \mathrm{curl}\,\underline{\eta} - \rho\,\Box_4\underline{\chi} \, , \\
&(\Box_2\Box_4 + 4\alpha^2\nabla^2)\underline{H} = 2\alpha\rho\,\mathrm{curl}\,\underline{\chi} - J\,\Box_2\underline{\eta} \, .
\end{aligned}
\right.
$$

We shall consider first the propagation of thermoelastic waves in an unbounded space.

If the quantities $\sigma, \underline{\chi}, \underline{\eta}$ and the initial conditions of the functions $\Gamma, \underline{\Psi}, \underline{H}$ are equal to zero, then in an unbounded elastic space only dilatational waves will propagate. Eq. $(2.8)_1$ describing the waves is identical with that obtained for the elastic classical medium (with no couple-stresses). These waves are attenuated and dispersed.

Since

$$u_i = \Phi_{,i} \ , \quad \varphi_i = 0 \ , \quad \gamma_{ji} = \Phi_{,ji} \ , \quad \varkappa_{ji} = 0 \ , \qquad (2.9)$$

we have

$$\sigma_{ij} = 2\mu(\Phi_{,ij} - \delta_{ij}\Phi_{,kk}) + \rho\delta_{ij}(\ddot{\Phi} - \vartheta) \ , \qquad \sigma_{<ij>} = 0 . \quad (2.10)$$

If $Q, \vartheta, \underline{\chi}, \underline{\eta}$ are equal to zero and the initial conditions of the functions $Q, \Phi, \underline{\Psi}, \underline{H}$ are homogeneous, then in an infinite medium only microrotational waves, described by eq. $(2.8)_3$ propagate. We have namely

$$u_i = 0 \ , \ \varphi_i = \Gamma_{,i} \ , \ \varkappa_{ji} = \Gamma_{,ij} \ , \ \gamma_{(ji)} = 0 \ , \ \gamma_{<ji>} = -\epsilon_{kij}\Gamma_{,k} \ . \qquad (2.11)$$

Couple-stresses and force-stresses will appear in the medium forming the symmetric tensor μ_{ij}

$$\mu_{(ij)} = 2\gamma\Gamma_{,ij} + \beta\delta_{ij}\Gamma_{,kk} \ , \ \mu_{<ij>} = 0 \ , \ \sigma_{ij} = 0 \ , \ \operatorname{div} \underline{u} = 0 \ , \qquad (2.12)$$

and the asymmetric tensor σ_{ij}

$$\sigma_{<ij>} = 2\mu\,\gamma_{<ij>} \ , \qquad \sigma_{(ij)} = 0 \ .$$

The propagation of these waves is not accompanied by a temperature field.

Finally, in the case when the quantities Q, ϑ, σ are equal to zero and the initial conditions of the functions

Φ, Γ, Q are homogeneous, only transverse waves propagate in an infinite space (described by the eqs. $(2.8)_4$ and $(2.8)_5$). In an infinite medium these waves are not accompanied by any temperature field. Since $\operatorname{div}\underline{u}=0$ they do not induce any changes in the volume of the body.

In a finite medium all three kinds of waves discussed here appear. Eqs. $(2.8)_1$ and $(2.8)_5$ are coupled by means of the boundary conditions.

The second method of separation of the system (2.10) is analogous to that applied by Galerkin [12] to the classical elastostatics and by M. Iacovache to the classical elastokinetics [13]. Functions of this type, suitable for asymmetric elasticity were established by N. Sandru [14].

The functions of this kind for the case of the dynamic problems of thermoelasticity have been devised by W. Nowacki [15]. These functions have been derived by another method by J. Stefaniak [16].

Below we give the final result of the separation. We represent the vectors $\underline{u}, \underline{\varphi}$ and the temperature ϑ by means of two vector functions $\underline{F}$, $\underline{G}$ and one scalar function L

$$(2.13)\quad \begin{cases} \underline{u} = \Box_4 M\underline{F} - \operatorname{grad}\operatorname{div} N\underline{F} - 2\alpha \operatorname{curl} \Box_3 \underline{G} + \nu \operatorname{grad} L \,, \\ \underline{\varphi} = \Box_2 \Box_3 \underline{G} - \operatorname{grad}\operatorname{div} \Theta \underline{G} - 2\alpha \operatorname{curl} M\underline{F} \,, \\ \vartheta = \eta \operatorname{div} \partial_t \Omega \underline{F} + \Box_1 L \,, \end{cases}$$

where

$$\Omega = \square_2\square_4 + 4\alpha^2\nabla^2 \quad , \qquad M = \square_1 D - \nu\eta\partial_t\nabla^2 ,$$

$$N = \Gamma D - \nu\eta\partial_t\square_4 \quad , \qquad \Gamma = (\lambda + \mu - \alpha)\square_4 - 4\alpha^2 ,$$

$$\Theta = (\beta + \gamma - \varepsilon)\square_2 - 4\alpha^2 .$$

Inserting the relations (2.13) into the system of eqs. (2.4) we obtain the following wave repeated equations

$$\left.\begin{aligned} &\Omega M \underline{F} + \underline{X} = 0 , \\ &\Omega \square_3 \underline{G} + \underline{Y} = 0 , \\ &ML + Q/\varkappa = 0 . \end{aligned}\right\} \qquad (2.14)$$

We have obtained a system of equations in which the body forces, the body couples and heat sources appear separately. Let us notice, that in an infinite, elastic space, the assumption $\underline{X} = 0$ with homogeneous initial conditions for $\underline{F}$, leads to the conclusion that $\underline{F} = 0$ in the whole space. The same result holds for the function $\underline{G} = 0$ with $\underline{Y} = 0$ and $L = 0$ with $Q = 0$.

Eqs. (2.14) are particularly useful in the case of the singular solutions for an infinite, micropolar space. Such solutions have been obtained for the case of the action of concentrated forces, concentrated moments and heat sources harmonically varying in time [17] . Then eqs. (2.14) simplify to a simple system of elliptic equations.

In this case we obtain the system of equations

$$
(2.15)\quad \begin{cases} A(\nabla^2+k_1^2)(\nabla^2+k_2^2)(\nabla^2+\mu_1^2)(\nabla^2+\mu_2^2)\underline{F}^*+\underline{X}^*=0\,, \\ B(\nabla^2+k_1^2)(\nabla^2+k_2^2)(\nabla^2+k_3^2)\underline{G}^*+\underline{Y}^*=0\,, \\ C(\nabla^2+m^2)(\nabla^2+\mu_2^2)L^*+Q^*/\varkappa=0\,, \end{cases}
$$

where

$\underline{X}=\underline{X}^*(\underline{x})e^{-i\omega t}$, $\quad \underline{Y}=\underline{Y}^*(\underline{x})e^{-i\omega t}$, and so on.

Let us return to the fundamental equations of thermoelasticity (2.4). Passing to the cylindrical coordinate system (r, ϑ, z) and assuming that the deformation possesses the axial symmetry, we obtain two independent systems of equations.

In the first system of equations the following components of the vectors $\underline{u}$, $\underline{\varphi}$, $\underline{X}$, $\underline{Y}$ grad θ, appear

$$
(2.16)\quad \begin{gathered} \underline{u}\equiv(u_r, 0, u_z)\,,\quad \varphi\equiv(0,\varphi_\vartheta,0)\,,\quad \underline{X}\equiv(X_r,0,X_z)\,,\quad \underline{Y}\equiv(0,Y_\vartheta,0)\,, \\ \operatorname{grad}\theta\equiv\left(\frac{\partial\theta}{\partial r},\,0,\,\frac{\partial\theta}{\partial z}\right)\,. \end{gathered}
$$

Now the system of equations has the form

$$
(2.17)\quad \begin{cases} (\mu+\alpha)\left(\nabla^2-\frac{1}{r^2}\right)u_r+(\lambda+\mu-\alpha)\frac{\partial e}{\partial r}-2\alpha\frac{\partial\varphi_\theta}{\partial z}+X_r=\nu\frac{\partial\theta}{\partial r}+\rho\ddot{u}_r\,, \\ (\mu+\alpha)\nabla^2 u_z+(\lambda+\mu-\alpha)\frac{\partial e}{\partial z}+2\alpha\frac{1}{r}\frac{\partial}{\partial r}(r\varphi_\theta)+X_z=\nu\frac{\partial\theta}{\partial z}+\rho\ddot{u}_z\,, \\ (\gamma+\varepsilon)\left(\nabla^2-\frac{1}{r^2}\right)\varphi_\theta-4\alpha\varphi_\theta+2\alpha\left(\frac{\partial u_r}{\partial z}-\frac{\partial u_z}{\partial r}\right)+Y_\vartheta=J\ddot{\varphi}_\theta\,, \end{cases}
$$

$$\left(\nabla^2 - \frac{1}{r^2}\right)\vartheta - \frac{1}{\varkappa}\dot{\vartheta} - \eta\,\partial_t e = -\frac{Q}{\varkappa} \tag{2.17}$$

where

$$e = \frac{1}{r}\frac{\partial}{\partial r}(u_r r) + \frac{\partial u_z}{\partial z} \quad , \quad \nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \ .$$

In the second system of equations, characterized by the vectors

$$\underline{u} \equiv (0, u_\vartheta, 0) \ , \ \underline{\varphi} \equiv (\varphi_r, 0, \varphi_z) \ ; \ \underline{X} \equiv (0, X_\vartheta, 0) \ , \ \underline{Y} \equiv (Y_r, 0, Y_z) \ , \tag{2.18}$$

the thermal terms vanish, therefore the vectors $\underline{u}, \underline{\varphi}$ do not depend on the field of temperature.

The system of equations takes the following form

$$\left.\begin{aligned}
&(\gamma+\varepsilon)\left(\nabla^2 - \frac{1}{r^2}\right)\varphi_r - 4\alpha\varphi_r + (\beta+\gamma-\varepsilon)\frac{\partial \varkappa}{\partial r} - 2\alpha\frac{\partial u_\vartheta}{\partial z} + Y_r = I\ddot{\varphi}_r \ ,\\
&(\gamma+\varepsilon)\nabla^2\varphi_z - 4\alpha\varphi_z + (\beta+\gamma-\varepsilon)\frac{\partial \varkappa}{\partial z} + 2\frac{\alpha}{r}\frac{\partial}{\partial r}(r u_\vartheta) + Y_z = J\ddot{\varphi} \ ,\\
&(\mu+\alpha)\left(\nabla^2 - \frac{1}{r^2}\right)u_\vartheta + 2\alpha\left(\frac{\partial \varphi_r}{\partial z} - \frac{\partial \varphi_z}{\partial r}\right) + X_\vartheta = \rho\ddot{u}_\vartheta \ .
\end{aligned}\right\} \tag{2.19}$$

Similarly in the two-dimensional state of stress the system of equations (2.4) can be split into two independent sets of equations. Under the assumption that the deformation of a body does not depend on the independent variable x_3 the following vectors

$$\underline{u} = (u_1, u_2, 0) \ , \ \varphi = (0, 0, \varphi_3) \ , \ \underline{X} = (X_1, X_2, 0) \ ,$$
$$(2.20) \qquad \underline{Y} = (0, 0, Y_3) \quad , \quad \operatorname{grad} \theta = (\partial_1\theta, \partial_2\theta, 0) \ ,$$

occur in the first system.

Here the system of equations has the form

$$(2.21) \quad \begin{cases} [(\mu+\alpha)\nabla_1^2 - \rho\partial_t^2]u_1 + (\lambda+\mu-\alpha)\partial_1 e + 2\alpha\partial_2\varphi_3 + X_1 = \nu\partial_1\theta \ , \\ [(\mu+\alpha)\nabla_1^2 - \rho\partial_t^2]u_2 + (\lambda+\mu-\alpha)\partial_2 e - 2\alpha\partial_1\varphi_3 + X_2 = \nu\partial_1\theta \ , \\ [(\gamma+\varepsilon)\nabla_1^2 - 4\alpha - J\partial_t^2]\varphi_3 + 2\alpha(\partial_1 u_2 - \partial_2 u_1) + Y_3 = 0 \ . \end{cases}$$

$$\left(\nabla^2 - \frac{1}{\varkappa}\partial_t\right)\theta - \eta\partial_t e = -Q/\varkappa \ , \quad e = \partial_1 u_1 + \partial_2 u_2 \ .$$

The second system in which the components of the vectors

$$(2.22) \quad \underline{u} = (0, 0, u_3) \ , \ \underline{\varphi} = (\varphi_1, \varphi_2, 0) \ , \ \underline{X} = (0, 0, X_3) \ , \underline{Y} = (Y_1, Y_2, 0) \ ,$$

appear, is independent of the field of temperature.

We have already noticed that for an infinite space and P-wave the results for the micropolar and Hooke's continua are of the same form.

Thus the following question arises: does the same situation occur for certain bounded bodies? It is easy to observe that such cases concern the one-dimensional problems.

Thus if all the causes and effects depend only on the variables x_1 and t, then we obtain the following system of equations

$$\left.\begin{aligned} &[(\lambda+2\mu)\partial_1^2-\varrho\partial_t^2]u_1=\nu\partial_1\theta\ ,\\ &\left(\partial_1^2-\frac{1}{\varkappa}\partial_t\right)\theta-\eta\partial_t\partial_1 u_1=-\frac{Q}{\varkappa}\ , \end{aligned}\right\} \tag{2.23}$$

which is exactly of the same form for Hooke's and micropolar media.

If all the causes depend on the variables $r=(x_1^2+x_2^2)^{1/2}$ and t then the system of coupled equations

$$\left.\begin{aligned} &(\lambda+2\mu)\left(\nabla_r^2-\frac{1}{r^2}\right)u_r-\varrho\ddot{u}_r=\nu\frac{\partial\theta}{\partial r}\ ,\\ &\left(\nabla_1^2-\frac{1}{\varkappa}\partial_t\right)\theta-\eta\partial_t\left(\frac{1}{r}\frac{\partial}{\partial r}(r u_r)\right)=-\frac{Q}{\varkappa}\ ,\\ &\nabla_r^2=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}\ , \end{aligned}\right\} \tag{2.24}$$

has the same form both for Hooke's and micropolar media. Likewise the equations

$$\begin{aligned} &(\lambda+2\mu)\left(\nabla_R^2-\frac{1}{R^2}\right)u_R-\varrho\ddot{u}_R=\nu\frac{\partial\theta}{\partial R}\ ,\\ &\left(\nabla_R^2-\frac{1}{\varkappa}\partial_t\right)\theta-\eta\partial_t\frac{1}{R^2}\frac{\partial}{\partial R}(u_R R^2)=-\frac{Q}{\varkappa}\ ,\quad \nabla_R^2=\frac{\partial^2}{\partial R^2}+\frac{2}{R}\frac{\partial}{\partial r}\ , \end{aligned} \tag{2.25}$$

determining the strain symmetric with respect to a point, have the same form for both the media.

So far all more important general theorems have been derived, let us mention here the principle of virtual work, the theorems of energy, the reciprocity theorem, the generalized formulae of Somigliano and Green. We shall not dwell on these problems referring to the bibliography [10] .

We have given a short review of the results of the dynamical thermoelasticity. The theory can be considerably simplified by neglecting the coupling of the thermal conduction equation with the equations in displacements and rotations. We disregard the term $-\eta \operatorname{div} \dot{u}$ in the heat conduction equation. This simplification does not influence the magnitude of stresses and strains, however it does change the qualitative picture of the problem. From the wave equation $(2.8)_1$ it results that the P-wave is attenuated and dispersed; on the other hand when we disregard the term $\eta \operatorname{div} \dot{u}$ the P-wave consists only of the elastic part (which is neither attenuated nor dispersed) and the diffusive wave. Besides, we cannot obtain any information on the amount of the dissipated energy from the simplified theory.

In recent years the dynamical problems of thermoelasticity have been extended on the Cosserats' continuum of viscoelastic properties. D. Iesan [18] has given a few general theorems for such a continuum (the reciprocity theorem, the variational theorems and the uniqueness theorem). He also examined with full particulars the plane dynamical problem [25] . S. Kaliski [19] has given the fundamentals of the thermo-magneto-microelasticity. By this term we understand the coupling of the field of deformation (of the Cosserats' continuum) with the field of temperature and the electromagnetic field in the conductors.

3. Stationary Thermal Stress Problems

For a stationary heat flow the time derivative in the thermoelasticity equations disappears and all the quantities take the following form

$$\left.\begin{aligned}
&(\mu+\alpha)\nabla^2\underline{u} + (\lambda+\mu-\alpha)\,\mathrm{grad}\,\mathrm{div}\,\underline{u} + 2\alpha\,\mathrm{curl}\,\underline{\varphi} = \nu\,\mathrm{grad}\,\theta,\\
&(\gamma+\varepsilon)\nabla^2\underline{\varphi} - 4\alpha\underline{\varphi} + (\beta+\gamma-\varepsilon)\,\mathrm{grad}\,\mathrm{div}\,\underline{\varphi} + 2\alpha\,\mathrm{curl}\,\underline{u} = 0,\\
&\nabla^2\theta = -\frac{Q}{\varkappa}\ .
\end{aligned}\right\} \qquad (3.1)$$

The equation of thermal conduction has become an equation of the Poisson type. We determine the temperature Q from eq. $(3.1)_3$ and substitute it, as a known function, in eq. $(3.1)_1$. Only eqs. $(3.1)_1$ and $(3.1)_2$ are coupled. The easiest way to obtain the solution to the system of equations is to introduce the potential of thermoelastic displacement

$$\underline{u}' = \mathrm{grad}\,\Phi \qquad (3.2)$$

and to assume that $\underline{\varphi}' = 0$.

Substituting (3.2) in eq. $(3.1)_1$ and $(3.1)_2$ we reduce the system of equations to the Poisson equation

$$(3.3) \qquad \nabla^2\Phi = m\vartheta\,, \quad m = \frac{\nu}{\lambda+2\mu}\,, \quad \underline{\varphi}' = 0$$

We express the stresses by means of the function Φ :

$$(3.4) \qquad \sigma'_{ij} = \sigma'_{ji} = 2\mu(\Phi_{,ij} - \delta_{ij}\nabla^2\Phi) \qquad , \qquad \mu'_{ij} = 0$$

These quantities constitute the complete solution for an infinite space and are identical for Hooke's medium and for Cosserats' medium. If the region is bounded we add to the stresses σ'_{ij}, μ'_{ij} such stresses σ''_{ij}, μ''_{ij} that all the boundary conditions are satisfied. The stresses σ''_{ij}, μ''_{ij} are connected with such state of displacement and rotations $\underline{u}''$, $\underline{\varphi}''$ that satisfies the following homogeneous system of equations

$$(3.5) \qquad \begin{cases} (\mu+\alpha)\nabla^2\underline{u}'' + (\lambda+\mu-\alpha)\,\mathrm{grad\,div}\,\underline{u}'' + 2\alpha\,\mathrm{curl}\,\underline{\varphi}'' = 0\,, \\ (\gamma+\varepsilon)\nabla^2\underline{\varphi}'' - 4\alpha\underline{\varphi}'' + (\beta+\gamma-\varepsilon)\,\mathrm{grad\,div}\,\underline{\varphi}'' + 2\alpha\,\mathrm{curl}\,\underline{u}'' = 0. \end{cases}$$

Similarly as in the classical thermoelasticity here too we have the "body force analogy" [20], [21]. The principle of the virtual work is our point of departure

$$(3.6) \qquad \int_V (X_i\delta u_i + Y_i\delta\varphi_i)\,dV + \int_A (p_i\delta u_i + m_i\delta\varphi_i)\,dA = \int_V (\sigma_{ji}\delta\gamma_{ji} + \mu_{ji}\delta\varkappa_{ji})\,dV$$

It says that the virtual work of the external forces (body forces and moments, tractions and surface moments) is equal to the virtual work of internal forces. Substituting the generalized Duhamel-Neumann equations to the right-hand side of the equations we obtain

$$\int_V (X_i \delta u_i + Y_i \delta \varphi_i) dV + \int_A (p_i \delta u_i + m_i \delta \varphi_i) dA = \\ = \delta \mathcal{H} - \nu \int_V \theta \delta \gamma_{kk} dV \tag{3.7}$$

where

$$\mathcal{H} = \int W dV \,, \quad W = \mu \gamma_{(ij)} \gamma_{(ij)} + \frac{\lambda}{2} \gamma_{kk} \gamma_{nn} + \alpha \gamma_{<ij>} \gamma_{<ij>} + \\ + \gamma \varkappa_{(ij)} \varkappa_{(ij)} + \frac{\beta}{2} \varkappa_{kk} \varkappa_{nn} + \varepsilon \varkappa_{<ij>} \varkappa_{<ij>} .$$

Eq. (3.7) can be represented in the form

$$\delta \mathcal{H} = \int [(X_i - \nu \theta_{,i}) \delta u_i + Y_i \delta \varphi_i] dV + \\ + \int_A [(p_i + \nu \theta n_i) \delta u_i + m_i \delta \varphi_i] dA \,. \tag{3.8}$$

Now we shall consider an identical body (i.e., of the same form and material), but placed under isothermal conditions. Let the body-forces X_i^* and the body couples Y_i^* act on the body. The tensions p_i^* and moments m_i^* are assumed to be given on the surface A_σ, while displacement u_i^* and rotations φ_i^* on A_u. We ask the following question: what should be the quantities X_i^* and Y_i^* - expressing forces and couples acting inside the body - and, on the other hand, the quantities p_i^* and m_i^* - expressing the tensions and moments acting on the surface A_σ - with identical boundary conditions for A_u in order to obtain the same field of displacement u_i and rotations φ_i in both. viz., thermoelastic and isothermal problems. To get the answer, we shall compare (3.8) with the virtual work equation

$$\delta \mathcal{W}_e = \int_V (X_i^* \delta u_i + Y_i^* \delta \varphi_i) dV + \int_A (p_i^* \delta u_i + m_i^* \delta \varphi_i) dA . \tag{3.9}$$

In view of the identity of u_i and φ_i fields, the left-hand parts of eqs. (3.8) and (3.9) are identical too; thus, we obtain the following relations

$$\begin{cases} X_i^* = X_i - \nu \theta_{,i} \ , \quad Y_i^* = Y_i \ , & \underline{x} \in V , \\ p_i^* = p_i + \nu \theta n_i \ , \ m_i^* = m_i \ , & \underline{x} \in A_\sigma , \\ u_i^* = u_i \quad , \quad \varphi_i^* = \varphi_i \quad , & \underline{x} \in A_u . \end{cases} \tag{3.10}$$

Relations (3.10) represent the body forces analogy by means of which each steady-state problem can be reduced to the isothermal problem of the theory of asymmetric elasticity.

Now we may ask the question whether the solution to the stationary equations of thermoelasticity can be combined from two parts, the first of which is identical in the form with the solution of the classical thermoelasticity. The answer to this question is affirmative.

Following H. Schaefer [22], let us introduce the vector

$$\underline{\zeta} = \frac{1}{2} \operatorname{curl} \underline{u} - \underline{\varphi} \tag{3.11}$$

and eliminate the function $\underline{\varphi}$ from the system of thermoelasticity equations. We obtain

$$\left.\begin{aligned} &\mu\nabla^2\underline{u} + (\lambda+\mu)\operatorname{grad}\operatorname{div}\underline{u} - \nu\operatorname{grad}\theta = 2\alpha\operatorname{curl}\underline{\zeta}\ , \\ &[(\gamma+\varepsilon)\nabla^2 - 4\alpha]\underline{\zeta} + (\beta+\gamma-\varepsilon)\operatorname{grad}\operatorname{div}\underline{\zeta} = \frac{1}{2}(\gamma+\varepsilon)\nabla^2\operatorname{curl}\underline{u}. \end{aligned}\right\} \tag{3.12}$$

Let us combine the solution of this system of equations from two parts

$$\underline{u} = \underline{u}' + \underline{u}''\ , \quad \underline{\zeta} = \underline{\zeta}' + \underline{\zeta}''\ , \quad \underline{\zeta}' = 0\ .$$

Thus the system of equations (3.12) is split into two systems of equations

$$\mu\nabla^2\underline{u}' + (\lambda+\mu)\operatorname{grad}\operatorname{div}\underline{u}' = \nu\operatorname{grad}\theta,\ \nabla^2\operatorname{curl}\underline{u}' = 0\ , \tag{3.13}$$

and

$$\left.\begin{aligned} &\mu\nabla^2\underline{u}'' + (\lambda+\mu)\operatorname{grad}\operatorname{div}\underline{u}'' = 2\alpha\operatorname{curl}\underline{\zeta}''\ , \\ &[(\gamma+\varepsilon)\nabla^2 - 4\alpha]\underline{\zeta}'' + (\beta+\gamma-\varepsilon)\operatorname{grad}\operatorname{div}\underline{\zeta}'' = \frac{1}{2}(\gamma+\varepsilon)\nabla^2\operatorname{curl}\underline{u}\ . \end{aligned}\right\} \tag{3.14}$$

Let us note that the system of equations (3.13) is identical with the corresponding system of classical thermoelasticity. (However, the constants μ, λ occurring in eqs.

(3.13) refer to the micropolar body).

Therefore the solution $\mathbf{v}'$ can be taken from the classical thermoelasticity. This solution satisfies eq. (3.13) and the corresponding boundary conditions. Since $\underline{\zeta}' = 0$, which is equivalent to the assumption that $\gamma'_{\langle ij \rangle} = 0$, the tensor γ_{ij} is symmetric. Also the force stress tensor is symmetric

$$(3.15) \qquad \sigma'_{ij} = 2\mu\gamma_{(ij)} + (\lambda\gamma'_{kk} - \nu\theta)\delta_{ij} \ .$$

However the assumption $\underline{\zeta}' = 0$ is equivalent to the assumption $\underline{\varphi}' = \frac{1}{2}\operatorname{curl}\underline{u}'$. Since $\underline{\varphi}' \neq 0$ then there exists the tensor $\varkappa'_{ji} = \varphi'_{i,j}$ and the moment stresses

$$(3.16) \qquad \mu'_{ji} = 2\mu\varkappa'_{(ij)} + 2\varepsilon\varkappa'_{\langle ij \rangle} + \beta\varkappa'_{kk}\delta_{ij} \ .$$

The solution of the system of equation (3.13) satisfies only part of the boundary conditions. If, for example, the boundary is free from loadings, then the condition $\sigma'_{ji} n_j = 0$ is satisfied while condition $\mu'_{ji} n_j = 0$ is not satisfied.

In order to satisfy all the boundary conditions the solution $\underline{u}''$, $\underline{\zeta}''$ of the system of equations (3.14) is necessary. The boundary conditions connected with the system of equations (3.14) have the form

$$(3.17) \qquad \sigma''_{ji} n_j = 0 \quad , \quad (\mu'_{ji} + \mu''_{ji}) n_j = 0 \ .$$

E.Betti's theorem of reciprocity of work constitutes one of the most interesting theorems in the theory of elasticity. The theorem is very general and contains the possibility to derive the integration methods of the equations of the theory of elasticity by means of the Green function. We succeeded in generalizing the theorem in the case of micropolar thermoelasticity. For the case of the stationary problem it has the form

$$\int_V (X_i u_i' + Y_i \varphi_i')\,dV + \int_A (p_i u_i' + m_i \varphi_i')\,dA + \nu \int_V \vartheta \gamma_{kk}'\,dV =$$
$$= \int_V (X_i' u_i + Y_i' \varphi_i)\,dV + \int_A (p_i' u_i + m_i' \varphi_i)\,dA + \nu \int_V \vartheta' \gamma_{kk}\,dV. \tag{3.18}$$

We consider here two systems of "generalized forces" acting on an elastic body and the corresponding "generalized displacements". The first group includes the body forces X_i, the body moments Y_i, the tractions p_i, the surface moments m_i, and the temperature field. The displacement $\underline{u}$ the rotation $\underline{\varphi}$ and the temperature ϑ constitute the generalized displacements. The second system of forces and displacements can be distinguished from the first one by the "primes".

Consider a bounded body, rigidly clamped on A_u and free from loadings on A_σ. Assume that the heat sources act in the body, while the surface $A = A_u + A_\sigma$ is heated. We have to determine the displacements u and the rotations φ in the body.

In order to determine the displacements $\underline{u}(\underline{x})$,

$\underline{x} \in V$ we consider a body of the same form and the boundary conditions in the isothermal state. Let the concentrated force $X_i' = \delta(\underline{x} - \underline{\xi})\delta_{ik}$ act at a point $\underline{\xi} \in V$ and produce such a displacement field $U_i^{(k)}(\underline{x},\underline{\xi})$ that it satisfies the homogeneous boundary conditions ($\underline{u}=0$, $\underline{\varphi}=0$ on A_u, $p=0$, $\underline{m}=0$ on A_σ) . We obtain from eq. (3.18) for $Y_i'=0$, $Q'=0$

$$(3.19) \qquad u_k(\underline{\xi}) = \gamma \int_V \theta(\underline{x}) \frac{\partial U_i^k(\underline{\xi},\underline{x})}{\partial x_i} dV(\underline{x}), \quad \underline{x} \in V, \; k=1,2,3 .$$

The symbol $U_{i,i}^{(k)}(\underline{\xi},\underline{x})$ stands for the dilatation at the point $\underline{\xi}$ due to the action of the concentrated force X_i' situated at the point $\underline{\xi}$.

Let us place at point $\underline{\xi}$ the concentrated moment $Y_i' = \delta(\underline{x} - \underline{\xi})\delta_{ik}$ producing in the body the displacement field $\hat{U}_i^{(k)}(\underline{x},\underline{\xi})$. This field has to satisfy the homogeneous boundary conditions on A_u and A_σ .

Assuming that $X_i'=0$, $Q'=0$, we obtain

$$(3.20) \qquad \varphi_k(\underline{\xi}) = \gamma \int \theta(\underline{x}) \frac{\hat{U}_i^{(k)}(\underline{x},\underline{\xi})}{\partial x_i} dV(\underline{x}), \quad x \in V, \; k=1,2,3 .$$

Eqs. (3.19) and (3.20) constitute a generalization of Maysel's formulae [23] known in the classical thermoelasticity. Eqs. (3.19) and (3.20) are very simple; in order to determine the fields $\underline{u}, \underline{\varphi}$ it is sufficient to integrate eqs. (3.19) and (3.20) provided the Green functions $U_i^{(k)}$, $\hat{U}_i^{(k)}$ have been determined beforehand.

In the particular case of an infinite micropolar body, eqs. (3.19) and (3.20) take the following form

$$u_i(\underline{\xi}) = \frac{\nu}{4\pi(\lambda+2\mu)} \int_V \vartheta(\underline{x}) \frac{\partial}{\partial x_i} \frac{1}{R(\underline{x},\xi)} \, dV(\underline{x}), \tag{3.19}$$

$$\varphi_i(\underline{\xi}) = 0 , \tag{3.20}$$

or

$$\left.\begin{aligned} u_i(\underline{\xi}) &= -\frac{m}{4\pi} \int \vartheta(\underline{x}) \frac{\partial}{\partial \xi_i} \left(\frac{1}{R(\underline{\xi}, \underline{x})} \right) dV(\underline{x}) , \\ \varphi_i &= 0 . \end{aligned}\right\} \tag{3.21}$$

The above equations are identical with those of the classical thermoelasticity.

Now, consider a single-connected bounded body free from loadings on its surface and free from the body forces and moments inside. The deformations of the body are generated only by its heating. We wish to determine the integrals [24]

$$I_1 = \int_V \operatorname{div} \underline{u} \, dV \quad , \quad I_2 = \int_V \operatorname{div} \underline{\varphi} \, dV , \tag{3.22}$$

characterizing the deformation of the body. The first integral denotes the increase of the body volume, the second one is the mean value of the function $\underline{\varphi} \cdot \underline{n}$ on the body surface. If we assume that the "primed" state corresponds to the uniform tension of the micropolar body ($p'_i = 1 \cdot n_i$, $\sigma'_{ji} = 1 \cdot \delta_{ij}$) then we obtain from the theorem of reciprocity of works

$$(3.23) \qquad I_1 = 3\alpha_t \int_V \vartheta(\underline{x})\, dV(\underline{x}) \quad , \quad \int_V \sigma_{kk}\, dV = 0 \quad .$$

The increase of volume depends here on the temperature distribution in the body, and the mean value of the sum of normal stresses is equal to zero. This result is identical with that of the classical thermoelasticity. Now, assume that the "primed" state corresponds to the uniform torsion ($\mu'_{ji} = 1\, \delta_{ij}$, $m'_i = 1\, n_i$). From the theorem of reciprocity we obtain

$$(3.24) \qquad I_2 = \frac{1}{3\beta + 2\gamma} \int_V \epsilon_{kji} x_k \sigma_{jk}\, dV = \frac{1}{3\beta + 2\gamma} \int_V \mu_{kk}\, dV \,.$$

Let us note that this integral vanishes when the tensor σ_{jk} is symmetric.

So far a number of particular solutions have been obtained. Most of them concern the two-dimensional problems. It is a known fact that two plane strain state problems exist, the first one is characterized by the vectors $\underline{u} = (u_1, u_2, 0)$, $\underline{\varphi} = (0, 0, \varphi_3)$ while the second one by the vectors $\underline{u} = (0, 0, u_3)$ and $\underline{\varphi} = (\varphi_1, \varphi_2, 0)$. Here on the variable x_3. Only the first of these two problems is connected with the field of temperature. A few papers have been devoted to this problem, here we mention the papers by G. Iesan [26], J. Dyszlewicz [27] and W.

Nowacki [28] .

Of the two axially symmetric problems, only the first one, characterized by the vectors $\underline{u} = (u_r, 0, u_z)$, $\underline{\varphi} = (0, \varphi_\theta, 0)$ is connected with the field of temperature. P. Puri [29] and R.S. Dhaliwal [30] have discussed this problem.

Though the main framework of the linear micropolar elasticity has been worked out, a number of particular problems remain unsolved.

The new directions consisting in the incorporation of the further fields are also encouraging. There exists the possibility to construct the theory of thermodiffusion in the micropolar bodies, the coupling of the electrodynamic field with the field of strain, and so on.

An important practical meaning (for the case of action of elevated temperature), may have the theory of non-homogeneous thermoelasticity, which takes into account the material coefficients varying in position and temperature.

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