IUTAM SYMPOSIA VIENNA 1966

IRREVERSIBLE ASPECTS OF CONTINUUM MECHANICS

AND

TRANSFER OF PHYSICAL CHARACTERISTICS IN MOVING FLUIDS

EDITORS

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SPRINGER-VERLAG / WIEN · NEW YORK

Couple-Stresses in the Theory of Thermoelasticity

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Summary

The aim of the present paper is to generalize some theorems on the coupled thermoelasticity of a medium characterized by two vectors independent from each other: the displacement vector $\boldsymbol{u}$ and the rotation vector $\boldsymbol{\omega}$.

Basing on the thermodynamics of irreversible processes the constitutive equations and the expanded equation of heat conductivity for an isotropic medium are derived.

The author succeeded in obtaining a basic system of differential equations of coupled thermoelasticity. The propagation of thermoelastic waves in an unbounded medium is discussed.

Moreover, a generalization of the virtual work principle to dynamic problem of coupled thermoelasticity is advanced.

Finally, the reciprocity theorem is derived and some conclusions resulting from this theorem are discussed.

1. Introduction

The asymmetric theory of elasticity was first advanced in 1887 by Voigt [1], and then developed in 1909 by the brothers E. and F. Cosserat [2]. It is assumed in this theory that the entire action upon a material volume bounded by a surface is described completely in terms of the field of stress vectors and "couple-stress" vectors.

Recently we witness a further development of this theory, particularly since it proved useful in explaining some regularities of propagation of short acoustic waves in crystals, in polycrystalline structures as well as in high polymers.

Let us mention here the papers by Truesdell and Toupin [3], and by Aero and Kuvshinskii [4]. Grioli [5] and Toupin [6] succeeded in generalizing the theory of the Cosserat medium to finite deformations. Finally, Mindlin and Tiersten [7] presented an exhaustive discussion of the linear theory of a homogeneous, isotropic and centrosymmetric medium.

In subsequent years KUVSHINSKII and AERO (1963, [8]), PALMOV [9] and ERINGEN and SUHUBI [10] developed the theory of asymmetric elasticity describing the deformation of a body by the vectors of displacement $\boldsymbol{u}$ and rotation $\boldsymbol{\omega}$, mutually independent thus departing from the kinematic assumption $\boldsymbol{\omega} = \frac{1}{2}\operatorname{curl}\boldsymbol{u}$, which is the very basis of the theory of the Cosserat medium.

The purpose of the present work is to consider, within the framework of the theory of asymmetric elasticity, the interaction between the fields of displacements $\boldsymbol{u}$, rotation $\boldsymbol{\omega}$ and temperature θ.

Confining our considerations to the elastic, homogeneous, isotropic and centrosymmetric medium we derive constitutive equations based on the thermodynamics of irreversible processes. A complete set of differential equations of asymmetric thermoelasticity is given as well as equations of motion and an generalized equation of heat conductivity.

Finally, a variational theorem and a theorem on reciprocity are given, and some conclusions are derived from these theorems.

2. Equations of Motion. Energy Equation and Entropy Balance

The system of equations of motion consists of two equations: the equation of balance of linear momentum and the equation of angular momentum [4], [7]:

$$\sigma_{ji,j} + X_i - \varrho \ddot{u}_i = 0, \tag{2.1}$$

$$\varepsilon_{ijk}\sigma_{jk} + \mu_{ji,j} + Y_i - J\ddot{\omega}_i = 0, \qquad i, j, k = 1, 2, 3. \tag{2.2}$$

In these equations the symbol σ_{ij} denotes the asymmetric force-stress tensor, μ_{ji} the asymmetric couple-stress tensor. X_i and Y_i designate the volume density of the body forces and body couples; ε_{ijk} stands for the usual alternator. u_i are the components of the displacement vector $\boldsymbol{u}$, ω_i the components of the rotation vector.

The principle of conservation of energy written for an arbitrary volume V of a body, bounded by a surface A, has the form

$$\frac{d}{dt}\int_V \left[\frac{1}{2}(\varrho v_i v_i + J w_i w_i) + U\right] dV = \int_V (X_i v_i + Y_i \omega_i)\, dV + \\ + \int_A (p_i v_i + m_i w_i)\, dA - \int_A q_i n_i\, dA. \tag{2.3}$$

Here $v_i = \dot{u}_i$, $w_i = \dot{\omega}_i$. By U we denote the internal energy, and by q_i the components of the heat flux vector $\boldsymbol{q}$.

The quantities p_i and m_i are connected with tensors σ_{ij} and μ_{ji} by the following relations

$$p_i = \sigma_{ji} n_j, \qquad m_i = \mu_{ji} n_j, \tag{2.4}$$

where n_i are the components of the unit vector $\boldsymbol{n}$ on A.

The terms on the left hand side of (2.3) represent the rate of increase of the kinetic and internal energy, respectively, of the volume. The first term on the right-hand side represents the rate of work of the body forces and body couples, the second term-the rate of work of the surface tractions and couples. Finally, the last integral on the right-hand side of eq. (2.3) denotes the heat transferred to the volume V by heat conduction. Taking into consideration (2.1), (2.2) and (2.4) and making use of the divergence theorem, we obtain the following equation

$$\int_V \{\dot{U} - [\sigma_{ij}(v_{i,j} - \varepsilon_{kji} w_k) + \mu_{ji} w_{i,j}] + q_{i,i}\}\, dV = 0. \tag{2.5}$$

This expression holds for any arbitrary volume V. If the integrand is continuous, then the relation

$$\dot{U} = \sigma_{ji}\dot{\gamma}_{ji} + \mu_{ji}\dot{\varkappa}_{ji} - q_{i,i} \tag{2.6}$$

holds locally.

The following notations were introduced

$$\gamma_{ji} = u_{i,j} - \varepsilon_{kji}\omega_k, \qquad \varkappa_{ji} = \omega_{i,j}. \tag{2.7}$$

Here γ_{ji} is the asymmetric strain tensor, $\varkappa_{ji}$ the curvature-twist tensor.

The equation of entropy balance can be written in the form ([11], p. 29).

$$\int_V \dot{S}\, dV = -\int_A \frac{q_i n_i}{T}\, dA + \int_V \Theta\, dV. \tag{2.8}$$

The left-hand side of this equation represents the rate of increase of entropy. The first term on the right-hand side is the rate at which entropy is supplied to the volume across the surface. The second term on the right-hand side of (2.8) denotes the rate of entropy production, due to heat conduction. Using the divergence theorem we have

$$\int_V \left(\dot{S} - \Theta - \left(\frac{q_i}{T}\right)_{,i}\right) dV = 0, \tag{2.9}$$

and hence, since V is arbitrary we get

$$\dot{S} = \Theta - \frac{q_{i,i}}{T} + \frac{q_i T_{,i}}{T^2}, \tag{2.10}$$

which holds at each point of the body. In accordance with the postulate of thermodynamics of irreversible processes we have $\Theta \geqq 0$.

Eliminating $q_{i,i}$ from (2.6) and (2.10), and introducing the expression for the Helmholtz free energy $F = U - ST$, we obtain

$$\dot{F} = \sigma_{ji}\dot{\gamma}_{ji} + \mu_{ji}\dot{\varkappa}_{ji} - \dot{T}S - T\left(\Theta + \frac{q_i T_{,i}}{T^2}\right). \tag{2.11}$$

Since the free energy is a function of the independent variables γ_{ji}, $\varkappa_{ji}$, T, there is

$$\dot{F} = \frac{\partial F}{\partial \gamma_{ji}}\dot{\gamma}_{ji} + \frac{\partial F}{\partial \varkappa_{ji}}\dot{\varkappa}_{ji} + \frac{\partial F}{\partial F}\dot{T}. \tag{2.12}$$

Assuming that the functions Θ, q_i, .. σ_{ji}, μ_{ji} do not explicitly depend on the time derivatives of the functions γ_{ji}, $\varkappa_{ji}$, T and defining the entropy as $S = -\dfrac{\partial F}{\partial T}$, we obtain — after comparing (2.11) with (2.12) — the following relations

$$\sigma_{ji} = \frac{\partial F}{\partial \gamma_{ji}}, \quad \mu_{ji} = \frac{\partial F}{\varkappa_{ji}}, \quad S = -\frac{\partial F}{\partial T}, \quad \Theta + \frac{q_i T_{,i}}{T^2} = 0. \tag{2.13}$$

The second law of thermodynamics will be satisfied if $\Theta \geqq 0$, or if

$$-\frac{T_{,i}q_i}{T^2} \geqq 0. \tag{2.14}$$

The above inequality satisfies the Fourier law of thermal conductivity

$$-q_i = k_{ij}T_{,j} \qquad \text{or} \qquad -q_i = k_{ij}\theta_{,j} \quad T = T_0 + \theta \tag{2.15}$$

Here T_0 denotes the temperature of the body in its natural state, in which stresses and deformations are equal to zero (i. e., for $\gamma_{ji} = 0$, $\varkappa_{ji} = 0$, $T = T_0$). The quantities k_{ij} are the coefficients of thermal conductivity and form a symetric tensor.

From eq. (2.10) — taking into account the last relation of the group (2.13) — we have

$$T\dot{S} = -q_{i,i} = k_{ij}\theta_{,ij}. \tag{2.16}$$

For a homogeneous and isotropic body, we get

$$T\dot{S} = k\theta_{,jj} \tag{2.17}$$

where k is constant.

3. Constitutive Equations

Let us expand the expression for the free energy $F(\gamma_{ji}, \varkappa_{ji}, T)$ in the neighborhood of a natural state ($\gamma_{ji} = 0$, $\varkappa_{ji} = 0$, $T = T_0$) into a Taylor series, omitting powers higher than 2. This expansion has, for a centrosymmetric body, the form

$$F = \frac{\mu+\alpha}{2}\gamma_{ji}\gamma_{ji} + \frac{\mu-\alpha}{2}\gamma_{ji}\gamma_{ij} + \frac{\lambda}{2}\gamma_{kk}\gamma_{nn} + \frac{\gamma+\varepsilon}{2}\varkappa_{ji}\varkappa_{ji} + \frac{\gamma-\varepsilon}{2}\varkappa_{ji}\varkappa_{ij} + \frac{\beta}{2}\varkappa_{kk}\varkappa_{nn} - \nu\gamma_{kk}\theta - \chi\varkappa_{kk}\theta - \frac{m}{2}\theta^2. \tag{3.1}$$

This form results from the following considerations. Since the free energy is a scalar, each term on the right-hand side of expression (3.1) must be a scalar. Now, with the components of the asymmetric tensor γ_{ji} we can construct three independent quadratic invariants, namely $\gamma_{ji}\gamma_{ji}$, $\gamma_{ji}\gamma_{ij}$ and $\gamma_{kk}\gamma_{nn}$. The same holds for the tensor $\varkappa_{ji}$. However, the terms $\gamma_{ji}\varkappa_{ji}$, $\gamma_{ji}\varkappa_{ij}$ and $\gamma_{kk}\varkappa_{nn}$ cannot appear in eq. (3.1), since this would contradict the postulate of centrosymmetric isotropy. In the seventh and eighth term of relation (3.1) the invariants γ_{kk} and $\varkappa_{kk}$ appear. This results from the fact that with the components of the tensor γ_{ji} and $\varkappa_{ji}$ one can form invariants of the first kind only, namely γ_{kk} and $\varkappa_{kk}$.

Making use of relations (3.13) we have

$$\sigma_{ji} = (\mu+\alpha)\gamma_{ji} + (\mu-\alpha)\gamma_{ij} + (\lambda\gamma_{kk} - \nu\theta)\delta_{ji}, \tag{3.2}$$

$$\mu_{ji} = (\gamma+\varepsilon)\varkappa_{ji} + (\gamma-\varepsilon)\varkappa_{ij} + (\beta\varkappa_{kk} - \chi\theta)\delta_{ij} \tag{3.3}$$

$$S = \nu\gamma_{kk} + \chi\varkappa_{kk} + m\theta. \tag{3.4}$$

Relations (3.2), (3.3) may be rewritten in an equivalent form

$$\sigma_{ji} = 2\mu\gamma_{(ii)} + 2\alpha\gamma_{\langle ij\rangle} + (\lambda\gamma_{kk} - \chi\nu\theta)\delta_{ij}, \tag{3.5}$$

$$\mu_{ij} = 2\gamma\varkappa_{(ij)} + 2\varepsilon\varkappa_{\langle ij\rangle} + (\beta\varkappa_{kk} - \theta)\delta_{ij}. \tag{3.6}$$

Here μ, λ are Lame constants and α, γ, ε, β are new constants of elasticity referring to the isothermal state. The constants ν, χ depend on the mechanical as well as on the thermal properties of the body. The symbols ( ) and $\langle\rangle$ denote symmetric and skew-symmetric part of a tensor, respectively.

Solving eqs. (3.5) and (3.6) for γ_{ij} and $\varkappa_{ij}$ we obtain

$$\gamma_{ij} = \alpha_t \delta_{ij} \theta + 2\mu' \sigma_{(ij)} + 2\alpha' \sigma_{\langle ij \rangle} + \lambda' \delta_{ij} \sigma_{kk}, \tag{3.7}$$

$$\varkappa_{ij} = \beta_t \delta_{ij} \theta + 2\gamma' \mu_{(ij)} + 2\varepsilon' \mu_{\langle ij \rangle} + \beta' \delta_{ij} \mu_{kk}, \tag{3.8}$$

where

$$\mu' = \frac{1}{4\mu}, \quad \alpha' = \frac{1}{4\alpha}, \quad \gamma' = \frac{1}{4\gamma}, \quad \varepsilon' = \frac{1}{4\varepsilon}, \quad \lambda' = -\frac{\lambda}{6\mu K},$$

$$\beta' = -\frac{\beta}{6\gamma\Omega}, \quad \alpha_t = \frac{\nu}{3K}, \quad \beta_t = \frac{\chi}{3\Omega}, \quad K = 2\mu + 3\lambda, \quad \Omega = 2\gamma + 3\beta.$$

The symbol α_t appearing in relation (3.7) is the coefficient of thermal dilatation.

If the volume element, free from force-stresses and couple-stresses on its surface, produced temperature voluminal changes only, we obtain

$$\overset{\circ}{\gamma}_{ij} = \alpha_t \delta_{ij} \theta, \qquad \overset{\circ}{\varkappa}_{ij} = 0.$$

The coefficient β_t and, consequently, the quantity χ should be equal to zero. This assertion is also supported by another argument. Let us consider the problem of classical elastokinetics where an adiabatic process is assumed ($\dot{S} = 0$). In this case the temperature is given by

$$\theta = -\frac{1}{m} (\nu \gamma_{kk} + \chi \varkappa_{kk}). \tag{3.9}$$

Introducing (3.9) into relations (3.5) and (3.6) we obtain[1]

$$\sigma_{ij} = 2\mu \gamma_{(ij)} + 2\alpha \gamma_{\langle ij \rangle} + (\lambda^* \gamma_{kk} + \tau^* \varkappa_{kk}) \delta_{ij}, \tag{3.10}$$

$$\mu_{ij} = 2\mu \varkappa_{(ij)} + 2\varepsilon \varkappa_{\langle ij \rangle} + (\beta^* \varkappa_{kk} + \delta^* \gamma_{kk}) \delta_{ij} \tag{3.11}$$

where

$$\lambda^* = \lambda + \frac{\nu^2}{m}, \quad \beta^* = \beta + \frac{\chi^2}{m}, \quad \tau^* = \delta^* = \frac{\nu\chi}{m}.$$

It is easily seen that relations (3.10) and (3.11) are in contradiction with the formulae for an isotropic, centrosymmetric medium, where force stresses σ_{ij} are independent of $\varkappa_{ij}$ and couple-stresses are independent of γ_{ij} (cf., e. g., [7], [8]). Thus, we should have $\chi = 0$. Consequently, we have to put for the free energy $\chi = 0$ in (3.1).

---

[1] Observe that for a Cosserat medium, where $\omega_i = \frac{1}{2} \varepsilon_{ijk} u_{k,j}$, we have $\varkappa_{kk} = 0$.

In this way, the final forms of relations (3.2) (3.4) will be:

$$\sigma_{ij} = 2\mu\gamma_{(ij)} + 2\alpha\gamma_{\langle ij\rangle} + (\lambda\gamma_{kk} - \nu\theta)\,\delta_{ij}, \tag{3.12}$$

$$\mu_{ij} = 2\mu\varkappa_{(ij)} + 2\varepsilon\varkappa_{\langle ij\rangle} + \beta\varkappa_{kk}\delta_{ij}, \tag{3.13}$$

$$S = \nu\gamma_{kk} + m\theta. \tag{3.14}$$

Let us remark that in the last of the above relations the quantity m remains undefined. It will be determined from the following considerations.

To begin with, we shall consider the differential relation

$$dU = \sigma_{ji}d\gamma_{ji} + \mu_{ji}d\varkappa_{ji} + TdS. \tag{3.15}$$

Introducing into (3.15) the relation

$$dS = \left(\frac{\partial S}{\partial\gamma_{ji}}\right)_{\varkappa,T} d\gamma_{ji} + \left(\frac{\partial S}{\partial x_{ji}}\right)_{\gamma,T} d\varkappa_{ji} + \left(\frac{\partial S}{\partial T}\right)_{\varkappa,\gamma} dT, \tag{3.16}$$

and taking into account completeness conditions of dU, we arrive at the dependence

$$\left(\frac{\partial S}{\partial\gamma_{ji}}\right)_{\varkappa,T} - \nu\delta_{ij} = 0, \quad \left(\frac{\partial S}{\partial\varkappa_{ji}}\right)_{\gamma,T} = 0. \tag{3.17}$$

Substituting now the above formulae into (3.15) and (3.16) and taking into account $\left(\frac{\partial S}{\partial T}\right)_{\gamma,\varkappa} = \frac{c_\varepsilon}{T}$, where c_ε denotes the specific heat at constant deformation, we get

$$dU = \sigma_{ji}d\gamma_{ji} + \mu_{ji}d\varkappa_{ji} + \nu Td\gamma\varepsilon_{kk} + c_\varepsilon dT, \tag{3.18}$$

$$dS = \nu d\gamma_{kk} + c_\varepsilon\frac{dT}{T}. \tag{3.19}$$

Integrating (3.19) under the assumption that $S = 0$ for the natural state, we have

$$S = \nu\gamma_{kk} + c_\varepsilon\log\frac{T}{T_0}. \tag{3.20}$$

Assuming $\left|\frac{\theta}{T_0}\right| \ll 1$, expanding the logarithm into a series and retaining one term of this series only, we have

$$S = \gamma_{kk}\nu + c_\varepsilon\frac{\theta}{T_0}. \tag{3.21}$$

From a comparison of (3.14), and (3.21) it results that $m = \frac{c_\varepsilon}{T_0}$.

4. Differential Equations of Thermoelasticity

The constitutive relations (3.12), (3.13) enable us to express the equations of motion (2.1), (2.2) in terms of the displacement vector $\boldsymbol{u}$, the rotation vector $\boldsymbol{\omega}$ and temperature. We obtain the following differential equations

$$(\lambda + 2\mu)\,\mathrm{grad}\;\mathrm{div}\,\boldsymbol{u} - (\mu + \alpha)\,\mathrm{curl}\,\mathrm{curl}\,\boldsymbol{u} + 2\alpha\;\mathrm{curl}\,\boldsymbol{\omega} + \boldsymbol{X} = \varrho\,\ddot{\boldsymbol{u}} + \nu\,\mathrm{grad}\,\theta, \tag{4.1}$$

$$(\beta + 2\gamma)\,\mathrm{grad}\;\mathrm{div}\,\boldsymbol{\omega} - (\gamma + \varepsilon)\,\mathrm{curl}\;\mathrm{curl}\,\boldsymbol{\omega} + 2\alpha\,\mathrm{curl}\,\boldsymbol{u} - 4\,\alpha\boldsymbol{\omega} + \boldsymbol{Y} = J\ddot{\boldsymbol{\omega}}. \tag{4.2}$$

To eqs. (4.1), (4.2) we have to add the equation of heat conduction. For this purpose we consider the relations (2.17) and (3.20)

$$T\dot{S} = k\theta_{,jj} \tag{4.3}$$

$$T\dot{S} = \nu T\gamma_{kk} + c_\varepsilon \dot{T}. \tag{4.4}$$

It results from a comparison of these relations that

$$\theta_{,jj} - \frac{1}{\varkappa}\dot{\theta} - \eta_0\left(1 + \frac{\theta}{T_0}\right)\mathrm{div}\,\boldsymbol{u} = 0, \quad \varkappa = \frac{k}{c_\varepsilon}, \quad \eta_0 = \frac{\nu T_0}{k}. \tag{4.5}$$

When linearizing this equation, we assume that $\left|\frac{\theta}{T_0}\right| \ll 1$. Moreover, taking heat sources within the body into account and denoting by W the quantity of heat generated par unit volume and unit time, we obtain the following extended equation of thermal conduction

$$\theta_{,jj} - \frac{1}{\varkappa}\dot{\theta} - \eta_0\,\mathrm{div}\,\boldsymbol{u} = -\frac{Q}{\varkappa}, \qquad Q = \frac{W}{k}. \tag{4.6}$$

It is interesting to note that only the term derived from dilatation appears in eq. (4.6), consequently connected with the first deformation invariant γ_{kk}.

Equations (4.1), (4.2) and (4.6) constitute a set of equations of linear coupled thermoelasticity in the theory of asymmetric elasticity.

We have to supplement the differential equations of thermoelasticity with the boundary and initial conditions. If on the surface A, bounding the body of volume V, the force-stresses p_i, m_i and temperature θ are prescribed, boundary conditions will have the form

$$p_i(\boldsymbol{x}, t) = \sigma_{ji}(\boldsymbol{x}, t)\,n_j(\boldsymbol{x}), \quad m_i(\boldsymbol{x}, t) = \mu_{ji}(\boldsymbol{x}, t)\,n_j(\boldsymbol{x}),$$

$$\theta(\boldsymbol{x}, t) = h(\boldsymbol{x}, t), \qquad \boldsymbol{x}\in A, \qquad t > 0, \tag{4.7}$$

while the initial conditions will be expressed by the formulae

$$
\begin{aligned}
u_i(\boldsymbol{x}, 0) &= f_i(\boldsymbol{x}), \qquad \dot{u}_i(\boldsymbol{x}, 0) = g_i(\boldsymbol{x}),\\
\omega_i(\boldsymbol{x}, 0) &= l_i(\boldsymbol{x}), \qquad \dot{\omega}_i(\boldsymbol{x}, 0) = k_i(\boldsymbol{x}),\\
\theta(\boldsymbol{x}, 0) &= r(\boldsymbol{x}), \qquad \boldsymbol{x}\,\varepsilon V, \qquad t = 0.
\end{aligned}
\tag{4.8}
$$

The dynamic equations of thermoelasticity (4.1), (4.2) can be separated by decomposing the vectors $\boldsymbol{u}$ and $\boldsymbol{\omega}$ into their potential and solenoidal parts

$$
\begin{aligned}
\boldsymbol{u} &= \operatorname{grad}\varphi + \operatorname{curl}\boldsymbol{\Psi}, \qquad \boldsymbol{\omega} = \operatorname{grad}\zeta + \boldsymbol{\Phi},\\
&\operatorname{div}\boldsymbol{\Psi} = 0, \qquad \operatorname{div}\boldsymbol{\Phi} = 0.
\end{aligned}
\tag{4.9}
$$

Decomposing in a similar way the expressions for the body forces and the couple-body forces

$$
\boldsymbol{X} = \operatorname{grad}\vartheta + \operatorname{curl}\boldsymbol{\chi}, \qquad \boldsymbol{Y} = \operatorname{grad}\sigma + \boldsymbol{\Omega}, \tag{4.10}
$$

we obtain the following system of equations

$$
\Box_1^2\varphi = n\theta - \frac{1}{c_1^2}\vartheta, \tag{4.11}
$$

$$
\Box_3^2\zeta - s\zeta = -\frac{\sigma}{c_3^2}, \tag{4.12}
$$

$$
\Box_2^2\boldsymbol{\Psi} - r\boldsymbol{\Phi} = -\frac{1}{c_2^2}\boldsymbol{\chi}, \tag{4.13}
$$

$$
\Box_4^2\boldsymbol{\Phi} - p\nabla^2\boldsymbol{\Psi} - 2p\boldsymbol{\Phi} = -\frac{\boldsymbol{\Omega}}{c_1^2}. \tag{4.14}
$$

The notations introduced in (4.11) + (4.14) are as below

$$
\Box_\alpha^2 = \nabla^2 - \frac{1}{c_\alpha^2}\partial_t^2, \quad \alpha = 1, 2, 3, 4, \quad c_1 = \left(\frac{\lambda + 2\mu}{\varrho}\right)^{1/2}, \quad c_2 = \left(\frac{\mu + \alpha}{\varrho}\right)^{1/2},
$$

$$
c_3 = \left(\frac{\beta + 2\gamma}{J}\right)^{1/2}, \quad c_4 = \left(\frac{\gamma + \varepsilon}{J}\right)^{1/2}, \quad r = \frac{2\alpha}{\varrho c_2^2}, \quad s = \frac{4\alpha}{J c_3^2}, \quad p = \frac{2\alpha}{J c_4^2}.
$$

To eq. (4.11) and eq. (4.14) we have to add the equation of heat conduction

$$
D\theta - \eta_0\nabla^2\dot{\varphi} = -\frac{Q}{\varkappa}, \qquad D = \nabla^2 - \frac{1}{\varkappa}\partial_t. \tag{4.15}
$$

We shall consider first the propagation of thermoelastic waves in an infinite space. Observe that eqs. (4.11) and (4.15) are coupled. Eliminating the temperature from these equations we obtain the equation for the longitudinal wave

$$(\square_1^2 D - \eta_0 \partial_t \nabla^2)\varphi = -\frac{nQ}{\varkappa} - \frac{1}{c_1^2} D\vartheta, \qquad \partial_t = \frac{\partial}{\partial t}. \tag{4.16}$$

If the quantities χ, $\boldsymbol{\Omega}$, σ and the initial conditions of the functions ζ, $\boldsymbol{\Psi}$, $\boldsymbol{\Phi}$ are equal to zero, then in an unbounded elastic space only dilatational waves will propagate. Equation (4.16) describing these waves is identical with that obtained for the elastic classical medium (with no couple-stresses). It is known [12] that these waves are attenuated and dispersed. Since

$$u_i = \varphi_{,i}, \quad \omega_i = 0, \quad \gamma_{ji} = \varphi_{,ij}, \quad \varkappa_{ji} = 0,$$

we have

$$\sigma_{(ij)} = 2\mu(\varphi_{,ij} - \delta_{ij}\varphi_{,kk}) + \varrho(\ddot{\Phi} - \vartheta)\delta_{ij},$$
$$\sigma_{\langle ij\rangle} = 0, \quad \mu_{ji} = 0. \tag{4.17}$$

If Q, ϑ, χ, $\boldsymbol{\Omega}$ are equal to zero and the initial conditions of the functions θ, φ, $\boldsymbol{\Psi}$, $\boldsymbol{\Phi}$ are homogeneous, then in an infinite medium only torsional waves, described by eq. (4.12), propagate. We have namely

$$u_i = 0, \quad \omega_i = \zeta_{,i} \quad \varkappa_{ji} = \omega_{i,j} = \zeta_{,ij}, \quad \gamma_{ji} = 0.$$

Only couple-stresses will appear in the medium forming the symmetric tensor

$$\mu_{(ij)} = 2\gamma\zeta_{,ij} + \beta\delta_{ij}\nabla^2\zeta, \quad \mu_{\langle ij\rangle} = 0, \quad \sigma_{ij} = 0. \tag{4.18}$$

The propagation of these waves is not accompanied by a temperature field.

Finally, in the case when the quantities Q, ϑ, σ are equal to zero and the initial conditions of the functions φ, ζ, θ are homogeneous, only transverse waves propagate in an infinite space (described by the set of eqs. (4.13) and (4.14)). Eliminating from these equations separately the function $\boldsymbol{\Psi}$ and then $\boldsymbol{\Phi}$ we arrive at the following equations

$$[(\square_4^2 - 2p)\,\square_2^2 - pr\nabla^2]\boldsymbol{\Phi} = -\frac{p}{c_2^2}\nabla^2\chi - \frac{1}{c_4^2}\square_2^2\boldsymbol{\Omega}, \tag{4.19}$$

$$[(\square_4^2 - 2p)\,\square_2^2 - pr\nabla^2]\boldsymbol{\Psi} = -\frac{r}{c_4^2}\boldsymbol{\Omega} - \frac{1}{c_2^2}(\square_4^2 - 2p)\chi. \tag{4.20}$$

In an infinite medium these waves are not accompanied by a temperature field. Since $\operatorname{div}\boldsymbol{u} = 0$ they do not induce any changes in the volume of the body. The force-stresses σ_{ji} and couple-stresses μ_{ji} form an asymmetric tensor.

In a finite medium all three kinds of waves discussed here appear. Equations (4.11) $\div$ (4.15) are coupled by means of the boundary conditions.

5. Virtual Work Principle

It may be easily shown that the following equation holds

$$\int_V [(X_i - \varrho \ddot{u}_i)\delta u_i + (Y_i - J\ddot{\omega}_i)\delta\omega_i] dV + \int_A (p_i \delta u_i + m_i \delta\omega_i) dA = \\ = \int_V (\sigma_{ij}\delta\gamma_{ji} + \mu_{ji}\delta\varkappa_{ji}) dV. \tag{5.1}$$

The left-hand side of this equation represents the variation of work of external forces, while the right-hand side equals that of internal forces.

In the above equation symbols δu_i and $\delta\omega_i$ stand for the virtual changes of the components of displacement vector $\boldsymbol{u}$ and rotation vector $\boldsymbol{\omega}$.

Introducing into the right-hand side of eq. (5.1) the relations (3.12) and (3.13) we reduce eq. (5.1) to the form

$$\int_V [(X_i - \varrho \ddot{u}_i)\delta u_i + (Y_i - J\ddot{\omega}_i)\delta\omega_i] dV + \int_A (p_i \delta u_i + m_i \delta\omega_i) dA = \\ = \delta W - \nu \int_V \theta\gamma_{kk} dV, \tag{5.2}$$

where

$$\delta W = \int_V (2\mu\gamma_{(ij)}\delta\gamma_{(ij)} + 2\alpha\gamma_{\langle ij\rangle}\delta\gamma_{\langle ij\rangle} + 2\gamma\varkappa_{(ij)}\delta\varkappa_{(ij)} + 2\varepsilon\varkappa_{\langle ij\rangle}\delta\varkappa_{\langle ij\rangle} + \\ + \lambda\gamma_{kk}\delta\gamma_{kk} + \beta\varkappa_{kk}\delta\varkappa_{kk}) dV.$$

We have to supplement eq. (5.2) by a further equation, since only four causes, namely X_i, Y_i, p_i, m_i, appear in this equation in explicit form. Thus, we adjoin to eq. (5.2) the relation

$$-\nu \int_V \theta\delta\gamma_{kk} dV = \int_A \theta n_i \delta H_i dA + \frac{c_\varepsilon}{T_0}\int_V \theta\delta\theta dV + \\ + \frac{T_0}{k}\int_V \dot{H}_i \delta H_i dV, \tag{5.3}$$

derived from the heat equation by Biot [13]. The vector $\boldsymbol{H}$ in (5.3) is connected with the vector of heat flow $\boldsymbol{q}$ and the entropy S by the following relations

$$\boldsymbol{q} = T_0\dot{\boldsymbol{H}}, \quad S = -\mathrm{div}\,(\boldsymbol{H}). \tag{5.4}$$

Introducing (5.3) into eq. (5.2) we have

$$\delta(W+P+D)=\int_V [(X_i-\varrho\ddot{u}_i)\delta u_i+(Y_i-J\ddot{\omega}_i)\delta\omega_i]dV+$$
$$+\int_A (p_i\delta u_i+m_i\delta\omega_i)dA-\int_A \theta n_i\delta H_i dA. \qquad (5.5)$$

Here the heat potential $P=\frac{c_\varepsilon}{2T_0}\int_V \theta^2 dV$ and the dissipation function D were applied, where $\delta D=\frac{T_0}{k}\int_V \dot{H}_i\delta H_i dV$.

For $Y_i=0$, $m_i=0$, $\varkappa_{ji}=0$, $\alpha=\gamma=\varepsilon=\beta=0$, eq. (5.5) reduces to the variational equation of the coupled thermoelastic medium without couple-stresses.

The variational principle eq. (5.5) may be used for the derivation of the energy theorem, if we compare the functions u_i, ω_i, θ at point $\boldsymbol{x}$ and time t with those actually occurring in the same point after a time interval dt. Thus, introducing into eq. (5.5)

$$\delta u_i=v_i dt,\quad \delta\omega_i=w_i dt,\quad \delta\theta=\dot{\theta}dt,\quad \delta H_i=\dot{H}_i dt=-\frac{k}{T_0}\theta_{,i}dt,$$

and so on, we obtain the following formula

$$\frac{d}{dt}(K+W+P)+\chi_0=\int_V (X_i v_i+Y_i w_i)dV+\int_A (p_i v_i+$$
$$+m_i w_i)dA+\frac{k}{T_0}\int_A \theta n_i\theta_{,i}dA, \qquad (5.6)$$

where

$$K=\frac{\varrho}{2}\int_V v_i v_i dV+\frac{J}{2}\int_V w_i w_i dV,\quad \chi_0=\frac{dD}{dt}=\frac{k}{T_0}\int_V \theta_{,i}\theta_{,i}dV>0.$$

Here K denotes kinetic energy, and χ_0 is proportional to the source of entropy which is always a positive quantity.

The energy theorem (5.6) may be exploited to demonstrate the uniqueness theorem for a simply connected body. Such a demonstration may be carried out in a way similar to that indicated in [14] or [15].

6. The Reciprocity Theorem

Let us consider two systems of causes and effects acting on an elastic body of volume V bounded by the surface A.

We assign to the first group of causes the body forces X_i, body couples Y_i, heat sources Q, loadings p_i and m_i on the surface and the heating of this surface (the prescribed temperature or the heat flux on the surface A). The effects are: the components of the displacement vector $\boldsymbol{u}$, of the rotation vector $\boldsymbol{\omega}$ and temperature θ.

The second set of causes and effects will be distinguished from the first one by a prime. In the sequel we assume the initial conditions to be homogeneous.

Let us apply the one-sided Laplace transformation to the constitutive equations. We obtain the following relations

$$\bar{\sigma}_{ji} = (\mu + \alpha)\bar{\gamma}_{ji} + (\mu - \alpha)\bar{\gamma}_{ij} + (\lambda\bar{\gamma}_{kk} - \nu\bar{\theta})\delta_{ij}, \tag{6.1}$$

$$\bar{\mu}_{ji} = (\gamma + \varepsilon)\bar{\varkappa}_{ji} + (\gamma - \varepsilon)\bar{\varkappa}_{ij} + \beta\bar{\varkappa}_{kk}\delta_{ij}, \tag{6.2}$$

where

$$\bar{\sigma}_{ji}(\boldsymbol{x}, p) = L[\sigma_{ji}(\boldsymbol{x}, t)] = \int_0^\infty \sigma_{ji}(\boldsymbol{x}, t)e^{-pt}dt, \quad \text{e. c. t.}$$

and similar relations for $\bar{\sigma}'_{ji}$ and $\bar{\mu}'_{ji}$. It may easily be seen that the following identity holds

$$\bar{\sigma}_{ji}\bar{\gamma}'_{ji} + \bar{\mu}_{ji}\bar{\varkappa}'_{ji} - \bar{\sigma}'_{ji}\bar{\gamma}_{ji} - \bar{\mu}'_{ji}\bar{\varkappa}_{ji} = \nu(\bar{\theta}'\bar{\gamma}_{kk} - \bar{\theta}\bar{\gamma}'_{kk}). \tag{6.3}$$

Integrating relation (6.3) over the volume V we get

$$\int_V (\bar{\sigma}_{ji}\bar{\gamma}'_{ji} - \bar{\sigma}'_{ji}\bar{\gamma}_{ji} + \bar{\mu}_{ji}\bar{\varkappa}'_{ji} - \bar{\mu}'_{ji}\bar{\varkappa}_{ji})dV = \nu\int_V (\bar{\theta}'\bar{\gamma}_{kk} - \bar{\theta}\bar{\gamma}'_{kk})dV. \tag{6.4}$$

Now let us perform the Laplace transformation on the equations of motion

$$\bar{\sigma}_{ji,j} + \bar{X}_i = p^2\varrho\bar{u}_i, \quad \varepsilon_{ijk}\bar{\sigma}_{jk} + \bar{\mu}_{ji,j} + \bar{Y}_i = p^2 J\bar{\omega}_i, \tag{6.5}$$

as well as on the equations of motion for the state marked with primes. Taking advantage of relations (6.5) we reduce eq. (6.4) to the form

$$\begin{aligned}\int_V (\bar{X}_i\bar{u}'_i + \bar{Y}_i\bar{\omega}'_i)dV + \int_A (\bar{p}_i\bar{u}'_i + \bar{m}_i\bar{\omega}'_i)dA = \\ = \int_V (\bar{X}'_i\bar{u}_i + \bar{Y}'_i\bar{\omega}_i)dV + \int_A (\bar{p}'_i\bar{u}_i + \bar{m}'_i\bar{\omega}_i)dA + \\ + \nu\int_V (\bar{\theta}'\bar{\gamma}_{kk} - \bar{\theta}\bar{\gamma}'_{kk})dV.\end{aligned} \tag{6.6}$$

This is the first part of the reciprocity theorem. The second part is obtained by taking account of the heat conduction equations for both sets.

We apply the Laplace transformation to these equations assuming that the initial conditions are homogeneous

$$\bar{\theta}_{,jj} - \frac{p}{\varkappa}\bar{\theta} - \eta_0 p \bar{\gamma}_{kk} = -\frac{\bar{Q}}{\varkappa},$$
$$\bar{\theta}'_{,jj} - \frac{p}{\varkappa}\bar{\theta}' - \eta_0 p \bar{\gamma}'_{kk} = -\frac{\bar{Q}'}{\varkappa}. \tag{6.7}$$

Multiplying the first of eqs. (6.7) by θ', the second by θ, subtracting one from another, integrating over the region V and making use of Green's transformation theorem, we get

$$p\eta_0 \int_V (\bar{\gamma}_{kk}\bar{\theta}' - \bar{\gamma}'_{kk}\bar{\theta})\,dV + \frac{1}{\varkappa}\int_V (\bar{Q}'\bar{\theta} - \bar{Q}\bar{\theta}')\,dV - \int_A (\bar{\theta}'\bar{\theta}_{,n} - \bar{\theta}'_{,n}\bar{\theta})\,dA = 0. \tag{6.8}$$

Eliminating the common term from (6.6) and (6.8), we obtain the final form of the theorem on reciprocity

$$\frac{\eta_0\varkappa p}{\nu}\Big[\int_V (\bar{X}_i\bar{u}'_i - \bar{X}'_i\bar{u}_i + \bar{Y}_i\bar{\omega}'_i - \bar{Y}'_i\bar{\omega}_i)\,dV + \int_A (\bar{p}_i\bar{u}'_i - \bar{p}'_i\bar{u}_i + \bar{m}_i\bar{\omega}'_i - \bar{m}'_i\bar{\omega}_i)\,dA\Big] + \varkappa\int_A (\bar{\theta}\bar{\theta}'_{,n} - \bar{\theta}'\bar{\theta}_{,n})\,dA + \int_V (\bar{Q}'\bar{\theta} - \bar{Q}\bar{\theta}')\,dV = 0. \tag{6.9}$$

It is obvious that in this relation all causes and effects appear. Applying the inverse Laplace transformation on eq. (6.9) we have

$$\begin{aligned}
&\frac{\eta_0 x}{\nu}\Big\{\int_V dV(x)\int_0^t\Big[X_i(x,\tau)\frac{\partial u'_i(x,t-\tau)}{\partial\tau} - X'_i(x,t-\tau)\frac{\partial u_i(x,\tau)}{\partial\tau} + \\
&+ Y_i(x,\tau)\frac{\partial\omega'_i(x,t-\tau)}{\partial\tau} - Y'_i(x,t-\tau)\frac{\partial\omega_i(x,\tau)}{\partial\tau}\Big]d\tau + \\
&+ \int_A dA(x)\int_0^t\Big[p_i(x,\tau)\frac{\partial u'_i(x,t-\tau)}{\partial\tau} - p'_i(x,t-\tau)\frac{\partial u_i(x,\tau)}{\partial\tau} + \\
&+ m_i(x,\tau)\frac{\partial\omega'_i(x,t-\tau)}{\partial\tau} - m'_i(x,t-\tau)\frac{\partial\omega_i(x,\tau)}{\partial\tau}\Big]d\tau\Big\} + \\
&+ \varkappa\int_A dA(x)\int_0^t[\theta(x,\tau)\theta'_{,n}(x,t-\tau) - \theta'(x,t-\tau)\theta_{,n}(x,\tau)]d\tau + \\
&+ \int_V dV(x)\int_0^t[\theta(x,\tau)Q'(x,t-\tau) - \theta'(x,t-\tau)Q(x,\tau)]d\tau = 0.
\end{aligned} \tag{6.10}$$

With $Y_i = Y_i' = 0$, $m_i = m_i' = 0$ the theorem on reciprocity (6.10) goes over into the theorem on reciprocity for an elastic medium without couple-stresses, given by CAZIMIR-IONESCU [16]. For static loads and for stationary heat flow we get the system of equations

$$\int_V (X_i u_i' - X_i' u_i + Y_i \omega_i' - Y_i' \omega_i) dV + \int_A (p_i u_i' - p_i' u_i + \\ + m_i \omega_i' - m_i' \omega_i) dA + \nu \int_V (\theta \gamma_{kk}' - \theta' \gamma_{kk}) dV = 0, \tag{6.11}$$

$$\int_V (Q'\theta - Q\theta') dV + \varkappa \int_A (\theta'_{,n}\theta - \theta_{,n}\theta') dA = 0. \tag{6.12}$$

In the reciprocity equation (6.11) tempeatures θ and θ' are treated as known functions, obtained from the solution of the heat conductivity equations

$$\theta_{,jj} = -\frac{Q}{\varkappa}, \qquad \theta'_{,jj} = -\frac{Q'}{\varkappa}. \tag{6.13}$$

Equation (6.12) may be treated as a theorem of reciprocity for the problem of heat conduction.

7. Conclusions from the Theorem of Reciprocity

Let us consider first an infinite medium. Let a concentrated and instantaneous force $X_i = \delta(x - \xi)\delta(t)\delta_{ik}$, directed along the axis x_k, be acting at the point ξ of the medium. Denote by $U_i^{(k)}(x, \xi, t)$ the displacement caused by this force. Furthermore, let a concentrated and instantaneous force $X_i' = \delta(x - \eta)\delta(t)\delta_{ij}$, directed along the axis x_j, be acting at the point η. Denote the displacement caused by this force by $U_i^{(j)}(x, \eta, t)$. From the theorem of reciprocity (6.10), formulated for an infinite region, we have

$$\int_V dV(x) \int_0^t d\tau \Big[[\delta(x - \xi)\delta(\tau)\delta_{ik} \frac{\partial U_i^{(j)}(x, \eta, t - \tau)}{\partial \tau} - \\ - \delta(x - \eta)\delta(t - \tau)\delta_{ij} \frac{\partial U_i^{(k)}(x, \xi, \tau)}{\partial \tau} \Big] = 0, \tag{7.1}$$

and hence

$$\dot{U}_k^{(j)}(\xi, \eta, t) = \dot{U}_j^{(k)}(\eta, \xi, t).$$

After integration with respect to time, we finally get

$$U_k^{(j)}(\xi, \eta, t) = U_j^{(k)}(\eta, \xi, t). \tag{7.2}$$

Let a concentrated and instantaneous force $X_i = \delta(x - \xi)\delta(t)\delta_{ik}$ act at the point ξ of an infinite medium, and a concentrated and instantaneous source of heat $Q' = \delta(x - \eta)\delta(t)$ act at the point η. Denote by $\Theta^{(k)}(x, \xi, t)$ the temperature caused by the action of force X_i, and by $U_i(x, \eta, t)$ the displacement caused by the action of the source Q'. From eq. (6.10) we obtain the following relation

$$\int_V dV(x)\int_0^t d\tau\Big[\delta(x-\eta)\delta(t-\tau)\frac{\partial\Theta(x, \xi, \tau)}{\partial\tau} + \\ + \frac{\eta_0\varkappa}{\nu}\delta(x-\xi)\delta(\tau)\delta_{ik}\frac{\partial U_i(x, \eta, t-\tau)}{\partial\tau}\Big] = 0, \tag{7.3}$$

wherefrom

$$\Theta^{(k)}(\eta, \xi, t) = -\frac{\eta_0\varkappa}{\nu}\frac{\partial U_k(\xi, \eta, t)}{\partial t}. \tag{7.4}$$

Let a concentrated and instantaneous force $X_i = \delta(x - \xi)\delta(t)\delta_{ik}$ act at the point ξ of infinite medium, and the concentrated and instantaneous body couple $Y_i' = \delta(x - \eta)\delta(t)\delta_{ij}$ act at the point η. Denote by $\Omega_i^{(k)}(x, \xi, t)$ the rotation vector caused by the action of force X_i, and by $V_i^{(j)}(x, \eta, t)$ the displacement caused by the body couple Y_i'. From eq. (6.10) we get

$$V_k^{(j)}(\xi, \eta, \mathrm{t}) = \Omega^{(k)}(\eta, \xi, t). \tag{7.5}$$

Finally, let a body couple $Y_i = \delta(x - \xi)\delta(t)\delta_{ik}$ act at the point ξ, and a source of heat $Q' = \delta(x - \eta)\delta(t)$ act at the point η. Denote the temperature caused by the action of body couple by $\vartheta^{(k)}(x, \xi, t)$, and the rotation vector caused by the action of the source Q' by $\Omega_i(x, \eta, t)$. From the theorem of reciprocity (6.10) we obtain the following relation

$$\vartheta^{(k)}(\eta, \xi, t) = -\frac{\eta_0\varkappa}{\nu}\frac{\partial\Omega_k(\xi, \eta, t)}{\partial t}. \tag{7.6}$$

It can be shown that the relations (7.2), (7.4) — (7.6) hold for a finite body at homogeneous boundary conditions

Let us consider a finite body V and assume that the causes which set the medium in motion are defined by the boundary conditions. We wish to find expressions for the displacements u_i, rotation vectors ω_i and temperature θ at an internal point $x \varepsilon V$ by means of integrals on the surface A bounding the region V. These functions should satisfy the equations of motion, the extended equation of heat conduction and the boundary conditions. When deriving the formulae for the functions $u_i(x, t)$, $\omega_i(x, t)$, $\theta(x, t)$ we shall use the theorem of reciprocity (6.10). Assume, first, that quantities marked with primes refer to displacements $u'_i = U_i^{(k)}(x, \xi, t)$, rotation vector $\omega'_i = \Omega_i^{(k)}(x, \xi, t)$ and temperature $\theta' = \Theta^{(k)}(x, \xi, t)$, caused in an infinite medium by a concentrated and instanteneous force $X'_i = \delta(x - \xi)\,\delta(t)\,\delta_{ik}$, applied at the point ξ and directed along the axis x_k. Assuming absence of body forces ($X_i = 0$), body couples ($Y_i = Y'_i = 0$) and heat sources ($Q = Q' = 0$) we obtain from (6.10) the following expression

$$\begin{aligned} \dot{u}_k(x, t) = \int_A dA\,(\xi) \int_0^t d\tau \Big\{ p_i(\xi, \tau)\, \frac{\partial U_i^{(k)}(\xi, x, t-\tau)}{\partial \tau} - \\ - p_i^{(k)}(\xi, x, t-\tau)\,\frac{\partial u_i(\xi, \tau)}{\partial \tau} + m_i(\xi, \tau)\,\frac{\partial \Omega_i^{(k)}(\xi, x, t-\tau)}{\partial \tau} - \\ - m_i^{(k)}(\xi, x, t-\tau)\,\frac{\partial \omega_i(\xi, \tau)}{\partial \tau} + \frac{\nu}{\eta_0}\,[\theta(\xi, \tau)\,\Theta_{,n}^{(k)}(\xi, x, t-\tau) - \\ - \Theta^{(k)}(\xi, x, t-\tau)\,\theta_{,n}(\xi, \tau)]\Big\}, \quad x \varepsilon V,\ \xi \varepsilon A. \end{aligned} \tag{7.7}$$

Here we have introduced the notations

$$p_i^{(k)} = \sigma_{ji}^{(k)}(x, \xi, t)\, n_j(x), \qquad m_i^{(k)}(x, t) = \mu_{ji}^{(k)}(x, \xi, t)\, n_j(x),$$

where $\sigma_{ji}^{(k)}$ we understand stresses and by $\mu_{ji}^{(k)}$ couple-stresses caused by a concentrated force $X'_i = \delta(x - \xi)\,\delta(t)\,\delta_{ik}$.

Integration operations in the surface integrals are to be carried out with respect to the variable ξ. Formula (7.7) gives us the relation between the function $\dot{u}_k(x, t)$, $x \varepsilon V$, $t > 0$ and functions u_i, p_i, m_i, ω_i, θ, $\theta_{,n}$ on the surface A.

Now let us assume, in the system with "primes", a concentrated and instantaneous body couple $Y'_i = \delta(x - \xi)\,\delta(t)\,\delta_{ik}$ acting along the axis x_k. In an infinite medium the body couple will cause the displacement $u'_i = V_i^{(k)}(x, \xi, t)$, vector $\omega'_i = \Lambda_i^{(k)}(x, \xi, t)$, and temperature $\theta' = \vartheta^{(k)}(x, \xi, t)$. From the theorem of reciprocity (6.10), at $X_i = X'_i = 0$, $Y_i = 0$, $Q = Q' = 0$ we obtain the following formula:

$$\begin{aligned} \dot{\omega}_k(x, t) = \int_A dA\,(\xi) \int_0^t d\tau \Big\{ p_i(\xi, \tau)\, \frac{\partial V_i^{(k)}(\xi, x, t-\tau)}{\partial \tau} - \\ - \hat{p}_i^{(k)}(\xi, x, t-\tau)\,\frac{\partial u_i(\xi, \tau)}{\partial \tau} + m_i(\xi, \tau)\,\frac{\partial \Lambda_i^{(k)}(\xi, x, t-\tau)}{\partial \tau} - \\ - \hat{m}_i^{(k)}(\xi, x, t-\tau)\,\frac{\partial \omega_i(\xi, \tau)}{\partial \tau} + \frac{\nu}{\eta_0}\,[\theta(\xi, \tau)\,\vartheta_{,n}^{(k)}(\xi, x, t-\tau) - \\ - \vartheta^{(k)}(\xi, x, t-\tau)\,\theta_{,n}(\xi, \tau)]\Big\}, \quad x \varepsilon V,\ \xi \varepsilon A \end{aligned} \tag{7.8}$$

Here

$$\hat{p}_i^{(k)}(x, t) = \sigma_{ji}^{(k)}(x, \xi, t)\, n_j(x), \quad \hat{m}_i^{(k)} = \mu_{ji}^{(k)}(x, \xi, t)\, n_j(x).$$

We denote $\hat{\sigma}_{ji}^{(k)}$ and $\hat{\mu}_{ji}^{(k)}$ the force-stress tensor and the couple-stress tensor, respectively. Also the function $\dot{\omega}_k(x, t)$, $x \varepsilon V$, $t > 0$, is expressed here by the functions u_i, p_i, ω_i, m_i, θ, $\theta_{,n}$ on the surface A. Now let the system with "primes" in an infinite medium be limited to the action of a concentrated and instantaneous heat source $Q' = \delta(x - \xi)\,\delta(t)$ causing displacements $u_i' = U_i(x, \xi, t)$, rotation vector $\omega_i' = \Omega_i(x, \xi, t)$ and temperature $\theta' = \Theta(x, \xi, t)$.

From (6.10), assuming that $X_i = X_i' = 0$, $Y_i = Y_i' = 0$, $Q = 0$ we get the temperature at the point

$$\begin{aligned}\theta(x, t) = \varkappa \int_A dA(\xi) \int_0^t d\tau \Big\{ &\theta_{,n}(\xi, \tau)\, \Theta(\xi, x, t - \tau) - \\ &- \theta(\xi, \tau)\, \Theta_{,n}(\xi, x, t - \tau) - \frac{\eta_0}{\nu} \Big[p_i(\xi, \tau) \frac{\partial U_i(\xi, x, t - \tau)}{\partial \tau} - \\ &- p_i^*(\xi, x, t - \tau) \frac{\partial u_i(\xi, \tau)}{\partial \tau} + m_i(\xi, \tau) \frac{\partial \Omega_i(\xi, x, t - \tau)}{\partial \tau} - \\ &- m_i^*(\xi, x, t - \tau), \frac{\partial \omega_i(\xi, \tau)}{\partial \tau} \Big] \Big\}, \quad x \varepsilon V, \quad \xi \varepsilon A. \end{aligned} \tag{7.9}$$

Here

$$p_i^*(x, t) = \sigma_{ji}^*(x, \xi, t)\, n_j(x), \qquad m_i^*(x, t) = \mu_{ji}^*(x, \xi, t)\, n_j(x).$$

We denote by σ_{ji}^* and μ_{ji}^* the force-stress tensor and couple-stress tensor caused by the action of an instantaneous and concentrated heat source Q'.

Formulae (7.7)—(7.9) may be treated as an extension of Somigliana's formulae [17], to the problems of thermoelasticity. Some simplifications of these formulae can be obtained by taking into account the reciprocity relations (7.4)—(7.6).

If Green's functions $U_i^{(k)}$, $\Omega_i^{(k)}$, $\Theta^{(k)}$ etc. are selected in such a way as to satisfy on the surface A the homogeneous boundary conditions for displacements, rotation vector and temperature, then the formulae (7.7)—(7.9) will yield the solution of the first boundary problem, if functions u_i, ω_i and θ are given at the boundary.

Similarly, if we select Green's functions $U_i^{(k)}$, $\Omega_i^{(k)}$, $\Theta^{(k)}$ etc., in such a way that the boundary is free from loads and temperature, then the formulae (7.7)—(7.9) will yield the solution of the second boundary value problem, when on A loads p_i, m_i and temperature θ are given.

Let us still consider the stationary problems. Let the body contained within the region V and bounded by the surface A be subjected to heating. Let on the part A_u of the surface A, equal to zero, appear displacements u_i and vector of rotation ω_i, and on the part A_σ of the surface A, equal to zero, appear functions p_i and m_i. Moreover, let $X_i = 0$.

For determining the displacement $u_i(x)$, $x \varepsilon V$ consider a body of the same shape and the same boundary conditions. Let in this body $\theta' = 0$ and let a concentrated force $X_i' = \delta(x - \xi)\, \delta_{ik}$ be acting at the point ξ which is, consequently, directed along the axis x_k. This force will cause displacements $U_i^{(k)}(x, \xi)$ assuming that the functions $U_i^{(k)}(x, \xi)$ are selected such as to satisfy homogeneous boundary conditions on A_σ and A_u.

Making use of formula (6.11), we obtain

$$u_k(x) = \nu \int_V \theta(\xi)\, U_{i,i}^{(k)}(\xi, x)\, d\,V(\xi), \quad x \varepsilon V, \quad k = 1, 2, 3. \qquad (7.10)$$

Here $U_{i,i}^{(k)}(\xi, x)$ should be treated as a dilatation caused at the point ξ by a concentrated force X_i applied at the point x. The formula (7.10) may be treated as a generalization of MAYSEL's formula [18], for the problem of thermoelasticity with couple-stresses.

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Discussion

GRIOLI: How do you determine the coefficient J in the kinetic energy of rotations?

NOWACKI: J — ist eine spezielle dynamische Charakteristik des Mediums. Sie ist gleich dem Trägheitsmoment einer Partikel in bezug auf eine beliebige, durch ihren Schwerpunkt gehende Achse, multipliziert mit der Anzahl der Partikel in einer Volumeneinheit.

SAVIN: We have in this theory of elasticity two cases. For each there is a number of corresponding boundary conditions. In the first case we have five, in the second case we have six. I did not hear about boundary conditions.

NOWACKI: In dieser Theorie erfordert das Differentialgleichungssystem sechs Randbedingungen. Man kann an der Oberfläche entweder drei Komponenten des Verschiebungsvektors $\boldsymbol{u}$ und drei Komponenten des Rotationsvektors ω oder drei Komponenten des Spannungsvektors und drei Komponenten des Momentenspannungsvektors vorschreiben. In der Cosseratschen Theorie haben wir, wegen der Einschränkung $\omega = {}^1/_2 \operatorname{curl} \boldsymbol{u}$, nur 5 Randbedingungen zu befriedigen.

KALISKI: The medium has six degrees of freedom. The six degrees of freedom are important for very short waves or very high frequencies. In this case, however, I think that taking the viscosity of the body into account may also be important.

NOWACKI: Man kann analog, wie in der symmetrischen Elastizitätstheorie, auch hier eine asymmetrische Viskoelastizität aufbauen.

KALISKI: Only for very short waves the terms may be important.