BULLETIN
DE
L'ACADÉMIE POLONAISE DES SCIENCES

SÉRIE DES SCIENCES TECHNIQUES

VOLUME XIX
NUMÉRO 7-8

VARSOVIE 1971

BULLETIN DE L'ACADÉMIE
11. POLONAISE DES SCIENCES
Série des sciences techniques
Volume XIX, No. 7-8 — 1971

APPLIED MECHANICS
(E L A S T I C I T Y)

Axial-symmetric Problems in Micropolar Elasticity

by

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Presented on May 3, 1971

Summary. The solution of axial-symmetric problem of micropolar theory of elasticity based on the generalized Love's functions has been given by the present author in [1]. In this note we advance a markedly simpler solution of this problem, owing to the introduction of elastic potentials Φ, Ψ, ϑ. Differential equation is derived for the potentials as well as relations of compatibility connecting the potentials Φ and ϑ. The application of the elastic potential is exemplified by the elastic half-space.

In the final part of the note the thermo-elastic axial-symmetric problems are discussed. The application of potentials to the thermo-elastic problem is explained recurring to the example of elastic half-space heated on its bounding surface.

1. Introduction

The solution of basic equations (in displacements and rotations) of micropolar elasticity has been presented in [1]. The way to the solution consisted in introducing generalized Love's functions. Hereafter we advance the solution of these equations by using the elastic potentials.

The state of deformation of a micropolar body is generally described by a system of six equations. We write them here in the vector form [2—5].

$$(1.1)\qquad \begin{cases} (\mu+a)\,\nabla^2\,\mathbf{u}+(\lambda+\mu-a)\,\operatorname{grad}\operatorname{div}\mathbf{u}+2a\operatorname{rot}\boldsymbol{\varphi}+\mathbf{X}=0\,, \\ (\gamma+\varepsilon)\,\nabla^2\boldsymbol{\varphi}-4a\boldsymbol{\varphi}+(\beta+\gamma-\varepsilon)\operatorname{grad}\operatorname{div}\boldsymbol{\varphi}+2a\operatorname{rot}\mathbf{u}+\mathbf{Y}=0\,. \end{cases}$$

Here $\mathbf{u}$ denotes the displacement vector, $\boldsymbol{\varphi}$ stands for the rotation vector. $\mathbf{X}$ is the vector of body forces and $\mathbf{Y}$ denotes the vector of body moments. The quantities a, β, γ, ε, μ, λ are material constants.

Passing from Eqs. (1.1) to the system of cylindrical coordinates and assuming that the problem is axial-symmetric, we obtain from (1.1) two self-dependent systems of equations. In the first of them the following components of the vectors $\mathbf{u}, \boldsymbol{\varphi}, \mathbf{X}, \mathbf{Y}$ appear:

$$(1.2)\qquad \mathbf{u}\equiv(u_r,0,u_z)\,,\quad \boldsymbol{\varphi}\equiv(0,\varphi_\theta,0)\,,\quad \mathbf{X}\equiv(X_r,0,X_z)\,,\quad \mathbf{Y}\equiv(0,Y_\theta,0)\,.$$

In this case Eqs. (1.1) transform into equations

$$(1.3)\qquad \begin{cases} (\mu+a)\left(\nabla^2-\dfrac{1}{r^2}\right)u_r+(\lambda+\mu-a)\dfrac{\partial e}{\partial r}-2a\dfrac{\partial \varphi_\theta}{\partial z}+X_r=0, \\ (\mu+a)\nabla^2 u_z+(\lambda+\mu-a)\dfrac{\partial e}{\partial z}+2a\dfrac{1}{r}\dfrac{\partial}{\partial r}(r\varphi_\theta)+X_z=0, \\ (\gamma+\varepsilon)\left(\nabla^2-\dfrac{1}{r^2}\right)\varphi_\theta-4a\varphi_\theta+2a\left(\dfrac{\partial u_r}{\partial z}-\dfrac{\partial u_z}{\partial r}\right)+Y_\theta=0, \end{cases}$$

where

$$e=\frac{1}{r}\frac{\partial}{\partial r}(ru_r)+\frac{\partial u_z}{\partial z},\qquad \nabla^2=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial z^2}.$$

In the second self-dependent system of equations

$$(1.4)\qquad \begin{cases} (\gamma+\varepsilon)\left(\nabla^2-\dfrac{1}{r^2}\right)\varphi_r-4a\varphi_r+(\beta+\gamma-\varepsilon)\dfrac{\partial \kappa}{\partial r}-2a\dfrac{\partial u_\theta}{\partial r}+Y_r=0, \\ (\gamma+\varepsilon)\nabla^2\varphi_z-4a\varphi_z+(\beta+\gamma-\varepsilon)\dfrac{\partial \kappa}{\partial z}+\dfrac{2a}{r}\dfrac{\partial}{\partial r}(ru_\theta)+Y_z=0, \\ (\mu+a)\left(\nabla^2-\dfrac{1}{r^2}\right)u_\theta+2a\left(\dfrac{\partial \varphi_\theta}{\partial z}-\dfrac{\partial \varphi_z}{\partial r}\right)+X_\theta=0, \end{cases}$$

where $\kappa=\dfrac{1}{r}\dfrac{\partial}{\partial r}(r\varphi_r)+\dfrac{\partial \varphi_z}{\partial_2}$, the following vectors appear:

$$(1.5)\qquad \mathbf{u}\equiv(0,u_\theta,0),\quad \boldsymbol{\varphi}\equiv(\varphi_r,0,\varphi_z),\quad \mathbf{X}\equiv(0,X_\theta,0),\quad \mathbf{Y}\equiv(Y_r,0,Y_z).$$

In the sequel we shall be concerned with the system of equations (1.3).

2. Elastic potentials

Let us consider the homogeneous system of equations of equilibrium (1.3) and assume the solution of this system in the form:

$$(2.1)\qquad u_r=\frac{\partial \Phi}{\partial r}-\frac{\partial \Psi}{\partial z},\qquad u_z=\frac{\partial \Phi}{\partial z}+\frac{\partial \Psi}{\partial r}+\frac{\Psi}{r}.$$

In this way we obtain the system of equations

$$(2.2)\qquad \begin{cases} (\lambda+2\mu)\dfrac{\partial}{\partial r}\nabla^2\Phi-\dfrac{\partial}{\partial z}\left[(\mu+a)\left(\nabla^2-\dfrac{1}{r^2}\right)\Psi+2a\varphi_\theta\right]=0, \\ (\lambda+2\mu)\dfrac{\partial}{\partial z}\nabla^2\Phi-\left(\dfrac{\partial}{\partial r}+\dfrac{1}{r}\right)[(\mu+a)\nabla^2\Psi+2a\varphi_\theta]=0, \\ (\gamma+\varepsilon)\left(\nabla^2-\dfrac{1}{r^2}\right)\varphi_\theta-2a\left(\nabla^2-\dfrac{1}{r^2}\right)\Psi-4a\varphi_\theta=0. \end{cases}$$

Here we made use of the identity

$$\left(\nabla^2-\frac{1}{r^2}\right)\frac{\partial\Phi}{\partial r}=\frac{\partial}{\partial r}\nabla^2\Phi. \tag{2.3}$$

Let us introduce new functions Ψ and ϑ defined as below,

$$\Psi=-\frac{\partial\psi}{\partial r}, \qquad \varphi_\theta=\frac{\partial\vartheta}{\partial r}. \tag{2.4}$$

Substituting these functions into (2.2) and integrating with respect to r, we obtain

$$\begin{cases} (\lambda+2\mu)\nabla^2\Phi+\dfrac{\partial}{\partial z}[(\mu+a)\nabla^2\psi-2a\vartheta]=0, \\ (\lambda+2\mu)\dfrac{\partial}{\partial z}\nabla^2\Phi-\left(\nabla^2-\dfrac{\partial^2}{\partial z^2}\right)[(u+a)\nabla^2\psi-2a\vartheta]=0, \\ [(\gamma+\varepsilon)\nabla^2-4a]\vartheta+2a\nabla^2\psi=0. \end{cases} \tag{2.5}$$

Eliminating the term $\nabla^2\psi$ from the first two equations of the (2.5) group we obtain — taking profit of Eq. $(2.5)_3$ — the following equations of compatibility

$$\begin{cases} \nabla^2\Phi-\dfrac{2\mu}{\lambda+2\mu}\dfrac{\partial}{\partial z}D\vartheta=0, \\ \dfrac{\partial}{\partial z}\nabla^2\Phi+\dfrac{2\mu}{\lambda+2\mu}\left(\nabla^2-\dfrac{\partial^2}{\partial z^2}\right)D\vartheta=0, \end{cases} \tag{2.6}$$

where $D=l^2\nabla^2-1, \quad l^2=\dfrac{(\mu+a)(\gamma+\varepsilon)}{4\mu a}$.

From the system of equations (2.6) we can derive the following differential equations wherefrom we can determine the potentials Φ and ϑ

$$\nabla^2\nabla^2\Phi=0, \quad \nabla^2 D\vartheta=0. \tag{2.7}$$

The above equations should be supplemented by the relation $(2.5)_3$

$$\nabla^2\psi=-\frac{1}{2a}[(\gamma+\varepsilon)\nabla^2-4a]\vartheta. \tag{2.8}$$

The route to the solution of the axial-symmetric problem is the following: We construct the general solution of Eqs. (2.7) and recurring to the relation (2.8), we determine the function ψ.

In the solutions Φ and ϑ there appear four integration constants. To determine them we have three boundary conditions and the relations of compatibility (2.6). The knowledge of the functions Φ, ϑ and ψ permits to determine the displacements u_r, u_2 and the rotation φ_θ.

The state of stress is described by the following matrices

$$\boldsymbol{\sigma}=\begin{vmatrix}\sigma_{rr} & 0 & \sigma_{rz}\\ 0 & \sigma_{\theta\theta} & 0\\ \sigma_{zr} & 0 & \sigma_{zz}\end{vmatrix},\qquad \boldsymbol{\mu}=\begin{vmatrix}0 & \mu_{r\theta} & 0\\ \mu_{\theta r} & 0 & \mu_{\theta z}\\ 0 & \mu_{z\theta} & 0\end{vmatrix}, \tag{2.9}$$

bearing in mind that [1]:

$$\begin{cases}\sigma_{rr}=2\mu\dfrac{\partial u_r}{\partial r}+\lambda e, \qquad \sigma_{\theta\theta}=2\mu\dfrac{u_r}{r}+\lambda e, \qquad \sigma_{zz}=2\mu\dfrac{\partial u_z}{\partial z}+\lambda e,\\ \sigma_{rz}=\mu\left(\dfrac{\partial u_z}{\partial r}+\dfrac{\partial u_r}{\partial z}\right)-\alpha\left(\dfrac{\partial u_r}{\partial z}-\dfrac{\partial u_z}{\partial r}\right)+2\alpha\varphi_\theta,\\ \sigma_{zr}=\mu\left(\dfrac{\partial u_z}{\partial r}+\dfrac{\partial u_r}{\partial z}\right)+\alpha\left(\dfrac{\partial u_r}{\partial z}-\dfrac{\partial u_z}{\partial r}\right)-2\alpha\varphi_\theta,\end{cases} \tag{2.10}$$

and

$$\begin{cases}\mu_{r\theta}=\gamma\left(\dfrac{\partial\varphi_\theta}{\partial r}-\dfrac{\varphi_\theta}{r}\right)+\varepsilon\left(\dfrac{\partial\varphi_\theta}{\partial r}+\dfrac{\varphi_\theta}{r}\right),\\ \mu_{\theta r}=\gamma\left(\dfrac{\partial\varphi_\theta}{\partial r}-\dfrac{\varphi_\theta}{r}\right)-\varepsilon\left(\dfrac{\partial\varphi_\theta}{\partial r}+\dfrac{\varphi_\theta}{r}\right),\\ \mu_{z\theta}=(\gamma+\varepsilon)\dfrac{\partial\varphi_\theta}{\partial z}, \qquad \mu_{\theta z}=(\gamma-\varepsilon)\dfrac{\partial\varphi_\theta}{\partial z}.\end{cases} \tag{2.11}$$

The above components of the state of stress should be now expressed in potentials Φ, Ψ, ϑ. We obtain successively

$$\begin{cases}\sigma_{rr}=2\mu\dfrac{\partial^2}{\partial r^2}\left(\Phi+\dfrac{\partial\psi}{\partial z}\right)+\lambda\nabla^2\,\Phi, \qquad \sigma_{\theta\theta}=2\mu\dfrac{1}{r}\,\dfrac{\partial}{\partial r}\left(\Phi+\dfrac{\partial\psi}{\partial z}\right)+\lambda\nabla^2\,\Phi,\\ \sigma_{zz}=2\mu\dfrac{\partial}{\partial z}\left[\dfrac{\partial\Phi}{\partial z}-\left(\nabla^2-\dfrac{\partial^2}{\partial z^2}\right)\psi\right]+\lambda\nabla^2\,\Phi,\\ \sigma_{zr}=\dfrac{\partial}{\partial r}\left\{\mu\left[2\dfrac{\partial\Phi}{\partial z}-\left(\nabla^2-2\dfrac{\partial^2}{\partial z^2}\right)\psi\right]+\alpha\nabla^2\,\psi-2\alpha\vartheta\right\},\\ \sigma_{rz}=\dfrac{\partial}{\partial r}\left\{\mu\left[2\dfrac{\partial\Phi}{\partial z}-\left(\nabla^2-\dfrac{\partial^2}{\partial z^2}\right)\psi\right]-\alpha\nabla^2\,\psi+2\alpha\vartheta\right\},\end{cases} \tag{2.12}$$

and

$$\begin{cases}\mu_{r\theta}=(\gamma+\varepsilon)\dfrac{\partial^2\vartheta}{\partial r^2}-(\gamma-\varepsilon)\dfrac{1}{r}\,\dfrac{\partial\vartheta}{\partial r},\\ \mu_{\theta r}=(\gamma-\varepsilon)\dfrac{\partial^2\vartheta}{\partial r^2}-(\gamma+\varepsilon)\dfrac{1}{r}\,\dfrac{\partial\vartheta}{\partial r},\\ \mu_{z\theta}=(\gamma+\varepsilon)\dfrac{\partial^2\vartheta}{\partial r\partial z}, \qquad \mu_{z\theta}=(\gamma-\varepsilon)\dfrac{\partial^2\,\vartheta}{\partial r\,\partial z},\end{cases} \tag{2.13}$$

taking into account that

$$\sigma_{rr}+\sigma_{\theta\theta}+\sigma_{zz}=(3\lambda+2\mu)\,\nabla^2\,\Phi. \tag{2.14}$$

The way to the solution of the axial-symmetric problem of micropolar elasticity will be shown on a simple example.

Let us suppose that in the plane $z=0$ of the elastic half-space $z\geqslant 0$ the axial-symmetric loading $p(r)$ is acting along the axis z. The boundary conditions take now the form

$$\sigma_{zz}(r,0)=-p(r),\quad \sigma_{zr}(r,0)=0,\quad \mu_{z\theta}(r,0)=0. \tag{2.15}$$

Now we require the displacements u_r, u_z and the rotation φ_θ to vanish for $|r^2+z^2|\to\infty$.

To solve the Eqs. (2.7) we introduce the Hankel's transformation

$$\begin{aligned}\bar{f}(\zeta,z)&=\int_0^\infty f(r,z)\,rJ_0(\zeta r)\,dr,\\ f(r,z)&=\int_0^\infty \bar{f}(\zeta,z)\,\zeta J_0(\zeta r)\,dr.\end{aligned} \tag{2.16}$$

We assume the solutions of the transformed Eqs. (2.7)

$$\begin{aligned}&(\partial_z^2-\zeta^2)^2\,\overline{\Phi}=0,\quad (\partial_z^2-\zeta^2)(\partial_z^2-\eta^2)\,\bar{\vartheta}=0,\\ &\eta=\left(\zeta^2+\frac{1}{l^2}\right)^{1/2},\end{aligned} \tag{2.17}$$

in the form

$$\overline{\Phi}=(A+B\zeta z)\,e^{-\zeta z},\quad \bar{\vartheta}=Ce^{-\zeta z}+De^{-\eta z}. \tag{2.18}$$

Let us carry out the Hankel's transformation on the compatibility equations (2.6). We get

$$\begin{aligned}(\partial_z^2-\zeta^2)\,\widetilde{\Phi}&=\frac{2\mu l^2}{\lambda+2\mu}\,\partial_z\,(\partial_z^2-\eta^2)\,\bar{\vartheta},\\ \partial_z\,(\partial_z^2-\zeta^2)\,\widetilde{\Phi}&=\frac{2\mu l^2}{\lambda+2\mu}\,\zeta^2\,(\partial_z^2-\eta^2)\,\bar{\vartheta}.\end{aligned} \tag{2.19}$$

From both conditions of (2.19) we obtain the same relation

$$C=-\frac{\lambda+2\mu}{\mu}\,\zeta\,B, \tag{2.20}$$

while from the third transformed boundary condition (2.15) we have

$$D=-\frac{\zeta}{\eta}\,C. \tag{2.21}$$

Carrying out Hankel's transformation on Eq. (2.8), we are able to determine the particular integral from this equation

$$(2.22)\qquad (\partial_z^2-\zeta^2)\,\widetilde{\psi}=-2\nu^2\left(\partial_z^2-\zeta^2-\frac{1}{\nu^2}\right)\widetilde{\psi},\qquad \nu^2=\frac{\gamma+\varepsilon}{4a}.$$

There is

$$(2.23')\qquad \widetilde{\psi}=-\frac{C}{\zeta^2}\left(z\zeta e^{-\zeta z}-2\tau^2\,l^2\,\zeta^2\,\frac{\zeta}{\eta}\,e^{-\eta z}\right),\qquad \tau^2=\frac{\nu^2}{l^2}-1,$$

or

$$(2.3'')\qquad \widetilde{\psi}=\frac{\lambda+\mu}{\mu\zeta}B\left(\frac{\lambda+2\mu}{\lambda+\mu}z\zeta e^{-\zeta z}+2a_0\,\zeta^2\,e^{-\eta z}\frac{\zeta}{\eta}\right),$$

where $a_0=\dfrac{(\gamma+\varepsilon)(\lambda+2\mu)}{4\mu(\lambda+\mu)}$.

Now, taking into account the relations (2.12), we obtain from the boundary conditions $(2.15)_{1,2}$

$$(2.24)\qquad \begin{aligned} A&=-\frac{\lambda+\mu}{\mu}\left(1-2a_0\,\zeta^2\frac{\zeta}{\eta}\right)B,\\ B&=\frac{\tilde{p}(\zeta)}{2\zeta^2\,\Delta_0\,(\lambda+\mu)},\end{aligned}$$

where $\Delta_0=1+2a_0\zeta^2\left(1-\dfrac{\zeta}{\eta}\right)$, $\quad \tilde{p}(\zeta)=\displaystyle\int_0^\infty p(r)\,rJ_0(\zeta r)\,dr$.

The integration constants being known we, determine the displacements u_r, u_z and the rotation φ_θ from the formulae (2.1).

They read as follows:

$$(2.25)\qquad \begin{aligned} u_r&=\frac{1}{2\mu_0}\int_0^\infty\frac{\tilde{p}(\zeta)}{\Delta_0(\zeta)}\left[\left(\zeta z-\frac{\mu}{\lambda+\mu}\right)e^{-\zeta z}+2a_0\,\zeta^2\left(e^{-\eta z}-\frac{\zeta}{\eta}\,e^{-\zeta z}\right)\right]J_1(\zeta r)\,d\zeta,\\ u_z&=\frac{1}{2\mu}\int_0^\infty\frac{\tilde{p}(\zeta)}{\Delta_0(\zeta)}\left[\left(\zeta z+\frac{\lambda+2\mu}{\lambda+\mu}\right)e^{-\zeta z}+2a_0\,\zeta^2\frac{\zeta}{\eta}\left(e^{-\zeta z}-e^{-\zeta z}\right)\right]J_0(\zeta r)\,d\zeta,\\ \varphi_\theta&=\frac{2a_0}{\gamma+\varepsilon}\int_0^\infty\frac{\zeta\tilde{p}(\zeta)}{\Delta_0(\zeta)}\left(e^{-\zeta z}-\frac{\zeta}{\eta}\,e^{-\eta z}\right)J_1(\zeta r)\,d\zeta.\end{aligned}$$

The stresses will be defined from the formulae (2.12) and (2.13). We have

$$(2.26)\qquad \left\{\begin{aligned} \sigma_{zz}&=-\int_0^\infty\frac{\zeta\tilde{p}(\zeta)}{\Delta_0(\zeta)}\left[(1+\zeta z)\,e^{-\zeta z}+2a_0\,\zeta^2\left(e^{-\eta z}-\frac{\zeta}{\eta}\,e^{-\zeta z}\right)\right]J_0(\zeta r)\,d\zeta,\\ \sigma_{zr}&=-\int_0^\infty\frac{\zeta\tilde{p}(\zeta)}{\Delta_0(\zeta)}\left[\zeta z e^{-\zeta z}+2a_0\,\zeta^2\frac{\zeta}{\eta}\,e^{-\eta z}-e^{-\zeta z})\right]J_1(\zeta r)\,d\zeta,\\ \mu_{z\theta}&=-2a_0\int_0^\infty\frac{\zeta^2\,\tilde{p}(\zeta)}{\Delta_0(\zeta)}\,(e^{-\zeta z}-e^{-\eta z})\,J_1(\zeta r)\,d\zeta.\end{aligned}\right.$$

For the Hooke's medium the above formulae simplify markedly. In this particular case we have to put $\zeta=\eta$, $\Delta_0=1$, $a=1$. The formulae (2.25) and (2.26) reduce to the known formulae of the classical theory of elasticity.

3. Problems in thermoelasticity

The equations of micropolar thermoelasticity assume for the axial-symmetric problem the following form [5]:

$$
(3.1)\qquad \begin{cases} (\mu+a)\left(\nabla^2-\dfrac{1}{r^2}\right)u_r+(\lambda+\mu-a)\dfrac{\partial e}{\partial r}-2a\dfrac{\partial \varphi_\theta}{\partial z}=\nu\dfrac{\partial T}{\partial r}, \\ (\mu+a)\nabla^2 u_z+(\lambda+\mu-a)\dfrac{\partial e}{\partial z}+\dfrac{2a}{r}\dfrac{\partial}{\partial r}(r\varphi_\theta)=\nu\dfrac{\partial T}{\partial z}, \\ (\gamma+\varepsilon)\left(\nabla^2-\dfrac{1}{r^2}\right)\varphi_\theta-4a\varphi_\theta+2a\left(\dfrac{\partial u_r}{\partial z}-\dfrac{\partial u_z}{\partial r}\right)=0. \end{cases}
$$

Symbol Γ denotes the increase of the temperature above that of natural state of the body. Next we have $\nu=(3\lambda+2\mu)\,a_t$, where a_t is the coefficient of linear thermal extension.

Let us observe that thermal terms appear in the system of Eqs. (3.1) only if $\mathbf{u}\equiv(u_r, 0, u_z)$ and $\varphi\equiv(0, \varphi_\theta, 0)$.

The quantities u_θ, φ_r, φ_z appearing in the second axial-symmetric problem (1.4) are independent of the temperature Γ.

Proceeding similarly as in Sec. **2**, i.e. introducing the potentials Φ, ψ, ϑ, we arrive at the following system of equations

$$
(3.2)\qquad \nabla^2\nabla^2\Phi-m\nabla^2 T=0, \qquad \nabla^2 D\vartheta=0.
$$

The functions ϑ and ψ are connected with one another by the equation

$$
(3.3)\qquad \nabla^2\psi=-2(\nu^2\nabla^2-1)\,\vartheta, \qquad \nu^2=\frac{\gamma+\varepsilon}{4a}.
$$

These equations should be supplemented by the relations of compatibility

$$
(3.4)\qquad \begin{aligned} &\nabla^2\Phi-mT-\frac{2\mu}{\lambda+2\mu}\frac{\partial}{\partial z}D\vartheta=0, \\ &\frac{\partial}{\partial z}(\nabla^2\Phi-mT)+\frac{2\mu}{\lambda+2\mu}\left(\nabla^2-\frac{\partial^2}{\partial z^2}\right)D\vartheta=0. \end{aligned}
$$

We solve the system of Eqs. (3.2) applying the superposition of the solutions

$$
(3.5)\qquad \Phi=\Phi'+\Phi'', \qquad \vartheta=\vartheta'+\vartheta''.
$$

We choose the functions Φ' and ϑ' so as to have

$$
(3.6)\qquad \nabla^2\Phi'-mT=0, \qquad \vartheta'=0.
$$

Thus Eqs. (3.2) and the relations of compatibility (3.4) are satisfied. The function Φ' becomes the particular integral of Eq. (3.2). Making use of the functions Φ' and ϑ', we determine the stresses. We obtain successively

$$
(3.7)\qquad \begin{cases}
\sigma'_{rr}=2\mu\dfrac{\partial u_r}{\partial r}+\lambda e'-\nu T=2\mu\left(\dfrac{\partial^2 \Phi'}{\partial r^2}-\nabla^2\Phi'\right),\\[2mm]
\sigma'_{\theta\theta}=2\mu\dfrac{u'_r}{r}+\lambda e'-\nu T=2\mu\left(\dfrac{1}{r}\,\dfrac{\partial \Phi'}{\partial r}-\nabla^2\Phi'\right),\\[2mm]
\sigma_{zz}=2\mu\dfrac{\partial u'_z}{\partial z}+\lambda e'-\nu T=2\mu\left(\dfrac{\partial^2 \Phi'}{\partial z^2}-\nabla^2\Phi'\right),\\[2mm]
\sigma'_{zr}=\sigma'_{rz}=2\mu\dfrac{\partial^2\Phi'}{\partial r\,\partial z},\\[2mm]
\mu'_{r\theta}=\mu'_{\theta r}=\mu'_{z\theta}=\mu'_{\theta z}=0.
\end{cases}
$$

The functions Φ'' and ϑ'' will be determined from the homogeneous equations

$$
(3.8)\qquad \nabla^2\nabla^2\Phi''=0,\qquad \nabla^2 D\vartheta''=0.
$$

In the sequel we take into account the relation

$$
(3.9)\qquad \nabla^2\psi''=-2(\nu^2\nabla^2-1)\,\vartheta'',
$$

and the relations of compatibility

$$
(3.10)\quad \nabla^2\Phi''-\frac{2\mu}{\lambda+2\mu}\,\frac{\partial}{\partial z}D\vartheta''=0,\qquad \frac{\partial}{\partial z}\nabla^2\Phi''+\frac{2\mu}{\lambda+2\mu}\left(\nabla^2-\frac{\partial^2}{\partial z^2}\right)D\vartheta''=0.
$$

Let us consider, exempli modo, the problem of the elastic half-space $z\geqslant 0$ heated in the plane $z=0$. Accordingly, we derive the following boundary conditions

$$
(3.11)\qquad T(r,0)=f(r),\quad \sigma_{zz}(r,0)=0,\quad \sigma_{zr}(r,0)=0,\quad \mu_{z\theta}(r,0)=0.
$$

The solution of the equation of heat conductivity $\nabla^2 T$ under the boundary condition $(3.11)_1$ is the function

$$
(3.12)\qquad T(r,z)=\int_0^\infty \tilde f(\zeta)\,e^{-\zeta z}\,\zeta J_0(\zeta r)\,d\zeta,\qquad \tilde f(\zeta)=\int_0^\infty f(r)\,rJ_0(\zeta r)\,dr.
$$

The solution of Eq. (3.6) taking into account — for the sake of comfort of the solution — the boundary condition $\partial\Phi'/\partial z|_{z=0}=0$ is the function

$$
(3.13)\qquad \Phi'=-\frac{m}{2}\int_0^\infty \frac{\tilde f(\zeta)}{\zeta}(1+\zeta z)\,e^{-\zeta z}J_0(\zeta r)\,d\zeta.
$$

We determine the stresses σ'_{ij} from the formulae (3.7). We obtain successively

$$
(3.14)\qquad \begin{cases}
\sigma'_{zz}=-\mu m\displaystyle\int_0^\infty \tilde f(\zeta)\,\zeta\,(1+\zeta z)\,e^{-\zeta z}J_0(\zeta r)\,d\zeta,\\[2mm]
\sigma'_{zr}=-\mu m z\displaystyle\int_0^\infty \zeta^2\tilde f(\zeta)\,e^{-\zeta z}J_1(\zeta r)\,d\zeta,\\[2mm]
\mu'_{z\theta}=0,\qquad \text{etc}.
\end{cases}
$$

We have now to solve Eqs. (3.8) with boundary conditions

$$\sigma'_{zz}+\sigma''_{zz}=0, \quad \sigma'_{zr}+\sigma''_{zr}=0, \quad \mu'_{z\theta}+\mu''_{z\theta} \quad \text{for} \quad z=0. \tag{3.15}$$

Submitting these boundary conditions to the Hankel's transformation, we get

$$\bar{\sigma}''_{zz}=\mu m\tilde{f}, \quad \bar{\sigma}''_{zr}=0, \quad \bar{\mu}''_{z\theta}=0 \quad \text{for} \quad z=0. \tag{3.16}$$

This problem has been solved already in Sec. **2**.

Substituting $\tilde{p}=-\mu m\tilde{f}$ into the formulae (2.25)—(2.27) and adding the stresses σ'_{ij}, μ'_{ij} to the stresses σ''_{ij}, u''_{ij} we obtain the final values of the stresses

$$\sigma_{zz}=-\mu m\int\limits_0^\infty \zeta\tilde{f}(\zeta)\left[\left(1-\frac{1}{\Delta_0}\right)(1+\zeta z)\,e^{-\zeta z}-\right.$$
$$\left.-\frac{2a_0}{\Delta_0}\zeta^2\left(e^{-\zeta z}-\frac{\zeta}{\eta}e^{-\zeta z}\right)\right]J_0(\zeta r)\,d\zeta,$$

$$\sigma_{zr}=-\mu m\int\limits_0^\infty \zeta\tilde{f}(\zeta)\left[\left(1-\frac{1}{\Delta_0}\right)\zeta z e^{-\zeta z}-\right. \tag{3.17}$$
$$\left.-\frac{2a_0}{\Delta_0}\zeta^2\frac{\zeta}{\eta}\left(e^{-\eta z}-\frac{\zeta}{\eta}e^{-\zeta z}\right)\right]J_1(\zeta r)\,d\zeta,$$

$$\mu_{z\theta}=2a_0\mu m\int\limits_0^\infty \frac{\zeta^2\tilde{f}(\zeta)}{\Delta_0(\zeta)}(e^{-\zeta z}-e^{-\eta z})\,J_1(\zeta r)\,d\zeta, \text{ etc.}$$

We obtained a solution coinciding with that obtained by other methods by Puri [6].

Let us observe that when passing to the Hooke's body, i.e., with $a=0$, $\eta=\zeta$, $\Delta_0=1$ the stresses σ_{zz} and σ_{zr} vanish [7]. The state of stresses becomes plane. The couple-stresses are equal to zero.

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В. Новацки, **Осе-симметрическая задача в микрополярной теории упругости**

Содержание. В работе [1], автор представил решение осе-симметрической задачи в микрополярной теории упругости, обоснованное на обобщенных функциях Ловэ. В настоящей работе предлагается значительно более простое решение этой задачи путем введения упругих потенциалов Φ, Ψ, ϑ. Выведено дифференциальное уравнение для потенциалов, а также даны отношения совместности, связывающие потенциалы Φ и ϑ. На простом примере упругого полупространства объясняется применение упругих потенциалов. В последней части работы рассмотрены термоупругие осе-симметрические задачи. Применение потенциалов к термоупругой задаче объясняется на примере упругого полупространства, обогретого на ограничивающей его плоскости.