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APPLIED MECHANICS
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Plane Problems of Micropolar Elasticity

by

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Summary. In this paper a new way is proposed for solving the system of Eqs. (1.7) of the plane problem of micropolar elasticity. It consists in introducing two elastic potentials Φ and Ψ. This way is particularly convenient if the displacements u_1, u_2 and the rotation φ_3 are searched for. Singular solution of the system of Eqs. (1.7) is given, the problem of elastic half-space is considered and the problem of stationary thermoelasticity is discussed.

1. Introduction

Let us consider a micropolar, elastic, homogeneous, isotropic and centrosymmetric body. Under the effect of loadings the body undergoes deformation described by two vectors, namely the deformation vector $\mathbf{u}(\mathbf{x})$ and the rotation vector $\boldsymbol{\varphi}(\mathbf{x})$.

Using these two vectors we construct two asymmetric tensors, the deformation tensor γ_{ji} and the curvature-twist tensor κ_{ji}. There is

$$\gamma_{ji}=u_{i,j}-\varepsilon_{kji}\,\varphi_k, \quad \kappa_{ji}=\varphi_{i,j} \quad i,j,k=1,2,3. \tag{1.1}$$

The state of stress is characterized by two asymmetric tensors, the tensor of force stresses σ_{ji} and that of moment stresses μ_{ji}. These tensors are connected with the tensors γ_{ji} and κ_{ji} by the following constitutive equations

$$\begin{cases}\sigma_{ji}=(\mu+a)\,\gamma_{ji}+(\mu-a)\,\gamma_{ij}+\lambda\gamma_{kk}\,\delta_{ji}, \\ \mu_{ji}=(\gamma+\varepsilon)\,\kappa_{ji}+(\gamma-\varepsilon)\,\kappa_{ij}+\beta\kappa_{kk}\,\delta_{ji},\end{cases} \quad i,j,k=1,2,3. \tag{1.2}$$

Here the symbols $a, \beta, \gamma, \varepsilon, \mu, \lambda$, denote the material constants.

Introducing Eqs. (1.2) and (1.1) into the equations of equilibrium

$$\sigma_{ji,j}+X_i=0, \quad \varepsilon_{ijk}\,\sigma_{jk}+\mu_{ji,j}+Y_i=0, \tag{1.3}$$

we obtain a system of equations in displacements and rotations [1—4]:

$$\begin{cases}(\mu+a)\,\nabla^2\,\mathbf{u}+(\lambda+\mu-a)\,\text{grad div}\,\mathbf{u}+2a\,\text{rot}\boldsymbol{\varphi}+\mathbf{X}=0, \\ [(\gamma+\varepsilon)\,\nabla^2-4a]\boldsymbol{\varphi}+(\beta+\gamma-\varepsilon)\,\text{grad div}\boldsymbol{\varphi}+2a\,\text{rot}\,\mathbf{u}+\mathbf{Y}=0.\end{cases} \tag{1.4}$$

In the above equations the magnitudes **X** and **Y** are the vectors of body forces and moments, respectively.

Eqs. (1.4) have to be supplemented by boundary conditions. If on the surface A bounding the body, the loadings (forces **p** and moments **m**) are prescribed, the boundary conditions will assume the form:

$$p_i=\sigma_{ji}\,n_j, \qquad m_i=\mu_{ji}\,n_j, \tag{1.5}$$

where n_j are the components of the vector of unit normal **n**.

Now, passing to the plane problems of micropolar elasticity, we assume all the causes and effects to depend solely on the variables x_1, x_2. In this case the system of six Eqs. (1.4) will dissociate into two systems of equations independent of each other.

The first of them, wherein the following components of the vectors **u** and **φ** appear

$$\mathbf{u}\equiv(u_1, u_2, 0), \qquad \boldsymbol{\varphi}\equiv(0, 0, \varphi_3) \tag{1.6}$$

is of the form

$$\begin{aligned}
&(\mu+a)\,\nabla_1^2\,u_1+(\lambda+\mu-a)\,\partial_1\,e+2a\partial_2\,\varphi_3+X_1=0,\\
&(\mu+a)\,\nabla_1^2\,u_2+(\lambda+\mu-a)\,\partial_2\,e-2a\partial_1\,\varphi_3+X_2=0,\\
&[(\gamma+\varepsilon)\,\nabla_1^2-4a]\,\varphi_3+2a\,(\partial_1\,u_2-\partial_2\,u_1)+Y_3=0.
\end{aligned} \tag{1.7}$$

The following notations have been introduced into the above equations

$$\nabla_1^2=\partial_1^2+\partial_2^2, \qquad e=\partial_1\,u_1+\partial_2\,u_2.$$

In the second system of equations the following components of the **u** and **φ** vectors appear

$$\mathbf{u}\equiv(0, 0, u_3), \qquad \boldsymbol{\varphi}\equiv(\varphi_1, \varphi_2, 0). \tag{1.8}$$

The system of equations has the form:

$$\begin{aligned}
&[(\gamma+\varepsilon)\,\nabla_1^2-4a]\,\varphi_1+(\beta+\gamma-\varepsilon)\,\partial_1\,\kappa+2a\,\partial_2\,u_3+Y_1=0,\\
&[(\gamma+\varepsilon)\,\nabla_1^2-4a]\,\varphi_2+(\beta+\gamma-\varepsilon)\,\partial_2\,\kappa-2a\partial_1\,u_3+Y_2=0,\\
&(\mu+a)\,\nabla_1^2\,u_3+2a\,(\partial_1\,\varphi_2-\partial_2\,\varphi_1)+X_3=0.
\end{aligned} \tag{1.9}$$

The following notations have been introduced

$$\kappa=\partial_1\,\varphi_1+\partial_2\,\varphi_2, \qquad \nabla_1^2=\partial_1^2+\partial_2^2.$$

We shall be concerned here with the system of Eqs. (1.7). This system has been studied by Mindlin [5] and Schaefer [6], both of them using the generalized Airy's functions. This way is convenient, provided the loadings on the edge are prescribed while the stresses and strains are searched for. In what follows we shall propose a different way of solving the system of Eqs. (1.7) which may prove convenient if the displacements and rotations are searched for. We shall consider the problem of the elastic half-space and give a singular solution. We shall discuss also the problem of thermoelasticity. The way of solving the system of Eqs. (1.7) will be similar to that applied by the present author when solving the system of Eqs. (1.9) [7].

2. The solution of the system of Eqs. (1.7)

Let us consider the homogeneous system of equations of equilibrium (1.7) assuming its solution in the form of decomposition of the vector $\mathbf{u}\equiv(u_1, u_2, 0)$ into potential and solenoidal parts

$$u_1=\partial_1 \Phi+\partial_2 \Psi, \quad u_2=\partial_2 \Phi-\partial_1 \Psi. \tag{2.1}$$

Here Φ and Ψ play the role of static potentials of elasticity.

Introducing (2.1) into (1.7), we arrive at the system

$$\begin{aligned} &(\lambda+2\mu)\,\partial_1 \nabla_1^2 \Phi+\partial_2 [(\mu+a)\nabla_1^2 \Psi+2a\,\varphi_3]=0,\\ &(\lambda+2\mu)\,\partial_2 \nabla_1^2 \Phi-\partial_1 [(\mu+a)\nabla_1^2 \Psi+2a\varphi_3]=0,\\ &[(\gamma+\varepsilon)\nabla_1^2-4a]\,\varphi_3-2a\nabla_1^2 \Psi=0. \end{aligned} \tag{2.2}$$

Making use of the third of the above equations, we eliminate the term $\nabla_1^2 \Psi$ from the first two of them. In this way we obtain the following Cauchy—Riemann relations:

$$\begin{aligned} &\partial_1 \nabla_1^2 \Phi+\frac{2\mu}{\lambda+2\mu}\partial_2 (l^2 \nabla_1^2-1)\,\varphi_3=0,\\ &\partial_2 \nabla_1^2 \Phi-\frac{2\mu}{\lambda+2\mu}\partial_1 (l^2 \nabla_1^2-1)\,\varphi_3=0, \qquad l^2=\frac{(\mu+a)(\gamma+\varepsilon)}{4\mu a}. \end{aligned} \tag{2.3}$$

The functions Φ and φ_3 have to verify the following equations

$$\nabla_1^2 \nabla_1^2 \Phi=0, \quad \nabla_1^2 (l^2 \nabla_1^2-1)\,\varphi_3=0. \tag{2.4}$$

The function Ψ is connected with the function φ_3 by the following dependence resulting from Eq. $(2.2)_3$.

$$\nabla_1^2 \Psi=\frac{1}{2a}[(\gamma+\varepsilon)\nabla_1^2-4a]\,\varphi_3. \tag{2.5}$$

The procedure of solving the plane problem is as follows. First we solve Eq. (2.4); in the relevant solutions four integration constants will appear. These will be determined from three boundary conditions and from a supplementary relation — Cauchy—Riemann relation (2.3).

Let us now express the stresses by the elastic potentials Φ and Ψ and the rotation φ_3. We have successively

$$\left\{\begin{aligned} &\sigma_{11}=2\mu\partial_1 u_1+\lambda e=(\lambda+2\mu)\nabla_1^2 \Phi+2\mu(\partial_1 \partial_2 \Psi-\partial_2^2 \Phi),\\ &\sigma_{22}=2\mu\partial_2 u_2+\lambda e=(\lambda+2\mu)\nabla_1^2 \Phi+2\mu(-\partial_1 \partial_2 \Psi-\partial_1^2 \Phi),\\ &\sigma_{33}=\lambda e=\lambda\nabla_1^2 \Phi,\\ &\sigma_{12}=(\mu+a)(\partial_1 u_2-\varphi_3)+(\mu-a)(\partial_2 u_1+\varphi_3)=\mu[2\partial_1 \partial_2 \Phi+\\ &\qquad\qquad +(\partial_2^2-\partial_1^2)\Psi]-a\nabla_1^2 \Psi-2a\varphi_3,\\ &\sigma_{21}=\mu[2\partial_1 \partial_2 \Phi+(\partial_2^2-\partial_1^2)\Psi]+a\nabla_1^2+2a\,\varphi_3,\\ &\mu_{13}=(\gamma+\varepsilon)\,\partial_1 \varphi_3, \quad \mu_{23}=(\gamma+\varepsilon)\,\partial_2 \varphi_3,\\ &\mu_{31}=(\gamma-\varepsilon)\,\partial_1 \varphi_3, \quad \mu_{32}=(\gamma-\varepsilon)\,\partial_2 \varphi_3. \end{aligned}\right. \tag{2.6}$$

Let us observe that

$$\sigma_{11}+\sigma_{22}=2(\lambda+\mu)\nabla_1^2\Phi,\quad \sigma_{21}-\sigma_{12}=2a\nabla_1^2\Psi+4a\varphi_3=(\gamma+\varepsilon)\nabla_1^2\varphi_3.$$

It may be easily checked that the conditions of compatibility in stresses will be satisfied

$$(2.7)\quad \begin{cases} \partial_2^2\sigma_{11}+\partial_1^2\sigma_{22}-\dfrac{\lambda}{2(\lambda+\mu)}\nabla_1^2(\sigma_{11}+\sigma_{22})=\partial_1\partial_2(\sigma_{12}+\sigma_{21}), \\ (\partial_2^2-\partial_1^2)(\sigma_{12}+\sigma_{21})+\dfrac{\mu}{a}\nabla_1^2(\sigma_{12}-\sigma_{21})=2\partial_1\partial_2(\sigma_{22}-\sigma_{11})- \\ \qquad\qquad -\dfrac{4\mu}{\gamma+\varepsilon}(\partial_1\mu_{13}+\partial_2\mu_{23}), \\ \partial_1\mu_{23}-\partial_2\mu_{13}=0. \end{cases}$$

The substitution of stresses (2.6) into (2.7) leads to Eqs. (2.4) and (2.5).

Let us consider, exempli modo, the way of solving the following static problem. Let the half-space $x_1\geqslant 0$ be loaded at the edge $x_1=0$ by a load $p(x_2)$ directed along the positive x_1-axis. The boundary conditions, (1.5), reduce to the following

$$(2.8)\qquad \sigma_{11}(0,x_2)=-p(x_2),\quad \sigma_{12}(0,x_2)=0,\quad \mu_{13}(0,x_2)=0.$$

Making use of the relations (2.6), we shall express these conditions by the potentials Φ, Ψ and the rotation φ_3. The solution of Eqs. (2.4) will be given in the form of Fourier integrals

$$(2.9)\quad \begin{aligned} \Phi&=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(A+Bx_1\zeta)\,e^{-x_1\zeta-ix_2\zeta}\,d\zeta, \\ \Psi&=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(Ce^{-x_1\zeta}+De^{-\eta x_1})\,e^{-ix_2\eta}\,d\zeta,\qquad \eta=\left(\zeta^2+\frac{1}{l^2}\right)^{\frac{1}{2}}. \end{aligned}$$

From the Cauchy—Riemann equations, (2.3), we obtain the following dependence

$$(2.10)\qquad C=i\zeta^2\frac{\lambda+2\mu}{\mu}B.$$

From the boundary condition $(2.8)_3$ we get $D=-\dfrac{\zeta}{\eta}C$. In boundary conditions $(2.8)_{1,2}$ the function Ψ appears which has to be determined from Eq. (2.5). After some simple calculations we obtain

$$(2.11)\qquad \Psi=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{C}{\zeta^2}\left[\zeta x_1\,e^{-\zeta x_1}-2\kappa^2 l^2\zeta^2\frac{\zeta}{\eta}e^{-\eta x_1}\right]e^{-ix_2\zeta}\,d\zeta,$$

$$\kappa^2=\frac{\nu^2}{l^2}-1,\qquad \nu^2=\frac{\gamma+\varepsilon}{4a}.$$

With the help of boundary conditions $(2.8)_{1,2}$ we obtain the full set of the integration constants

$$(2.12)\quad \begin{cases} B=\dfrac{\tilde{p}}{2(\lambda+\mu)\,\zeta^2\,\Delta_0}, \quad \Delta_0=1+2\zeta^2 a_0\left(1-\dfrac{\zeta}{\eta}\right), \quad a_0=\dfrac{(\gamma+\varepsilon)(\lambda+2\mu)}{4\mu(\lambda+\mu)}, \\ A=-\dfrac{\lambda+\mu}{\mu}B\left(1-2a_0\,\zeta^2\dfrac{\zeta}{\eta}\right), \quad C=i\zeta^2\dfrac{\lambda+2\mu}{\mu}B, \quad D=-\dfrac{\zeta}{\eta}C. \end{cases}$$

Here

$$\tilde{p}(\zeta)=\frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty} p(x_2)\,e^{ix_2\zeta}\,dx_2 .$$

Making use of the relations (2.6), we obtain the following formulae for the stresses

$$(2.13)\quad \begin{cases} \sigma_{11}=-\dfrac{1}{\sqrt{2\pi}}\displaystyle\int\limits_{-\infty}^{\infty}\dfrac{\tilde{p}}{\Delta_0}\left[(1+x_1\,\zeta)\,e^{-\zeta x_1}+2a_0\,\zeta^2\left(e^{-\eta x_1}-\dfrac{\zeta}{\eta}e^{-\zeta x_1}\right)\right]e^{-ix_2\zeta}\,d\zeta, \\ \sigma_{12}=-\dfrac{i}{\sqrt{2\pi}}\displaystyle\int\limits_{-\infty}^{\infty}\dfrac{\tilde{p}}{\Delta_0}\left[x_1\,\zeta\,e^{-\zeta x_1}+2a_0\,\zeta^2\dfrac{\zeta}{\eta}(e^{-\eta x_1}-e^{-\zeta x_1})\right]e^{-ix_2\zeta}\,d\zeta, \\ \mu_{13}=-\dfrac{2ia_0}{\sqrt{2\pi}}\displaystyle\int\limits_{-\infty}^{\infty}\dfrac{\zeta\tilde{p}}{\Delta_0}(e^{-\zeta x_1}-e^{-\eta x_1})\,e^{-ix_2\zeta}\,d\zeta, \text{ etc.} \end{cases}$$

Obviously, we arrive at the same formulae applying the method of the stress function [5, 6].

3. Plane problem of thermoelasticity

The way of solution outlined above may be generalized so as to comprise the stationary and quasi-static problem of thermoelasticity.

The relevant system of equations in displacements and rotations reads as below:

$$(3.1)\quad \begin{aligned} &(\mu+a)\,\nabla_1^2\,u_1+(\lambda+\mu-a)\,\partial_1\,e+2a\,\partial_2\,\varphi_3=\gamma_0\,\partial_1\,\theta, \\ &(\mu+a)\,\nabla_1^2\,u_2+(\lambda+\mu-a)\,\partial_2\,e-2a\,\partial_1\,\varphi_3=\gamma_0\,\partial_2\,\theta, \\ &[(\gamma+\varepsilon)\,\nabla_1^2-4a]\,\varphi_3+2a\,(\partial_1\,u_2-\partial_2\,u_1)=0. \end{aligned}$$

Here $\gamma_0=(3\lambda+2\mu)\,a_t$, where a_t the coefficient of linear thermal expansion. Introducing into Eqs. (3.1) the potentials Φ, Ψ described by the relations

$$(3.2)\quad u_1=\partial_1\,\Phi+\partial_2\,\Psi, \quad u_2=\partial_2\,\Phi-\partial_1\,\Psi,$$

we obtain the following Cauchy—Riemann relations:

$$(3.3)\quad \begin{aligned} &\partial_1(\nabla_1^2\,\Phi-m\theta)+\frac{2\mu}{\lambda+2\mu}\partial_2\,(l^2\,\nabla_1^2-1)\,\varphi_3=0, \\ &\partial_2(\nabla_1^2\,\Phi-m\theta)-\frac{2\mu}{\lambda+2\mu}\partial_1\,(l^2\,\nabla_1^2-1)\,\varphi_3=0, \\ &m=\frac{\gamma_0}{\lambda+2\mu}. \end{aligned}$$

We have made use of Eq. $(3.1)_3$ which — taking into acount (3.2) — will assume the form:

$$\nabla_1^2 \Psi = 2(\nu^2 \nabla_1^2 - 1)\varphi_3, \qquad \nu^2 = \frac{\gamma+\varepsilon}{4a}. \tag{3.4}$$

Taking profit of the relations (3.3), we arrive at the following equations

$$\nabla_1^2 \nabla_1^2 \Phi - m \nabla_1^2 \theta = 0, \quad \nabla_1^2 (l^2 \nabla_1^2 - 1)\varphi_3 = 0. \tag{3.5}$$

The solution of these equations may be composed of two parts

$$\Phi = \Phi' + \Phi'', \quad \varphi_3 = \varphi_3' + \varphi_3''. \tag{3.6}$$

We assume $\varphi_3' = 0$, Φ' being the particular integral of the equation

$$\nabla_1^2 \Phi' - m\theta = 0. \tag{3.7}$$

We may assume at the edge $\Phi' = 0$, provided the body is bounded. The Φ' function is connected with the stresses σ_{ji}', $\mu_{ji}' = 0$, the σ_{ji}' stresses generating the symmetric tensor. There is

$$\sigma_{ji}' = 2\mu(\Phi_{,ij}' - \delta_{ji} \nabla_1^2 \Phi'), \quad \mu_{ji}' = 0. \tag{3.8}$$

The functions Φ'' and φ_3'' have to satisfy the following homogeneous equations

$$\nabla_1^2 \nabla_1^2 \Phi'' = 0, \quad \nabla_1^2 (l^2 \nabla_1^2 - 1)\varphi_3'' = 0 \tag{3.9}$$

as well as the Cauchy—Riemann relations

$$\begin{aligned} &\partial_1 \nabla_1^2 \Phi'' + \frac{2\mu}{\lambda+2\mu} \partial_2 (l^2 \nabla_1^2 - 1)\varphi_3'' = 0, \\ &\partial_2 \nabla_1^2 \Phi'' - \frac{2\mu}{\lambda+2\mu} \partial_1 (l^2 \nabla_1^2 - 1)\varphi_3'' = 0. \end{aligned} \tag{3.10}$$

Eq. (3.4) will now assume the form

$$\nabla_1^2 \Psi'' = 2(\nu^2 \nabla_1^2 - 1)\varphi_3''. \tag{3.11}$$

If we wish to have the edge free of stresses, the following conditions should be satisfied

$$(\sigma_{\beta\alpha}' + \sigma_{\beta\alpha}'') n_\beta = 0, \quad (\mu_{\beta 3}' + \mu_{\beta 3}'') n_\beta = 0, \quad \alpha, \beta = 1, 2. \tag{3.12}$$

The stresses σ_{ji}'' connected with the potentials Φ'' and Ψ'' and the rotation φ_3'' are given by the formulae (2.6).

Let us now consider the case of a simply connected cylinder, infinitely long, heated at its side wall. Let the cylinder axis coincide with the x_3-axis. Assuming a stationary heat flow, we have $\nabla^2 \theta = 0$. Eqs. (3.5) become homogeneous.

The solution of the system of Eqs. (3.5) under homogeneous boundary conditions leads to the trivial solution

$$\Phi \equiv 0, \quad \varphi_3 \equiv 0, \quad \Psi \equiv 0, \tag{3.13}$$

the temperature θ being the sole constant at the cylinder cross-section. This is due to the fact that in this case the conditions (3.3) will be also satisfied. The only stress different from zero will be the stress σ_{33}. We have

$$\sigma_{33}=\lambda\gamma_{kk}-\gamma\theta=\frac{2\gamma_0\,\theta}{2(\lambda+\mu)}-\gamma_0\,\theta=-\frac{\mu(3\lambda+2\mu)}{\lambda+\mu}\,a_t\,\theta,$$

which concords with the result obtained in [8] with the use of Airy—Mindlin function of stresses.

4. Singular solutions

Let us now consider the singular solution of the system of Eqs. (1.7). The body forces will be written in the form

$$X_1=\rho\,(\partial_1\,\vartheta+\partial_2\,\chi),\qquad X_2=\rho\,(\partial_2\,\vartheta-\partial_1\,\chi). \tag{4.1}$$

The above expression is equivalent to the decomposition of the body force into the potential and solenoidal parts.

Introducing into (1.7) the Eqs. (4.1) and the relations

$$u_1=\partial_1\,\Phi+\partial_2\,\Psi,\qquad u_2=\partial_2\,\Phi-\partial_1\,\Psi, \tag{4.2}$$

we obtain the following system of equations

$$\begin{aligned}
&\partial_1\,[(\lambda+2\mu)\,\nabla_1^2\,\Phi+\rho\vartheta]+\partial_2\,[(\mu+a)\,\nabla_1^2\,\Psi+2a\varphi_3+\rho\chi]=0,\\
&\partial_2\,[(\lambda+2\mu)\,\nabla_1^2\,\Phi+\rho\vartheta]-\partial_1\,[(\mu+a)\,\nabla_1^2\,\Psi+2a\,\varphi_3+\rho\chi]=0,\\
&\nabla_1^2\,\Psi=\frac{1}{2a}\,[(\gamma+\varepsilon)\,\nabla_1^2-4a]+\frac{1}{2a}\,Y_3\,.
\end{aligned} \tag{4.3}$$

Eliminating the term $\nabla_1^2\,\Psi$, we get the following Cauchy—Riemann relations:

$$\begin{aligned}
&\partial_1\left(\nabla_1^2\,\Phi+\frac{\rho}{\lambda+2\mu}\,\vartheta\right)+\frac{2\mu}{\lambda+2\mu}\partial_2\left[(l^2\,\nabla^2-1)\,\varphi_3+\frac{\mu+a}{4\mu a}\,Y_3+\frac{1}{2\mu}\,\rho\chi\right]=0,\\
&\partial_2\left(\nabla_1^2\,\Phi+\frac{\rho}{\lambda+2\mu}\,\vartheta\right)-\frac{2\mu}{\lambda+2\mu}\partial_1\left[(l^2\,\nabla^2-1)\,\varphi_3+\frac{\mu+a}{4\mu a}\,Y_3+\frac{1}{2\mu}\,\rho\chi\right]=0.
\end{aligned} \tag{4.4}$$

The above relations lead to the following differential equations

$$\nabla_1^2\,\nabla_1^2\,\Phi+\frac{\rho}{\lambda+2\mu}\,\nabla_1^2\,\vartheta=0,\qquad \nabla_1^2(l^2\,\nabla_1^2-1)\,\varphi_3+\frac{\mu+a}{4a\mu}\,\nabla_1^2\,Y_3+\frac{1}{2\mu}\rho\,\nabla^2\,\chi=0. \tag{4.5}$$

Bearing in mind that

$$\partial_1\,X_1+\partial_2\,X_2=\rho\nabla_1^2\,\vartheta,\qquad \partial_2\,X_1-\partial_1\,X_2=\rho\,\nabla_1^2\,\chi,$$

we obtain the final system of equations

$$\begin{aligned}
&\nabla_1^2\,\nabla_1^2\,\Phi+\frac{1}{\lambda+2\mu}(\partial_1\,X_1+\partial_2\,X_2)=0,\\
&\nabla_1^2(l^2\,\nabla_1^2-1)\,\varphi_3+\frac{\mu+a}{4a\mu}\,\nabla_1^2\,Y_3+\frac{1}{2\mu}(\partial_2\,X_1-\partial_1\,X_2)=0.
\end{aligned} \tag{4.6}$$

These equations should be supplemented by the relation $(4.3)_3$:

$$\nabla_1^2 \Psi = 2(\nu^2 \nabla_1^2 - 1)\, \varphi_3 + \frac{Y_3}{2a}. \tag{4.7}$$

The system of Eqs. (4.6) will be solved with the help of the double exponential Fourier transformation. We get

$$\begin{aligned} \Phi &= \frac{i}{2\pi(\lambda+2\mu)} \int\limits_{-\infty}^{\infty}\int \frac{1}{\boldsymbol{\zeta}^2 \cdot \boldsymbol{\zeta}^2} (\zeta_1 \tilde{X}_1 + \zeta_2 \tilde{X}_2)\, e^{-ix_k \zeta_k}\, d\zeta_1\, d\zeta_2, \\ \varphi_3 &= \frac{1}{4\pi\mu} \int\limits_{-\infty}^{\infty}\int \left[\frac{i(\zeta_2 \tilde{X}_1 - \zeta_1 \tilde{X}_2)}{\boldsymbol{\zeta}^2 (l^2 \boldsymbol{\zeta}^2 + 1)} + \frac{\mu+a}{2a} \frac{\tilde{Y}_3}{(l^2 \boldsymbol{\zeta}^2 + 1)} \right] e^{-ix_k \zeta_k}\, d\zeta_1\, d\zeta_2, \end{aligned} \tag{4.8}$$

where $\boldsymbol{\zeta}^2 = \zeta_1^2 + \zeta_2^2$.

Now, from Eq. (4.7) we have

$$\Psi = \frac{1}{4\pi\mu} \int\limits_{-\infty}^{\infty}\int \left[\frac{\tilde{Y}_3}{\boldsymbol{\zeta}^2 (l^2 \boldsymbol{\zeta}^2 + 1)} + \frac{2i(\nu^2 \boldsymbol{\zeta}^2 + 1)}{\boldsymbol{\zeta}^2 \boldsymbol{\zeta}^2 (l^2 \boldsymbol{\zeta}^2 + 1)} (\zeta_2 \tilde{X}_1 - \zeta_1 \tilde{X}_2) \right] e^{-ix_k \zeta_k}\, d\zeta_1\, d\zeta_2. \tag{4.9}$$

For $X_1 = 0$ and $X_2 = 0$, $Y_3 \neq 0$ we have, respectively, $\Phi = 0$ and $\Psi \neq 0$, $\varphi_3 \neq 0$.

We pass now to the particular case of action of the concentrated moment $Y_3 = M\delta(x_1)\,\delta(x_2)$. Then

$$\begin{aligned} \Phi &= 0, \qquad \varphi_3 = \frac{M(\mu+a)}{16\pi^2 \mu a l^2} \int\limits_{-\infty}^{\infty}\int \frac{e^{-ix_k \zeta_k}\, d\zeta_1\, d\zeta_2}{\boldsymbol{\zeta}^2 + \frac{1}{l^2}}, \\ \Psi &= \frac{M}{8\pi^2 \mu l^2} \int\limits_{-\infty}^{\infty}\int \frac{e^{-ix_k \zeta_k}\, d\zeta_1\, d\zeta_2}{\boldsymbol{\zeta}^2 (l^2 \boldsymbol{\zeta}^2 + 1)}. \end{aligned} \tag{4.10}$$

The above integrals do not exist as improper integrals; neither can we ascribe them the Cauchy main value. We are able, however, to separate what is called the finite part [9].

Applying the method described in [10] we obtain:

$$\begin{aligned} \Phi &= 0, \qquad \varphi_3 = \frac{M}{2\pi(\gamma+\varepsilon)} K_0\left(\frac{r}{l}\right), \\ \Psi &= -\frac{M}{4\pi\mu} \left(\ln r + K_0\left(\frac{r}{l}\right) \right), \qquad r = (x_1^2 + x_2^2)^{1/2}. \end{aligned} \tag{4.11}$$

Here $K_0(z)$ is the (modified) Bessel's function of the third kind. The displacements u_1 and u_2 will be given by the formulae

$$u_1 = \partial_2 \Psi, \qquad u = -\partial_1 \Psi.$$

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В. Новацки, **Плоские задачи микфополярной упругости**

Содержание. Предложен новый путь решения системы уравнений (1.7) плоской задачи микрополярной упругости, состоящий в введении двух упругих потенциалов Φ и Ψ. Этот путь решения особенно удобен в случае, когда ищем перемещений u_1 и u_2 и оборота φ_3. Приведено особое решение для системы уравнений (1.7), рассмотрена проблема упругого полупространства, а также продискутирована задача стационарной термоупругости.