BULLETIN
DE
L'ACADÉMIE POLONAISE DES SCIENCES

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VOLUME XVIII
NUMÉRO 4

VARSOVIE 1970

BULLETIN DE L'ACADÊMIE
POLONAISE DES SCIENCES
Série des sciences techniques
Volume XVIII, No. 4 — 1970

APPLIED MECHANICS

Wave-type Equations of Thermo-Magneto-Microelasticity

by

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Presented on December 27, 1969

1. Introduction

Papers [1, 2 and 3] consider the formulation of wave-type equations of heat conduction, thermoelasticity and thermo-electro-magnetoelasticity. These equations are characterized by a modified Fourier law of heat conduction and imply finite propagation speed of thermal perturbations and perturbations of fields coupled with thermal ones, which is in agreement with the reality.

The effect of finite propagation velocity of thermal perturbations plays an essential role in a number of processes, in which we are concerned with velocities of a body or systems cooperating with it, approaching the sound velocity and also for high-frequency periodic fields [4].

In connection with the theory of microelasticity theories of thermo-micro-elasticity and thermo-magneto-microelasticity have been developed ([5, 6], respectively).

It appears to be of interest to generalize the equations of thermo-magneto--microelasticity to obtain wave-type equations of thermo-magneto-microelasticity in which the velocity of propagation of thermal and coupled perturbation would be finished. This problem is the subject-matter of the present communication. In the case of thermo-microelasticity of the wave-type such equations are given in [7]. Since in the case of an isotropic medium the temperature field is not coupled with a moment field, the wave-type equations of thermo-magneto-microelasticity can be obtained directly by combining the results of [6] and [3]. In the case of an anisotropic body the coupling is a very simple one.

Bearing this in mind we shall give the above equations in their final form disregarding the thermodynamical considerations, which would be identical with those of [3] and [6]. A detailed list of the literature on the problem can be found in [8].

In Sec. **2** an anisotropic medium will be considered in a general. Sec. **3** will be devoted to the particular case of an isotropic medium.

2. Equations for an anisotropic body

The RMKS system of measures will be adopted. Linearized equations will be considered. In agreement with the results of [3] and [6], the set of equations of the problem will have the following form, the notations being the same as in [3, 6].

2.1. Equations of electrodynamics

$$\operatorname{rot} \mathbf{h}=\mathbf{j}+\dot{\mathbf{D}}; \qquad \operatorname{rot} \mathbf{E}=-\dot{\mathbf{b}}; \qquad \operatorname{div} \mathbf{b}=0; \qquad \operatorname{div} \mathbf{D}=\rho_e, \tag{2.1}$$

where

$$\mathbf{D}=\mathbf{D}_0-\frac{1}{c^2}(\mathbf{u}\times\mathbf{H}_0); \quad \mathbf{E}=\mathbf{E}_0-(\dot{\mathbf{u}}\times\mathbf{B}_0); \quad \mathbf{j}=\mathbf{j}_0+\rho_e\dot{\mathbf{u}}\approx\mathbf{j}_0, \tag{2.2}$$

where $\mathbf{H}_0, \mathbf{B}_0$ — the primary magnetic field and magnetic induction, $\mathbf{E}_0, \mathbf{D}_0, \mathbf{j}$ — quantities connected with the medium.

2.2. Equations of motion

$$\begin{aligned} &\varepsilon_{ijk}\sigma_{jk}+\mu_{ji,j}+M_i=m\ddot{\varphi}_i, \\ &\sigma_{ji,j}+F_i+P_i=\rho\ddot{u}_i, \end{aligned} \tag{2.3}$$

where

$$F_i=\rho_e E_i+(\mathbf{j}\times\mathbf{B}_0)_i, \tag{2.4}$$

and P_i, M_i — the field of mass forces and moments, respectively, u_i, φ_i — the displacement and rotation vector, respectively, σ_{ik}, μ_{ik} — the stress and micromoment tensor, respectively.

2.3. Heat equation

$$T_0\beta_0(\tau\ddot{T}+\dot{T})+T_0 a_{ij}(\tau\ddot{e}_{ij}+\dot{e}_{ij})+T_0\nu_{ik}(\tau\ddot{\varphi}_{i,j}+\dot{\varphi}_{i,j})- \\ -(k_{ij}T_{,j})_{,i}+(\pi_{ik}j_k)_{,i}=f; \tag{2.5}$$

the notations being the same as in [3].

In addition we have the equation of the current [3],

$$j_i+\tau\dot{j}_i+\kappa_{ik}k_{kj}^{-1}\pi_{js}(j_s+\tau j_s)= \\ =\eta_{ik}\{E_k+\tau\dot{E}_k+[(\dot{\mathbf{u}}+\varepsilon\ddot{\mathbf{u}})\times\mathbf{B}_0]_k\}+\kappa_{ik}[k_{kj}^{-1}(\pi_{js}j_s-k_{js}T_{,s})]; \tag{2.6}$$

the notations being also the same as in [3]. The symmetry (Onsager) relations for the above-mentioned coefficients are the same as in [3] and will not be quoted.

To these equations we add the equations of state.

If we assume, after [6], the thermodynamic potential in the form

$$F=F(e_{ij},\varphi_{i,j},D_{0i},B_i,T)=A_{iklm}e_{ik}e_{lm}+B_{iklm}e_{ik}\varphi_{l,m}+D_{iklm}\varphi_{i,k}\varphi_{l,m}+ \\ +\bar{\varepsilon}_{ik}D_{0i}D_{0k}+\bar{\mu}_{ik}B_iB_k-a_{ik}T_{ik}-\nu_{ik}T\varphi_{i,k}-\frac{m}{2}T^2, \tag{2.7}$$

we obtain the following equations of state

$$\sigma_{ij}=\frac{\partial F}{\partial e_{ij}}; \quad \mu_{ij}=\frac{\partial F}{\partial\varphi_{i,j}}; \quad E_{0i}=\frac{\partial F}{\partial D_{0i}}; \quad H_i=\frac{\partial F}{\partial B_i}; \quad S=-\frac{\partial F}{\partial T}. \tag{2.8}$$

The systems of Eqs. (2.1) to (2.8) constitute the complete system of equations of the problem.

In these equations, in addition to the Onsager equations quoted after [3], we should determine, by physical analysis, the interrelation and variability ranges of tensor coefficients. In the case of general equations of the micromoment theory this is, in general, neither possible nor unequivocal. In [6] for instance, the term involving $\varphi_{i,k}$ was rejected in the heat equation. In the present case it is preserved although its order of magnitude is difficult to estimate. Such an estimation can be attempted only for particular crystals or symmetry classes. It is for these reasons that the equations for an isotropic medium are much more illustrative and will be quoted now.

3. Equations for an isotropic medium

In the case of an isotropic centrally symmetric medium the expression (2.7) for F will have the form, [5, 6]:

$$F=\frac{\mu+a}{2}e_{ij}e_{ij}+\frac{\mu-a}{2}e_{ij}e_{ji}+\frac{\lambda}{2}e_{kk}e_{ll}-\nu_1 e_{kk}T-\frac{n}{2}T^2+ \\ +\frac{\gamma+\varepsilon}{2}\varphi_{i,j}\varphi_{i,j}+\frac{\gamma-\varepsilon}{2}\varphi_{i,j}\varphi_{j,i}+\frac{\beta}{2}\varphi_{k,k}\varphi_{l,l}+\frac{D_{0l}^2}{2\varepsilon}+\frac{B_{1l}^2}{2\mu_0}. \tag{3.1}$$

Hence

$$\begin{gathered}\sigma_{ij}=(\mu+a)e_{ij}+(\mu-a)e_{ji}+(\lambda e_{kk}-\nu_1 T)\delta_{ij},\\ \mu_{ij}=(\gamma+\varepsilon)\varphi_{i,j}+(\gamma-\varepsilon)\varphi_{j,i}+\beta\varphi_{k,k}\delta_{ij},\\ S=\nu_1 e_{kk}+nT,\\ E_{0i}=\frac{1}{\varepsilon}D_{0i};\qquad H_i=\frac{1}{\mu_0}B_i.\end{gathered} \tag{3.2}$$

On substituting into the equations of magnetic field, motion and heat conduction, we obtain:

3.1. Equations of electrodynamics

$$\begin{gathered}\operatorname{rot}\mathbf{h}=\mathbf{j}+\varepsilon\dot{\mathbf{E}}+\frac{\varepsilon\mu c^2-1}{c^2}(\ddot{\mathbf{u}}\times\mathbf{H}_0);\\ \operatorname{rot}\mathbf{E}=-\mu_0\dot{\mathbf{h}};\qquad \operatorname{div}\mathbf{H}=0;\qquad \operatorname{div}\mathbf{D}=\rho_e.\end{gathered} \tag{3.3}$$

3.2. Equations of motion

$$\begin{gathered}\rho\ddot{\mathbf{u}}=(\mu+a)\nabla^2\mathbf{u}+(\lambda+\mu-a)\operatorname{grad}\operatorname{div}\mathbf{u}+\\ +2a\operatorname{rot}\boldsymbol{\varphi}+\mathbf{P}-\nu_t\operatorname{grad}T+(\mathbf{j}\times\mathbf{B}_0)+\rho_e\mathbf{E},\\ m\ddot{\boldsymbol{\varphi}}_i=(\gamma+\varepsilon)\nabla^2\boldsymbol{\varphi}+(\beta+\gamma-\varepsilon)\operatorname{grad}\operatorname{div}\mathbf{u}+2a\operatorname{rot}\mathbf{u}-4a\boldsymbol{\varphi}+\mathbf{M}.\end{gathered} \tag{3.4}$$

3.3 Heat equation

$$\nabla^2 T-\left((\eta_1\operatorname{div}\dot{\mathbf{u}}+\frac{1}{\kappa}\dot{T}\right)(1+\tau\partial_t)-\pi_0\operatorname{div}\mathbf{j}=0, \tag{3.5}$$

by

$$\pi_0 = \frac{\pi}{k}; \qquad \kappa = \frac{k}{c_\varepsilon}; \quad \eta_1 = \nu_1 T_0/k;$$

c_ε — denoting the specific heat with constant strain and

$$\mathbf{j} + \tau \dot{\mathbf{j}} \left(1 + \frac{\kappa\pi}{k}\right) = \eta \{\mathbf{E}(1+\tau\partial_t) + [\dot{\mathbf{u}}(1+\tau\partial_t) \times \mathbf{B}_0]\} - \kappa \nabla T. \tag{3.6}$$

The system of equations (3.3) to (3.6) constitutes the complete system of equations of the problem of wave theory of thermo-magneto-microelasticity for an isotropic centrally symmetric body.

The boundary conditions preserve the same form as in the non-wave equations, therefore they will not be discussed. Simpler forms of the equations quoted, for perfect conductors, for instance, can be obtained according to known principles, therefore they will not be dealt with either.

In conclusion let us observe that, similarly to [7], we can introduce here also a series of potentials, by means of which the system of equations can be separated. The problems involved are more difficult, however, than in [7] and will be analyzed separately as will some simpler solutions of the equations obtained.

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С. КАЛИСКИЙ и В. НОВАЦКИЙ, **ВОЛНОВЫЕ УРАВНЕНИЯ ТЕРМО-МАГНИТО-МИКРОУПРУГОСТИ**

В работе выведены волновые уравнения термо-магнито-микроупругости, обеспечивающие конечную скорость распространения термических возмущений и сопряженных с ними. Уравнения выведены в общем виде для анизотропной среды, а для случая изотропной среды в подробном виде.