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APPLIED MECHANICS
(THERMOELASTICITY)

Wave Equations of Thermo-microelasticity

by

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1. Introduction

Wave equations of heat conductivity and thermoelasticity in an elastic Hooke's medium have been derived in [1] and [2], while in [3] basic equations and relations of linear thermoelasticity in a micropolar Cosserat medium were given.

In this paper we are going to generalize the wave equations of thermoelasticity on the micropolar medium.

Since in the isothermal micropolar medium the temperature field does not couple with moment fields, we may arrive at the wave equations of thermo-microelasticity by joining directly the results obtained in [1] and [3]. This is why we shall give the wave equations of thermo-microelasticity in their final form refraining from thermodynamic considerations which, besides, would be identical with those stated in [1] and [3].

In the next, second section we derive the basic relations and equations of thermo-microelasticity, in the third section the elastic potentials are given and, finally, the fourth section is devoted to the generalization of Galerkin's stress functions.

2. Equations of thermo-microelasticity

Let us consider a micropolar, homogeneous, isotropic and centrisymmetric body. Under the effect of external forces and heating the body will suffer a deformation characterized by two asymmetric tensors, namely the strain tensor e_{ji} and the curvature-twist tensor κ_{ji}. These tensors are defined as follows

$$e_{ji}=\mathbf{u}_{i,j}-\varepsilon_{kji}\boldsymbol{\varphi}_k, \qquad \kappa_{ji}=\varphi_{i,j}. \tag{2.1}$$

Here the symbol $\mathbf{u}$ denotes the displacement vector, $\boldsymbol{\varphi}$ stands for the rotation vector, while ε_{kji} is the known Cartesian alternator.

The state of stress is described also by two asymmetric tensors, namely the force-stress tensor σ_{ji} and the couple-stress tensor μ_{ji}.

The state of stress, that of strain and temperature are related to each other by the following constitutive equations [3]:

$$\sigma_{ji}=(\mu+a)\, e_{ji}+(\mu-a)\, e_{ij}+(\lambda e_{kk}-T\nu)\, \delta_{ji}, \tag{2.2}$$

$$\mu_{ji}=(\lambda+\varepsilon)\, \kappa_{ji}+(\gamma-\varepsilon)\, \kappa_{ij}+\beta\kappa_{kk}\, \delta_{ji}. \tag{2.3}$$

The symbols $\mu, \lambda, a, \beta, \gamma, \varepsilon$ stand for material constants $T=T'-T_0$, where T' is the absolute temperature, while T_0 denotes the temperature of the body when in natural state. Finally, $\gamma=(3\lambda+2\mu)\, a_t$, where a_t is the coefficient of linear thermal dilatation.

The above equations have to be supplemented by the equations of motion

$$\sigma_{ji,j}+P_i=\rho\ddot{u}_i, \tag{2.4}$$

$$\varepsilon_{ijk}\, \sigma_{jk}+\mu_{ji,j}+M_i=I\ddot{\varphi}_i, \tag{2.5}$$

and the wave equation of heat conductivity [1]

$$\left[\nabla^2-\frac{1}{\kappa}\, \partial_t(1+\tau\partial_t)\right]-\eta\partial_t(1+\tau\partial_t)\, u_{k,k}=-\frac{Q}{\kappa}. \tag{2.6}$$

The following notations have been used in Eqs. (2.4)—(2.6): The symbol $\mathbf{P}$ denotes the vector of body forces, $\mathbf{M}$ the vector of body moments, ρ density, I the rotational inertia. $\kappa=\lambda_0/c_\varepsilon$. where λ_0 is the coefficient of the heat conductivity and c_ε the specific heat of strain. Funally, $Q=W/c_\varepsilon$, where W denotes the heat quantity produced within a time and volume unit, $\eta=\dfrac{\nu T_0}{\lambda_0}$.

Making use of the relations (2.2) and (2.3) we may eliminate stresses and strains from Eqs. (2.4) and (2.5). In this way we obtain the following system of conjugate equations [4]:

$$\Box_2\, \mathbf{u}+(\lambda+\mu-a)\, \mathrm{grad\, div}\, \mathbf{u}+2a\, \mathrm{rot}\, \boldsymbol{\varphi}+\mathbf{P}=\nu\, \mathrm{grad}\, T, \tag{2.7}$$

$$\Box_4\, \boldsymbol{\varphi}+(\beta+\gamma-\varepsilon)\, \mathrm{grad\, div}\, \boldsymbol{\varphi}+2a\, \mathrm{rot}\, \mathbf{u}+\mathbf{Y}=0, \tag{2.8}$$

$$DT-\eta C\, \mathrm{div}\, \mathbf{u}=-\frac{Q}{\kappa}. \tag{2.9}$$

The following operators have been introduced in the above equations:

$$\Box_2=(\mu+a)\, \nabla^2-\rho\partial_t^2, \qquad \Box_4=(\gamma+\varepsilon)\, \nabla^2-4a-I\partial_t^2,$$

$$D=\nabla^2-\frac{1}{\kappa}\, \partial_t(1+\tau\partial_t), \qquad C=\partial_t(1+\tau\partial_t).$$

where

$$\partial_t=\frac{\partial}{\partial t}, \qquad \partial_t^2=\frac{\partial^2}{\partial t^2}, \qquad \nabla^2=\frac{\partial}{\partial x_i}\, \frac{\partial}{\partial x_i}.$$

Eqs. (2.7)—(2.9) form a complete set of wave equations of thermo-microelasticity. For $\tau\to 0$ these equations reduce to the system of equations considered in [3]. Eqs. (2.7)—(2.0) should be supplemented by boundary and initial conditions. As regards the boundary conditions for the functions $\mathbf{u}, \boldsymbol{\varphi}$, T and the initial conditions for the functions $\mathbf{u}, \boldsymbol{\varphi}$, they are the same as those given in [3].

For temperature the initial conditions are more diversified. Besides the condition $T(\mathbf{x}, 0) = h(\mathbf{x})$, we have to add the initial condition $\dot{T}(\mathbf{x}, 0) = l(\mathbf{x})$.

Let us perform the divergence operation on Eqs. (2.7) and (2.8). We shall obtain the wave equations

$$\Box_1 e_{kk} + \operatorname{div} \mathbf{P} = \nu \nabla^2 T, \tag{2.10}$$

$$\Box_3 \kappa_{kk} + \operatorname{div} \mathbf{M} = 0, \tag{2.11}$$

where

$$\Box_1 = (\lambda + 2\mu)\nabla^2 - \rho\partial_t^2, \qquad \Box_3 = (\beta + 2\gamma)\nabla^2 - 4a - I\partial_t^2,$$

$$e_{kk} = \operatorname{div} \mathbf{u}, \qquad \kappa_{kk} = \operatorname{div} \boldsymbol{\varphi}.$$

Eq. (2.10) is conjugated with the equation of heat conductivity.

Eliminating the temperature from Eqs. (2.9) and (2.10) we get:

$$(\Box_1 D - \nu\eta C \nabla^2) e_{kk} + \operatorname{div} D\mathbf{P} + \frac{\nu}{\kappa} \nabla^2 Q = 0. \tag{2.12}$$

Eq. (2.12) describes the dilatational wave, while Eq. (2.11) a microrotational one. As may be inferred from the analysis of the plane dilatational wave, the dilatational wave is subject to damping and dispersion. In an infinite micropolar medium the microrotational wave, Eq. (2.11), is not accompanied by the temperature field. The microrotational wave is undamped but it undergoes dispersion.

Let us perform on Eqs. (2.7) and (2.8) the operation of rotation. In this way we obtain the following equations

$$\Box_2 \boldsymbol{\Omega} + 2a \operatorname{rot} \boldsymbol{\Xi} = -\frac{1}{2} \operatorname{rot} \mathbf{P}, \qquad \boldsymbol{\Omega} = \frac{1}{2} \operatorname{rot} \mathbf{u},$$

$$\Box_4 \boldsymbol{\Xi} + 2a \operatorname{rot} \boldsymbol{\Omega} = -\frac{1}{2} \operatorname{rot} \mathbf{M}, \qquad \boldsymbol{\Xi} = \frac{1}{2} \operatorname{rot} \boldsymbol{\varphi}.$$

Eliminating from the equations the quantities $\boldsymbol{\Xi}$ and $\boldsymbol{\Omega}$, respectively, we arrive at the following formulae

$$(\Box_2 \Box_4 + 4a^2 \nabla^2) \boldsymbol{\Omega} = a \operatorname{rot} \operatorname{rot} \mathbf{M} - \frac{1}{2} \operatorname{rot} \Box_4 \mathbf{P}, \tag{2.13}$$

$$(\Box_2 \Box_4 + 4a^2 \nabla^2) \boldsymbol{\Xi} = a \operatorname{rot} \operatorname{rot} \mathbf{P} - \frac{1}{2} \operatorname{rot} \Box_2 \mathbf{M}. \tag{2.14}$$

Let us remark that in an infinite elastic space the waves $\boldsymbol{\Xi}$ and $\boldsymbol{\Omega}$ are not accompanied by the temperature field. The waves $\boldsymbol{\Xi}$ and $\boldsymbol{\Omega}$ undergo dispersion although they are not damped.

3. The potentials Φ, Ξ, Ψ, H

The system of Eqs. (2.7)—(2.9) may be separated in two ways. The first of them consists in decomposing the vectors **u** and **φ** into potential and solenoidal parts

$$\mathbf{u}=\operatorname{grad}\Phi+\operatorname{rot}\mathbf{\Psi}, \quad \operatorname{div}\mathbf{\Psi}=0, \tag{3.1}$$

$$\boldsymbol{\varphi}=\operatorname{grad}\Xi+\operatorname{rot}\mathbf{H}, \quad \operatorname{div}\mathbf{H}=0. \tag{3.2}$$

Introducing (3.1) and (3.2) into Eqs. (2.7)—(2.9) and expressing the quantities **P** and **M** in a similar manner, i.e.

$$\mathbf{P}=\rho(\operatorname{grad}\vartheta+\operatorname{rot}\boldsymbol{\chi}), \quad \operatorname{div}\boldsymbol{\chi}=0, \tag{3.3}$$

$$\mathbf{M}=I(\operatorname{grad}\sigma+\operatorname{rot}\boldsymbol{\eta}), \quad \operatorname{div}\boldsymbol{\eta}=0, \tag{3.4}$$

we obtain the following system of equations

$$\Box_1\Phi+\rho\vartheta=\nu T, \tag{3.5}$$

$$\Box_3\Xi+I\sigma=0, \tag{3.6}$$

$$\Box_2\mathbf{\Psi}+2a\operatorname{rot}\mathbf{H}+\rho\boldsymbol{\chi}=0, \tag{3.7}$$

$$\Box_4\mathbf{H}+2a\operatorname{rot}\mathbf{\Psi}+I\boldsymbol{\eta}=0, \tag{3.8}$$

$$DT-\eta C\nabla^2\Phi=-\frac{Q}{\kappa}. \tag{3.9}$$

Eq. (3.5) describes the longitudinal wave, while Eq. (3.6) represents the micro-rotational wave. The conjugate equations (3.7) and (3.8) describe, respectively, the modified transverse and twist waves. Eq. (3.9) is conjugated with Eq. (3.5). Performing on the above equations the operation of elimination, we obtain the following system of differential equations of hyperbolic type (wave equations):

$$(\Box_1 D-\nu\eta C\nabla^2)\Phi=-\rho D\vartheta-\frac{\nu}{\kappa}Q, \tag{3.10}$$

$$\Box_3\Xi=-I\sigma, \tag{3.11}$$

$$(\Box_2\Box_4+4a^2\nabla^2)\mathbf{\Psi}=2aI\operatorname{rot}\boldsymbol{\eta}-\rho\Box_4\boldsymbol{\chi}, \tag{3.12}$$

$$(\Box_2\Box_4+4a^2\nabla^2)\mathbf{H}=2a\rho\operatorname{rot}\boldsymbol{\chi}-I\Box_2\boldsymbol{\eta}, \tag{3.13}$$

$$(\Box_1 D-\nu\eta C\nabla^2)T=-\frac{1}{\kappa}\Box_1 Q-\rho\eta C\nabla^2\vartheta. \tag{3.14}$$

Let us now consider the propagation of waves in an infinite micropolar space.

First of all, let us assume the quantities $\boldsymbol{\chi}$, $\boldsymbol{\eta}$ and σ as equal to zero, the initial conditions of the functions $\mathbf{\Psi}$, **H** and Ξ being assumed homogeneous. In this case only the longitudinal wave will propagate in the micropolar space under the effect of action of heat sources and body forces $P=\rho\operatorname{grad}\vartheta$. This wave is damped and undergoes dispersion. The wave is accompanied by the temperature field described by Eq. (3.14).

Since

$$u_i=\Phi_{,i}, \quad \varphi_i=0, \quad e_{ji}=\Phi_{,ji}, \quad \kappa_{ji}=0$$

there is

$$\sigma_{ji}=\sigma_{(ji)}=2\mu(\Phi_{,ij}-\delta_{ji}\Phi_{kk})+\rho\delta_{ji}(\ddot{\Phi}-\vartheta). \quad \mu_{ji}=0. \tag{3.15}$$

The longitudinal waves induce a symmetric stress tensor. If the quantities Q, ϑ, η, χ are equal to zero and the initial conditions of the functions Φ, Ψ, H homogeneous, then — in the micropolar space — there is the microrotational wave Ξ.

Since

$$u_i=0, \quad \varphi_i=\Xi_{,i}, \quad e_{(ji)}=0, \quad e_{\langle ji\rangle}=-\varepsilon_{kij}\,\Xi_{,k}, \quad \kappa_{ji}=\Xi_{,ji},$$

there is

$$\begin{aligned}&\sigma_{(ji)}=0, \quad \sigma_{\langle ji\rangle}=2a\varepsilon_{kij}\,\Xi_{,k},\\ &\mu_{(ji)}=2\gamma\Xi_{,ji}+\beta\delta_{ji}\,\Xi_{,kk}, \quad \mu_{\langle ji\rangle}=0.\end{aligned} \tag{3.16}$$

The Ξ wave is indamped, it undergoes, however, dispersion.

If the quantities Q, ϑ and σ are equal to zero and if so are the initial conditions of the functions Φ and Ξ, then — in an infinite space — the $\mathbf{\Psi}$ and $\mathbf{H}$ waves propagate. They are undamped, subject to dispersion. They are not accompanied by the temperature field.

The character of the $\mathbf{\Psi}$ wave is that of a transverse wave (for $a\to 0$ it transforms into the transverse wave in Hooke's medium, as may be inferred from Eq. (3.7)). We shall call it the modified transverse wave. The $\mathbf{H}$ wave is a transverse rotational wave.

4. The stress functions Ψ, λ, ω

The second method of separating Eqs. (2.7)—(2.9) resembles that used by Galerkin [5] with respect to elastostatics and by Jacovache [6] with respect to the elastokinetics of Hooke's body.

In what follows we shall give the final result of the separation of Eqs. (2.7)—(2.9), the necessary proceedings being derived in [7].

Let us express the vectors $\mathbf{u}$ and $\boldsymbol{\varphi}$ by two vector functions $\boldsymbol{\psi}$ and $\boldsymbol{\lambda}$, respectively, while the temperature T will be expressed by the scalar function ω

$$\mathbf{u}=\Box_4 M\boldsymbol{\psi}-\operatorname{grad}\operatorname{div}N\boldsymbol{\psi}-2a\operatorname{rot}\Box_3\boldsymbol{\lambda}+\nu\operatorname{grad}\omega, \tag{4.1}$$

$$\boldsymbol{\varphi}=\Box_2\Box_3\boldsymbol{\lambda}-\operatorname{grad}\operatorname{div}\Theta\boldsymbol{\lambda}-2a\operatorname{rot}M\boldsymbol{\psi}, \tag{4.2}$$

$$T=\eta\operatorname{div}C\Omega\boldsymbol{\psi}+\Box_1\omega. \tag{4.3}$$

The following rotations have been introduced in the above equations

$$\begin{aligned}&M=\Box_1 D-\nu\eta C\nabla^2, && C=\partial_t(1+\tau\partial_t),\\ &N=D\Gamma-\nu\eta C\Box_4, && \Gamma=(\lambda+\mu-a)\Box_4-4a^2,\\ &\Theta=(\beta+\gamma-\varepsilon)\Box_2-4a^2, && \Omega=\Box_2\Box_4+4a^2\nabla^2.\end{aligned}$$

Introducing the expressions (4.1)—(4.3) into Eqs. (2.7)—(2.9) we obtain the following separated equations wherefrom we are able to determine the functions $\boldsymbol{\psi}, \boldsymbol{\lambda}, \omega$. They read:

$$(\square_2 \square_4 + 4a^2 \nabla^2)(\square_1 D - \nu\eta C \nabla^2)\boldsymbol{\psi} + \mathbf{P} = 0, \tag{4.4}$$

$$(\square_2 \square_4 + 4a^2 \nabla^2)\square_3 \boldsymbol{\lambda} + \mathbf{M} = 0, \tag{4.5}$$

$$(\square_1 D - \nu\eta C \nabla^2)\omega + \frac{Q}{\kappa} = 0. \tag{4.6}$$

The above equations are of particular importance for determining the basic solutions in an infinite micropolar space. This regards the determination of the functions $\mathbf{u}, \boldsymbol{\varphi}, T$ induced by the action of concentrated forces and moments and of heat sources.

Let us consider the homogeneous equations (4.4)—(4.6).

$$\Omega M \boldsymbol{\psi} = 0, \quad \Omega \square_3 \boldsymbol{\lambda} = 0, \quad M\omega = 0. \tag{4.7}$$

Let us compose the particular solutions of these equations of two parts.

In accordance with a theorem due to T. Boggio, the functions $\boldsymbol{\psi}', \boldsymbol{\lambda}', \boldsymbol{\psi}'', \boldsymbol{\lambda}''$ have to verify the following equations

$$M\boldsymbol{\psi}' = 0, \quad \Omega\boldsymbol{\psi}'' = 0, \quad \square_3 \boldsymbol{\lambda}' = 0, \quad \Omega\boldsymbol{\lambda}'' = 0. \tag{4.8}$$

Substituting (4.8) into Eqs. (4.1)—(4.3), we get

$$\mathbf{u} = \square_4 M\boldsymbol{\psi}'' - \operatorname{grad}\operatorname{div} N(\boldsymbol{\psi}' + \boldsymbol{\psi}'') - 2a \operatorname{rot} \square_3 \boldsymbol{\lambda}'' + \nu \operatorname{grad}\omega, \tag{4.9}$$

$$\boldsymbol{\varphi} = \square_2 \square_3 \boldsymbol{\lambda}'' - \operatorname{grad}\operatorname{div} \Theta(\boldsymbol{\lambda}' + \boldsymbol{\lambda}'') - 2a \operatorname{rot} M\boldsymbol{\psi}''. \tag{4.10}$$

$$T = \eta \partial_t \operatorname{div} \Omega \boldsymbol{\psi}' + \square_1 \omega. \tag{4.11}$$

Now, making use of the following relations

$$\Omega = \square_1 \square_4 - \nabla^2 \Gamma = \square_2 \square_3 - \nabla^2 \Theta, \quad \square_4 M - \nabla^2 N = D\Omega,$$

$$\operatorname{rot}\operatorname{rot}\mathbf{u} = \operatorname{grad}\operatorname{div}\mathbf{u} - \nabla^2 \mathbf{u},$$

we obtain

$$\mathbf{u} = -\operatorname{grad}(\operatorname{div} N\boldsymbol{\psi}' - \nu\omega) - \operatorname{rot}(\operatorname{rot} N\boldsymbol{\psi}'' + 2a \square_3 \boldsymbol{\lambda}''), \tag{4.12}$$

$$\boldsymbol{\varphi} = -\operatorname{grad}\operatorname{div}\Theta\boldsymbol{\lambda}' - \operatorname{rot}(\operatorname{rot}\Theta\boldsymbol{\lambda}'' + 2aN\boldsymbol{\psi}''). \tag{4.13}$$

Comparing (4.12) and (4.13) with (3.1) and (3.2), respectively, we arrive at the relations

$$\begin{aligned} \Phi &= -\operatorname{div} N\boldsymbol{\psi}' + \nu\omega, \\ \Xi &= -\operatorname{div}\Theta\boldsymbol{\lambda}', \\ \boldsymbol{\Psi} &= -\operatorname{rot} N\boldsymbol{\psi}'' - 2a \square_3 \boldsymbol{\lambda}'', \\ \mathbf{H} &= -\operatorname{rot}\Theta\boldsymbol{\lambda}'' - 2aM\boldsymbol{\psi}''. \end{aligned} \tag{4.14}$$

In this way we succeeded in establishing the connection between the elastic potentials Φ, Ξ, $\mathbf{\Psi}$, $\mathbf{H}$ and the generalized stress functions ψ, λ, ω.

We have only to check whether the functions Φ, Ξ, $\boldsymbol{\psi}$, $\mathbf{H}$ as given by the relations (4.14) verify the homogeneous wave equations (3.10)—(3.13). The answer is positive, as is easily seen from (4.8).

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REFERENCES

[1] S. Kaliski, *Wave equation of heat conduction*, Bull. Acad. Polon. Sci., Sér. Sci. Techn., **13** (1965), 211 [353].

[2] — , *Wave equations of thermoelasticity*, ibid., **13** (1965), 253 [409].

[3] W. Nowacki, *Couple-stresses in the theory of thermoelasticity. III*, ibid., **14** (1966), 505 [801].

[4] — , *Couple-stresses in the theory of elasticity*, Proc. JUTAM Symposia, Vienna, June 22—28, 1965, Springer, Wien—New York, 1968.

[5] B. Galerkin, C. R. Acad. Sci., Paris, **190** (1903), 1047.

[6] M. Jacovache, Bull. St. Acad. RPR, **1** (1949), 593.

[7] W. Nowacki, *On the completeness of stress functions in asymmetric elasticity*, Bull. Acad. Polon. Sci., Sér. Sci. Techn., **16** (1968), 309 [549].

С. КАЛИСКИЙ и В. НОВАЦКИЙ, **ВОЛНОВЫЕ УРАВНЕНИЯ ТЕРМО-МИКРО-УПРУГОСТИ**

В настоящей работе волновые уравнения термоупругости, выведенные в [1] и [2] для среды Гука, обобщаются на микрополярную среду Коссератов.

Приводятся основные соотношения и уравнения волновой термо-микроупругости, а также упругие потенциалы и функции напряжений. Кроме того, выведены соотношения между потенциалами и функциями напряжений.