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APPLIED MECHANICS
(ELASTICITY)

The "Second" Plane Problem of Micropolar Elasticity

by

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1. Introduction

Let us consider a micropolar body, elastic and homogeneous, isotropic and centrosymmetric. Under the effect of static loadings the body suffers deformation described by two vectors, namely the displacement vector $\mathbf{u}(\mathbf{x})$ and the rotational, vector $\boldsymbol{\varphi}(\mathbf{x})$.

Two asymmetric tensors — deformation tensor γ_{ji} and curvature-twist tensor κ_{ji} — are formed from these vectors. Both these tensors are connected with the $\mathbf{u}$ and $\boldsymbol{\varphi}$ vectors by the following relations

$$\gamma_{ji} = u_{i,j} - \epsilon_{kji}\varphi_k, \quad \kappa_{ji} = \varphi_{i,k}, \quad i, j, k = 1, 2, 3. \tag{1.1}$$

The state of stress is characterized by two asymmetric tensors, the tensor of force-stresses σ_{ji}, and that of couple-stresses, μ_{ji}. They are connected with the tensors γ_{ji}, κ_{ji} by the constitutive equations as below:

$$\sigma_{ji} = (\mu + a)\gamma_{ji} + (\mu - a)\gamma_{ij} + \lambda\gamma_{kk}\delta_{ji}, \quad \mu_{ji} = (\gamma + \varepsilon)\kappa_{ji} + (\gamma - \varepsilon)\kappa_{ij} + \beta\kappa_{kk}\delta_{ji}. \tag{1.2}$$

where the symbols μ and λ denote Lamé's constants, while a, β, γ and ε stand for other material constants.

Substituting (1.2) and (1.1) into the equations of equilibrium

$$\sigma_{ji,j} + X_i = 0, \quad \epsilon_{ijk}\sigma_{jk} + \mu_{ji,j} + Y_i = 0, \tag{1.3}$$

we obtain a system of equations in displacements and rotations [1—3]. It reads as follows:

$$\begin{aligned} &(\mu + a)\nabla^2\mathbf{u} + (\lambda + \mu - a)\,\mathrm{grad\,div}\,\mathbf{u} + 2a\,\mathrm{rot}\,\boldsymbol{\varphi} + \mathbf{X} = 0, \\ &(\gamma + \varepsilon)\nabla^2\boldsymbol{\varphi} - 4a\boldsymbol{\varphi} + (\beta + \gamma - \varepsilon)\,\mathrm{grad\,div}\,\boldsymbol{\varphi} + 2a\,\mathrm{rot}\,\mathbf{u} + \mathbf{Y} = 0. \end{aligned} \tag{1.4}$$

Here the symbols X_i and Y_i represent the components of body forces and body-couple vectors, respectively.

Eqs. (1.4) have to be supplemented by the appropriate boundary conditions. Under the assumption that on the surface A bounding the body the loadings (forces p_i and moments m_i) are prescribed, boundary conditions will take the form:

$$p_i=\sigma_{ji}\,n_j, \quad m_i=\mu_{ji}\,n_j, \quad i,j=1,2,3. \tag{1.5}$$

where the symbol n_j denotes the components of the unitary $\mathbf{n}$ normal to the surface A.

Now — passing to the plane problem of the theory of micropolar elasticity — let us assume that all causes and effects depend solely on two variables, x_1 and x_2. In this case the system of six equations (1.4) becomes partitioned into two systems of equations independent of each other.

The first of them, wherein the following components

$$\mathbf{u}\equiv(u_1,u_2,0), \quad \boldsymbol{\varphi}\equiv(0,0,\varphi_3) \tag{1.6}$$

of the vectors $\mathbf{u}$ and $\boldsymbol{\varphi}$ appear, has the form:

$$\begin{aligned}
&(\mu+a)\,\nabla_1^2 u_1+(\lambda+\mu-a)\,\partial_1 e+2a\,\partial_2\varphi_3+X_1=0,\\
&(\mu+a)\,\nabla_1^2 u_2+(\lambda+\mu-a)\,\partial_2 e-2a\,\partial_1\varphi_3+X_2=0,\\
&[(\gamma+\varepsilon)\,\nabla_1^2-4a]\,\varphi_3+2a\,(\partial_1 u_2-\partial_2 u_1)+Y_3=0.
\end{aligned} \tag{1.7}$$

In the above system of equations the following notations have been introduced:

$$e=\partial_1 u_1+\partial_2 u_2, \quad \nabla_1^2=\partial_1^2+\partial_2^2.$$

In the second system of equations, (1.9), the following components of the $\mathbf{u}$ and $\boldsymbol{\varphi}$ vectors appear

$$\mathbf{u}\equiv(0,0,u_3), \quad \boldsymbol{\varphi}\equiv(\varphi_1,\varphi_2,0). \tag{1.8}$$

The system has the form:

$$\begin{aligned}
&[(\gamma+\varepsilon)\,\nabla_1^2-4a]\,\varphi_1+(\beta+\gamma-\varepsilon)\,\partial_1\kappa+2a\,\partial_2 u_3+Y_1=0,\\
&[(\gamma+\varepsilon)\,\nabla_1^2-4a]\,\varphi_2+(\beta+\gamma-\varepsilon)\,\partial_2\kappa-2a\,\partial_1 u_3+Y_2=0,\\
&(\mu+a)\,\nabla_1^2 u_3+2a\,(\partial_1\varphi_2-\partial_2\varphi_1)+X_3=0,
\end{aligned} \tag{1.9}$$

where $\kappa=\partial_1\varphi_1+\partial_2\varphi_2$.

The deformation of the body described by the system of Eqs. (1.9) will be called the "second" plane problem of micropolar elasticity. It will be the object of our subsequent considerations. We shall attempt to give the general solution of the system of Eqs. (1.9) making use of appropriately chosen functions (elastic potentials). We shall give also the singular solution of this system of equations for the infinite elastic space.

2. The solution of the system of Eqs. (1.9)

It follows from (1.8) and from the definition of deformation tensors (1.1) that — as regards our "second" plane state of deformation — there appear the following components of the tensors γ_{ji} and κ_{ji}:

$$\begin{aligned}
&\gamma_{31}=-\varphi_2, \quad \gamma_{32}=\varphi_1, \quad \gamma_{13}=\partial_1 u_3+\varphi_2, \quad \gamma_{23}=\partial_2 u_3-\varphi_1,\\
&\kappa_{11}=\partial_1\varphi_1, \quad \kappa_{22}=\partial_2\varphi_2, \quad \kappa_{12}=\partial_1\varphi_2, \quad \kappa_{21}=\partial_2\varphi_1.
\end{aligned} \tag{2.1}$$

Applying the relations (1.2), we see that the components of the force- and couple-stress tensors, i.e. σ_{ji} and μ_{ji}, are described by the matrices

$$\boldsymbol{\sigma}=\begin{vmatrix} 0 & 0 & \sigma_{13} \\ 0 & 0 & \sigma_{32} \\ \sigma_{31} & \sigma_{32} & 0 \end{vmatrix}, \qquad \boldsymbol{\mu}=\begin{vmatrix} \mu_{11} & \mu_{12} & 0 \\ \mu_{21} & \mu_{22} & 0 \\ 0 & 0 & \mu_{33} \end{vmatrix}, \tag{2.2}$$

where

$$\begin{aligned} &\sigma_{13}=(\mu+a)\,\partial_1 u_3+2a\,\varphi_2, \qquad \sigma_{31}=(\mu-a)\,\partial_1 u_3-2a\,\varphi_2, \\ &\sigma_{23}=(\mu+a)\partial_2 u_3-2a\,\varphi_1, \qquad \sigma_{32}=(\mu-a)\,\partial_2 u_3+2a\,\varphi_1, \\ &\mu_{11}=2\gamma\partial_1 \varphi_1+\beta(\partial_1\varphi_1+\partial_2\varphi_2), \qquad \mu_{22}=2\gamma\,\partial_2\varphi_2+\beta(\partial_1\varphi_1+\partial_2\varphi_2), \\ &\mu_{12}=(\gamma+\varepsilon)\,\partial_1\varphi_2+(\gamma-\varepsilon)\,\partial_2\varphi_1, \qquad \mu_{21}=(\gamma+\varepsilon)\,\partial_2\varphi_1+(\gamma-\varepsilon)\,\partial_1\varphi_2, \\ &\mu_{33}=\beta(\partial_1\varphi_1+\partial_2\varphi_2). \end{aligned} \tag{2.3}$$

The equations of equilibrium, Eqs. (1.4), become here reduced to the three equations

$$\begin{aligned} &\sigma_{23}-\sigma_{32}+\partial_1\mu_{11}+\partial_2\mu_{21}+Y_1=0, \\ &\sigma_{31}-\sigma_{13}+\partial_1\mu_{12}+\partial_2\mu_{22}+Y_2=0, \\ &\partial_1\sigma_{13}+\partial_2\sigma_{23}+X_3=0. \end{aligned} \tag{2.4}$$

The solution of the system of homogeneous equations (1.9) will be searched in the form

$$\varphi_1=\partial_1\Phi+\partial_2\Psi, \qquad \varphi_2=\partial_2\Phi-\partial_1\Psi, \tag{2.5}$$

where Φ and Ψ denote the elastic potentials.

Introducing (2.5) into Eqs. (1.9), we get the following system of equations:

$$\begin{cases} \partial_1[(\beta+2\gamma)\nabla_1^2-4a]\,\Phi+\partial_2[((\gamma+\varepsilon)\nabla_1^2-4a)\,\Psi+2au_3]=0, \\ \partial_2[(\beta+2\gamma)\nabla_1^2-4a]\,\Phi-\partial_1[((\gamma+\varepsilon)\nabla_1^2-4a)\,\Psi+2au_3]=0, \\ (\mu+a)\nabla_1^2 u_3-2a\nabla_1^2\Psi=0, \end{cases} \tag{2.6}$$

or

$$\begin{cases} -\partial_1(\nu^2\nabla_1^2-1)\,\Phi=\dfrac{\mu}{\mu+a}\partial_2(l^2\nabla_1^2-1)\,\Psi, \\ \partial_2(\nu^2\nabla_1^2-1)\,\Phi=\dfrac{\mu}{\mu+a}\partial_1(l^2\nabla^2-1)\,\Psi, \\ \nabla_1^2 u_3-\dfrac{2a}{\mu+a}\nabla_1^2\Psi=0, \end{cases} \tag{2.7}$$

where

$$\nu^2=\frac{\beta+2\gamma}{4a}, \qquad l^2=\frac{(\gamma+\varepsilon)(\mu+a)}{4a\mu}.$$

Let us remark that the relations $(2.7)_1$ and $(2.7)_2$ represent the Cauchy—Riemann conditions for the functions $P=(\nu^2 \nabla_1^2-1)\ \Phi$ and $Q=(l^2 \nabla_1^2-1)\ \Psi$. Taking profit of the relations $(2.7)_1$ and $(2.7)_2$ we obtain the following equations:

$$\nabla_1^2 (\nu^2 \nabla_1^2-1)\ \Phi=0, \tag{2.8}$$

$$\nabla_1^2 (l^2 \nabla_1^2-1)\ \Psi=0. \tag{2.9}$$

Solving the above equations — boundary conditions being taken into account — we obtain the functions Φ and Ψ searched for.

Introducing the functions Φ and Ψ into the relations (2.3) we obtain:

$$\begin{gathered}\sigma_{13}=2a\,\partial_2\,\Phi, \quad \sigma_{23}=-2a\partial_1\,\Phi,\\ \sigma_{31}=-2a\partial_2\,\Phi+\frac{4a\mu}{\mu+a}\partial_1\,\Psi, \qquad \sigma_{32}=2a\partial_1\,\Psi+\frac{4a\mu}{\mu+a}\,\partial_2\,\Psi,\end{gathered} \tag{2.10}$$

and

$$\begin{gathered}\mu_{11}=(\beta+2\gamma)\,\nabla_1^2\,\Phi-2\gamma\partial_2^2\,\Phi+2\gamma\partial_1\,\partial_2\,\Psi,\\ \mu_{22}=(\beta+2\gamma)\,\nabla_1^2\,\Phi-2\gamma\partial_1^2\,\Phi-2\gamma\,\partial_1\,\partial_2\,\Psi,\\ \mu_{33}=\beta\nabla_1^2\,\Phi, \quad \mu_{12}=-(\gamma+\varepsilon)\,\nabla_1^2\,\Psi+2\gamma\,\partial_2^2\,\Psi+2\gamma\,\partial_1\,\partial_2\,\Phi,\\ \mu_{21}=(\gamma+\varepsilon)\,\nabla_1^2\,\Psi-2\gamma\partial_1^2\,\Psi+2\gamma\,\partial_1\,\partial_2\,\Phi.\end{gathered} \tag{2.11}$$

Let us remark that

$$\mu_{11}+\mu_{22}=2(\gamma+\beta)\,\nabla_1^2\,\Phi, \quad \mu_{21}-\mu_{12}=2\varepsilon\nabla_1^2\,\Psi. \tag{2.12}$$

Now, we can easily derive from the relations (2.1) the following conditions of compatibility

$$\begin{gathered}\partial_1^2\,\kappa_{22}+\partial_2^2\,\kappa_{11}=\partial_1\,\partial_2(\kappa_{12}+\kappa_{21}),\\ \partial_2^2\,\kappa_{12}-\partial_1^2\,\kappa_{21}=\partial_1\,\partial_2(\kappa_{22}-\kappa_{11}),\\ \partial_1\,(\gamma_{32}+\gamma_{23})=\partial_2\,(\gamma_{13}+\gamma_{31}).\end{gathered} \tag{2.13}$$

The above equations may be written in stresses, use being made of the relations (1.2)

$$\begin{gathered}\partial_2^2\,\mu_{11}+\partial_1^2\,\mu_{22}-\frac{\beta}{2(\gamma+\beta)}\,\nabla_1^2(\mu_{11}+\mu_{22})=\partial_1\,\partial_2(\mu_{12}+\mu_{21}),\\ (\partial_2^2-\partial_1^2)\,(\mu_{12}+\mu_{21})+\frac{\gamma}{\varepsilon}\,\nabla_1^2(\mu_{12}-\mu_{21})=\partial_1\,\partial_2(\mu_{22}-\mu_{11}),\\ \partial_1(\sigma_{23}+\sigma_{32})=\partial_2(\sigma_{13}+\sigma_{31}).\end{gathered} \tag{2.14}$$

Putting (2.10) and (2.11) into (2.14) we demonstrate that Eqs. (2.14) are satisfied identically.

It follows from the relations (2.14) — the equations of equilibrium (2.4) being taken into account — that the following equations hold true:

$$(\nu^2\,\nabla_1^2-1)\,(\mu_{11}+\mu_{22})=0, \tag{2.15}$$

$$(l^2\,\nabla_1^2-1)\,(\mu_{12}-\mu_{21})=0. \tag{2.16}$$

Introducing the dependencies (2.12) into the Eqs. (2.15) and (2.16), we return to the differential equations (2.8) and (2.9). The last equations have to be supplemented by boundary conditions. If on the edge of an infinite cylinder with the axis directed along the x_3 coordinate the loadings p_3 and the moments m_1 and m_2 are prescribed, the boundary conditions will assume the form:

$$p_3=\sigma_{j3}\, n_j, \quad m_1=\mu_{j1}\, n_j, \quad m_2=\mu_{j2}\, n_j, \quad j=1,2. \tag{2.17}$$

Let us consider the case of an elastic half-space $x_1 \geqslant 0$ loaded on the edge $x_1=0$ by the forces $p_3=f(x_2)$ acting within the plane x_2, x_3. At $\mathbf{n}=(1,0,0)$ boundary conditions are reduced in this case to the following ones:

$$\sigma_{13}(0,x_2)=f(x_2), \quad \mu_{11}(0,x_2)=0, \quad \mu_{12}(0,x_2)=0. \tag{2.18}$$

In view of the relations (2.10) and (2.11) the boundary conditions are expressed in the form of the functions Φ, Ψ.

The solutions of Eqs. (2.8) and (2.9) will be written in the form of Fourier integrals

$$\Phi=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(Ae^{-\zeta x_1}+Be^{-\eta x_1})\, e^{-i\zeta x_2}\, d\zeta, \tag{2.19}$$

$$\Psi=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(Ce^{-\zeta x_1}+De^{-\rho x_1})\, e^{-i\zeta x_2}\, d\zeta, \tag{2.20}$$

where

$$\eta=\left(\zeta^2+\frac{1}{\nu^2}\right)^{1/2}, \qquad \rho=\left(\zeta^2+\frac{1}{l^2}\right)^{1/2} \qquad i=\sqrt{-1}.$$

Since we know three boundary conditions — Eqs. (2.17) — and the Cauchy—Riemann relations — Eqs. $(2.7)_{1,2}$ — we are able to determine the constants A, B, C, D. Basing on (2.7), we can write

$$A=-\frac{i\mu}{\mu+a}C. \tag{2.21}$$

The functions Φ and Ψ being known, we determine the displacement u_3 — from Eq. $(2.7)_3$ — and the rotations φ_1 and φ_2 — from the formulae (2.5).

Let us now consider the case discussed above within the frame-work of the classical theory of elasticity. Assuming the function $u_3(x_1, x_2)$ as the sole unknown, we obtain the symmetric tensor of stresses with the components

$$\sigma_{13}=\sigma_{31}=\mu\partial_1 u_3, \quad \sigma_{23}=\sigma_{32}=\mu\partial_2 u_3. \tag{2.22}$$

Introducing (2.22) into the known equation $(2.4)_3$ we obtain the Poisson's equation which makes it possible to determine the quantity u_3. It reads as follows:

$$\mu\nabla^2 u_3+X_3=0. \tag{2.23}$$

In the case of an elastic half-space $x_1 \geqslant 0$ loaded in the plane $x_1=0$ by the forces $p_3(x_2)=f(x_2)$ we have to solve the homogeneous differential equation (2.23) with the boundary condition

$$\sigma_{13}(0, x_2)=\mu\, |\partial_1 u_3|_{x_1=0}=f(x_2). \tag{2.24}$$

The integral

$$u_3=-\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{\tilde{f}(\zeta)}{\zeta}e^{-\zeta x_1-i\zeta x_2}\,d\zeta. \tag{2.25}$$

is the solution of this problem.

3. Singular solutions in an infinite elastic space

Let us consider the non-homogeneous equations (1.8). The moments and the body forces will be expressed by the relations

$$Y_1=J(\partial_1 \sigma+\partial_2 \chi), \qquad Y_2=J(\partial_2 \sigma-\partial_1 \chi). \tag{3.1}$$

The above expression is equivalent to the decomposition of the vector $\mathbf{Y}\equiv \equiv(Y_1, Y_2, 0)$ into the potential and solenoidal parts. Introducing (3.1) into the system of Eqs. (1.9), we get:

$$\begin{gathered}(\mu+a)\nabla_1^2 u_3-2a\nabla_1^2\Psi+X_3=0,\\ \partial_1\{[(\beta+2\gamma)\nabla_1^2-4a]\Phi+J\sigma\}=-\partial_2\{[(\gamma+\varepsilon)\nabla_1^2-4a]\Psi+2au_3+J\chi\},\\ \partial_2\{[(\beta+2\gamma)\nabla_1^2-4a]\Phi+J\sigma\}=\partial_1\{[(\gamma+\varepsilon)\nabla_1^2-4a]\Psi+2au_3+J\chi\}.\end{gathered} \tag{3.2}$$

After some simple transformations we arrive at the following differential equations

$$\nabla_1^2(\nu^2\nabla_1^2-1)\Phi=-\frac{1}{4a}(\partial_1 Y_1+\partial_2 Y_2), \tag{3.3}$$

$$\nabla_1^2(l^2\nabla_1^2-1)\Psi=\frac{1}{2\mu}X_3-\frac{\mu+a}{4\mu a}(\partial_2 Y_1-\partial_1 Y_2), \tag{3.4}$$

$$\nabla_1^2(l^2\nabla_1^2-1)u_3=-\frac{1}{\mu}(\kappa^2\nabla_1^2-1)X_3-\frac{1}{2\mu}(\partial_2 Y_1-\partial_1 Y_2), \tag{3.5}$$

where $\kappa^2=\dfrac{\gamma+\varepsilon}{4a}$.

The solutions of Eqs. (3.3)—(3.5) will be obtained by applying the double exponential Fourier transformation. In this way we get

$$\Phi=\frac{i}{8a\nu^2\pi}\iint_{-\infty}^{\infty}\frac{(\zeta_1\tilde{Y}_1+\zeta_2\tilde{Y}_2)}{\zeta^2\left(\zeta^2+\frac{1}{\nu^2}\right)}e^{-i\zeta_k x_k}\,d\zeta_1\,d\zeta_2, \tag{3.6}$$

$$\Psi=\frac{1}{4\pi\mu l^2}\iint_{-\infty}^{\infty}\left[\tilde{X}_3+\frac{i(a+\mu)}{2a}(\zeta_2\tilde{Y}_1-\zeta_1\tilde{Y}_2)\right]\frac{e^{-i\zeta_k x_k}}{\zeta^2\left(\zeta^2+\frac{1}{l^2}\right)}\,d\zeta_1\,d\zeta_2, \tag{3.7}$$

$$(3.8)\qquad u_3 = \frac{1}{2\pi\mu l^2}\int\limits_{-\infty}^{\infty}\!\!\int \left[\kappa^2\left(\zeta^2\frac{1}{\kappa^2}\right)\tilde{X}_3 + \frac{i}{2}(\zeta_2\tilde{Y}_1 - \zeta_1\tilde{Y}_2)\right]\frac{e^{-i\zeta_k x_k}}{\zeta^2\left(\zeta^2+\frac{1}{l^2}\right)}\,d\zeta_1\,d\zeta_2,$$

where $\zeta^2=\zeta_1^2+\zeta_2^2$.

Let us consider the particular case of action of the concentrated force $X_3=$ $=\delta(x_1)\,\delta(x_2)\,P$ distributed uniformly along the x_3 axis. Then, $\tilde{X}_3=\frac{P}{2\pi}$, $\tilde{Y}_1=\tilde{Y}_2=0$. From Eqs. (3.6)—(3.8) we obtain

$$(3.9)\qquad \begin{aligned} &\Phi=0,\\ &\Psi=\frac{P}{8\pi^2\mu l^2}\int\limits_{-\infty}^{\infty}\!\!\int\frac{e^{-i\zeta_k x_k}}{\zeta^2\left(\zeta^2+\frac{1}{l^2}\right)}\,d\zeta_1\,d\zeta_2,\\ &u_3=\frac{P\kappa^2}{4\pi^2\mu l^2}\int\limits_{-\infty}^{\infty}\!\!\int\frac{\left(\zeta^2+\frac{1}{\kappa^2}\right)e^{-i\zeta_k x_k}}{\zeta^2\left(\zeta^2+\frac{1}{l^2}\right)}\,d\zeta_1\,d\zeta_2. \end{aligned}$$

The above integrals do not exist as improper integrals; nor can we assign to them the principal value after Cauchy. We can, however, separate out of them the part called "finite part", [4, 5]. Applying the method advanced in [5], we get

$$(3.10)\qquad \begin{aligned} &\Phi=0,\\ &\Psi=\frac{P}{4\pi\mu}\left(K_0\left(\frac{r}{l}\right)-\ln r\right),\qquad r=\sqrt{x_1^2+x_2^2},\\ &u_3=2\Psi-\frac{P\kappa^2}{2\pi\mu l^2}K_0\left(\frac{r}{l}\right), \end{aligned}$$

where $K_0(z)$ is the modified Bessel's function of the third kind. The function Ψ being known, we are able to determine the rotations φ_1 and φ_2 from the formulae

$$(3.11)\qquad \varphi_1=\partial_2\Psi,\qquad \varphi_2=-\partial_1\Psi.$$

Let us consider also the case of action of the concentrated moment $Y_1=$ $=M\delta(x_1)\,\delta(x_2)$ with the vector directed along the x_3 axis. In this case there is $\tilde{Y}_1=\frac{M}{2\pi}$, $\tilde{Y}_2=0$, $\tilde{Y}_3=0$. From (3.6)—(3.8) we obtain

$$(3.12)\qquad \Phi=\frac{M}{16\pi^2 a\nu^2}\frac{\partial\Omega}{\partial x_1},\qquad \Psi=-\frac{M(\mu+a)}{16\pi^2 l^2\mu a}\frac{\partial\Omega}{\partial x_2},\qquad u_3=\frac{2a}{\mu+a}\Psi,$$

where

$$\Omega=\int\limits_{-\infty}^{\infty}\!\!\int\frac{e^{-i\zeta_k \kappa_k}}{\zeta^2\left(\zeta^2+\frac{1}{b}\right)}\,d\zeta_1\,d\zeta_2=2\pi\nu^2\left[K_0\left(\frac{r}{b}\right)-\ln r\right].$$

Let us now pass to the classical theory of elasticity. We have the following equation:

$$\mu \nabla_1^2 u_3 + X_3 = 0. \tag{3.13}$$

The function

$$u_3 = \frac{1}{2\pi\mu} \int_{-\infty}^{\infty}\int \frac{\tilde{X}_3 e^{-i\zeta_k x_k}}{\zeta^2} d\zeta_1 d\zeta_2. \tag{3.14}$$

is the solution of Eq. (3.13). Hence,

$$u_3(x_1, x_2) = -\frac{1}{2\pi\mu} \int_A\int X_3(\eta_1, \eta_2) \ln[(x_1-\eta_1)^2+(x_2-\eta_2)^2]^{1/2} d\eta_1 d\eta_2. \tag{3.15}$$

The displacement u_3 being known, we are able to determine the stresses taking profit of the formulae:

$$\sigma_{13} = \mu\delta_1 u_3, \quad \sigma_{23} = \mu\delta_2 u_3.$$

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В. НОВАЦКИЙ, „ВТОРАЯ" ПЛОСКАЯ ПРОБЛЕМА МИКРОПОЛЯРНОЙ УПРУГОСТИ

В работе рассматривается двухмерная задача микрополярной упругости, в которой вектор смещения $\mathbf{u} \equiv (0, 0, u_3)$ и вектор вращения $\boldsymbol{\varphi} \equiv (\varphi_1, \varphi_2, 0)$ зависят единственно от переменных x_1 и x_2. Вводится потенциалы Φ, Ψ и при их использовании система трех дифференциальных уравнений проблемы (1.9) сводится к простым эллиптическим уравнениям (2.8) и (2.9). Даются основные решения дифференциальных уравнений для бесконечного упругого пространства.