BULLETIN
DE
L'ACADÉMIE POLONAISE DES SCIENCES

SÉRIE DES SCIENCES TECHNIQUES

Volume XVII, Numéro 6

VARSOVIE 1969

1. BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Série des sciences techniques
Volume XVII, No. 6 — 1969

APPLIED MECHANICS

Integral Theorems for the Wave-type Heat Conductivity Equation

by

S. KALISKI and W. NOWACKI

Presented on January 13, 1969

1. Introductory remarks

The classical heat equation is known to be parabolic it is characterized by infinite propagation velocity of thermal perturbations — a fairly paradoxical feature — from the physical point of view. In most practical cases, however, this paradoxical feature does not lead, to erroneous quantitative results, because a thermal perturbation dies out rapidly with increasing distance from the source. Nonetheless, in the case of marked non-stationariness or rapidly varying thermal processes the effect of finite propagation velocity of thermal perturbations may be essential for qualitative and sometimes also quantitative results. This concerns in particular the coupling effects between the thermal field and the elastic and electromagnetic fields. This is why, beginning with the dynamic (nonstationary) modification of the Fourier law and modified thermodynamical considerations, a wave-type hyperbolic heat equation was derived in [1]. In [2] a method for measuring the propagation velocity of thermal perturbation was suggested on the grounds of Čerenkov's effe t.

The classical wave equation is connected with a set of integral theorems established already very long ago. In the present communication an analogous set of integral theorems will be established for the wave-type heat equation [1].

2. The reciprocity theorem

Let us consider the generalized wave-type, heat equation as derived in [1]:

$$\left(\nabla^2 - \frac{1}{\varkappa}\,\partial_t - \beta^2\,\partial_t\right)\theta(x, t) = -\frac{Q(x, t)}{\varkappa}. \tag{2.1}$$

In this equation the wave term $\beta^2\,\partial_t^2\,\theta$ is involved in addition to those of the classical heat equation.

The notations in (2.1) are as follows: $\theta(x, t)$ — temperature; $\varkappa = \lambda/c_\varepsilon$, where λ is the coefficient of heat conductivity and c_ε the specific heat at constant strain; $\beta^2 = \tau c_\varepsilon/\lambda$, where τ is the relaxation time; $Q = \dfrac{W}{c_\varepsilon}$, where W is the quantity of heat produced per volume and time units.

For $\beta \to 0$ Eq. (2.1) becomes the classical heat equation. We shall establish first the reciprocity theorem and, on this basis, a number of other theorems constituting generalizations of those known from the theory of heat conduction, [3], and the theory of wave equation.

Let us consider two sets of causes and effects. The causes will include the action of heat sources Q, Q', the action of the mean temperature at the heat boundary A $(\theta, \theta', x \in A)$ or at the heat flow through that boundary $\left(\lambda_0 \frac{\partial \theta}{\partial n}, \lambda_0 \frac{\partial \theta'}{\partial n}, x \in A\right)$. To the effects belong the temperatures $(\theta, \theta', x \in V)$ at the point x and instant t. The first set of causes and effects is described by the differential Eq. (2.1). To this equation we add the boundary conditions

$$\theta(x,t) = h(x,t), \quad x \in A_u; \quad \lambda_0 \frac{\partial \theta(x,t)}{\partial n} = k(x,t), \quad x \in A_\sigma \tag{2.2}$$

and the initial conditions

$$\theta(x,0) = f(x), \quad \dot{\theta}(x,0) = g(x), \quad x \in V, \quad t = 0. \tag{2.3}$$

The symbol A_u denotes the part of the surface A, where the temperature is prescribed and A_σ the part of the surface A where the heat flow is prescribed.

The second set of causes and effects satisfies the equation

$$\left(\nabla^2 - \frac{1}{\varkappa}\partial_t - \beta^2 \partial_t^2\right)\theta'(x,t) = -\frac{Q'(x,t)}{\varkappa} \tag{2.4}$$

with boundary conditions

$$\theta'(x,t) = h'(x,t), \quad x \in A_u, \quad \lambda_0 \frac{\partial \theta'(x,t)}{\partial n} = k'(x,t), \quad x \in A_\sigma \tag{2.5}$$

and initial conditions

$$\theta'(x,0) = f'(x), \quad \dot{\theta}'(x,0) = g'(x). \tag{2.6}$$

In further considerations it will be more convenient to use Eqs. (2.1) and (2.4) after subjecting them to the Laplace integral transformation.

On introducing the notation

$$\bar{\theta}(x,p) = \mathcal{L}[\theta(x,t)] = \int_0^\infty \bar{e}^{pt}\, \theta(x,t)\, dt, \quad \text{etc}.$$

Eqs. (2.1) and (2.4) can be expressed — taking into consideration the initial conditions (2.3) and (2.6) — in the form

$$\left(\nabla^2 - \frac{p}{\varkappa} - \beta^2 p^2\right)\bar{\theta}(x,p) = -\frac{\bar{Q}}{\varkappa} - \left(\frac{1}{\varkappa} + \beta^2 p\right) f(x) - g(x)\,\beta^2; \tag{2.7}$$

$$\left(\nabla^2 - \frac{p}{\varkappa} - \beta^2 p^2\right)\bar{\theta}'(x,p) = -\frac{\bar{Q}'}{\varkappa} - \left(\frac{1}{\varkappa} + \beta^2 p\right) f'(x) - g'(x)\,\beta^2. \tag{2.8}$$

Let us multiply Eq. (2.7) by $\bar{\theta}'$ and Eq. (2.8) by $\bar{\theta}$, and subtract one of these equations from the other and integrate over the region V. Applying Green transformations, we have

$$\int_A \left(\bar{\theta}' \frac{\partial \bar{\theta}}{\partial n} - \bar{\theta} \frac{\partial \bar{\theta}'}{\partial n}\right) dA = -\frac{1}{\varkappa} \int_V (\bar{Q}\bar{\theta}' - \bar{Q}'\bar{\theta})\, dV + \\ -\left(\frac{1}{\varkappa} + p\beta^2\right) \int_V (\bar{\theta}' f - \bar{\theta} f')\, dV - \beta^2 \int_V (\bar{\theta}' g - \bar{\theta} g')\, dV. \tag{2.9}$$

Performing the inverse Laplace transformation on Eq. (2.9) and making use of the convolution theorem, we obtain the reciprocity theorem in the form

$$\int_0^t d\tau \int_A \left[\theta'(x, t-\tau) \frac{\partial \theta(x, \tau)}{\partial n} - \theta(x, \tau) \frac{\partial \theta'(x, t-\tau)}{\partial n}\right] dA(x) + \\ + \frac{1}{\varkappa} \int_0^t d\tau \int_V [\theta'(x, t-\tau)\, Q(x, \tau) - \theta(x, \tau)\, Q'(x, t-\tau)]\, dV(x) + \\ + \frac{1}{\varkappa} \int_V [(f(x) + \varkappa\beta^2 g(x))\, \theta'(x, t) - (f'(x) + \varkappa\beta^2 g'(x))\, \theta(x, t)]\, dV(x) + \\ + \beta^2 \frac{\partial}{\partial t} \int_V [f(x)\, \theta'(x, t) - f'(x)\, \theta(x, t)]\, dV(x) = 0. \tag{2.10}$$

In the case of an infinite body Eq. (2.10) will be essentially simplified, because if we assume that the heat sources and the initial conditions concern a bounded region, the surface integrals vanish.

Let us consider the particular case in which a concentrated source $Q = \delta(x + - \xi)\, \delta(t)$ acts at a point ξ of an infinite body and a concentrated source $Q' = \delta(x - \eta)\, \delta(t)$ acts at a point η of that body. From Eq. (2.10) we obtain

$$\int_0^t d\tau \int_V \delta(x, \xi)\, \delta(\tau)\, \theta'(x, \eta, t-\tau)\, dV(x) - \int_0^t dt \int_V \delta(x-\eta)\, \delta(t-\tau)\, \theta(x, \xi, \tau)\, dV(x).$$

Hence

$$\theta'(\xi, \eta, t) = \theta(\eta, \xi, t). \tag{2.11}$$

In the particular case in which the heat sources and the temperatures vary harmonically in time, that is if

$$\theta(x, t) = \theta^*(x)\, \bar{e}^{i\omega t}, \quad Q'(x, t) = \theta'^*(x)\, \bar{e}^{i\omega t}$$

the reciprocity relation becomes

$$\int_A \left(\theta'^* \frac{\partial \theta^*}{\partial n} - \theta^* \frac{\partial \theta'}{\partial n}\right) dA = -\frac{1}{\varkappa} \int_V (Q^* \theta'^* - Q'^* \theta^*)\, dV(x). \tag{2.12}$$

3. The modified retarded potential

Let us consider an infinite body, wherein some sources $Q(x, t)$ located within a bounded region V' are acting. The homogeneous initial conditions are assumed. The temperature θ satisfies Eq. (2.1). It will be determined, from the reciprocity theorem (2.10) assuming $\theta' = G(x, \xi, t)$, where the function G verifies the equation

$$\left(\nabla^2 - \frac{1}{\varkappa}\partial_t - \beta^2 \partial_t^2\right) G(x, \xi, t) = -\frac{1}{\varkappa}\delta(x-\xi)\,\delta(t), \tag{3.1}$$

and expresses the temperature produced by the action of the instantaneous concentrated heat source acting at the point ξ. It is assumed that the function G satisfies the homogeneous initial conditions. On subjecting (3.1) to Laplace integral transformation we find

$$\left[\nabla^2 - p\left(\frac{1}{\varkappa} + \beta^2 p\right)\right] \bar{G}(x, \xi, p) = -\frac{1}{\varkappa}\delta(x-\xi). \tag{3.2}$$

The particular integral of this equation is

$$\bar{G}(x, \xi, p) = \frac{1}{4\pi\varkappa R}\exp\left[-R\sqrt{\frac{p}{\varkappa}(1+\varkappa\beta^2 p)}\right], \tag{3.3}$$

where $R = |x-\xi|$.

The inverse transformation of the function (3.3) is, [4],

$$G(x, \xi, t) = \frac{1}{8\pi\varkappa^2\beta} e^{-\frac{t}{2\varkappa\beta^2}} \frac{I_1\left(\frac{1}{2\varkappa\beta^2}\sqrt{t^2 - R^2\beta^2}\right)}{\sqrt{t^2-R^2\beta^2}} H(t-R\beta) + \frac{1}{4\pi\varkappa R} e^{-\frac{t}{2\varkappa\beta^2}}\delta(t-R\beta). \tag{3.4}$$

From the reciprocity theorem (2.10) it follows, for $Q' = \delta(x-\xi)\,\delta(t)$, that

$$\begin{gathered} Q' = G(x, \xi, t); \quad f = g = f' = g' = 0, \\ \theta(\xi, t) = \int_0^t d\tau \int_V Q(x, t-\tau)\, G(x, \xi, \tau)\, dV(x). \end{gathered} \tag{3.5}$$

Hence, by virtue of (3.4), we have

$$\theta(\xi, t) = \frac{1}{4\pi\varkappa}\int_V \frac{Q(x, t-R\beta)}{R(x,\xi)} e^{-\frac{R}{2\varkappa\beta}} dV(x) + \frac{1}{8\pi\varkappa^2\beta}\int_0^t d\tau \int_V Q(x, t-\tau) \times e^{-\frac{\tau}{2\varkappa\beta^2}} \frac{I_1\left(\frac{1}{2\varkappa\beta^2}\sqrt{\tau^2-R^2\beta^2}\right)}{\sqrt{\tau^2-R^2\beta^2}} H(\tau - R\beta)\, dV(x). \tag{3.6}$$

This expression contains the retarded argument $t - R\beta\ (R \leqslant t/\beta)$. It occurs directly in the first integral without summing up with respect to time. Thus, we are concerned here with a retarded potential.

For $\beta \to 0$ we find from (3.3)

$$\bar{G}(x, \xi, p) = \frac{1}{4\pi\varkappa R} \exp\left(-R\sqrt{\frac{p}{\varkappa}}\right). \tag{3.7}$$

Hence

$$G(x, \xi, t) = \frac{1}{8\pi\varkappa\sqrt{\pi\tau^3}} \exp\left(-\frac{R^2}{4\varkappa t}\right) \tag{3.8}$$

and, on the basis of (3.4),

$$\theta(x, t) = \frac{1}{8\pi\varkappa} \int_0^t d\tau \int_V \frac{Q(x, t-\tau)}{\sqrt{\pi t^3}} \exp\left(-\frac{R^2}{4\pi\tau}\right) dV(x), \tag{3.9}$$

which coincides with the known expression of the theory of the classical heat equation.

From (2.10) we can also determine the temperature $\theta(x, t)$ resulting from the action of the initial conditions Eq. (2.3). For $Q' = \delta(x - \xi)\,\delta t$, $Q = 0$, $\theta' = G(x, \xi, t)$, $f' = g' = 0$ we have

$$\theta(\xi, t) = \frac{1}{\varkappa} \int_V [f(x) + \beta^2 \varkappa g(x)]\, G(x, \xi, t)\, dV(x) + \beta^2 \varkappa \int_V f(x) \frac{\partial G(x, \xi, t)}{\partial t} dV(x), \tag{3.10}$$

where the function G is given by Eq. (3.4).

Let us apply now the reciprocity theorem (2.10) to the body bounded by the surface A assuming that the initial conditions are homogeneous and that the boundary conditions (2.2) are prescribed on the boundary of A. Another set of causes and effects will be the instantaneous and concentrated source of heat $Q' = \delta(x - \xi)\,\delta(t)$ and the resulting temperature $\theta' = \hat{G}(x, \xi, t)$ distributed over the body. Green's function has to satisfy the differential equation

$$\left(\nabla^2 - \frac{1}{\varkappa}\partial_t - \beta^2 \partial_t^2\right) \hat{G}(x, \xi, t) = -\frac{\delta(x-\xi)\,\delta(t)}{\varkappa} \tag{3.11}$$

with the homogeneous initial conditions and the homogeneous boundary conditions

$$\hat{G}(x, \xi, t) = 0 \quad \text{on} \quad A_u \quad \text{and} \quad \frac{\partial \hat{G}(x, \xi, t)}{\partial n} = 0 \quad \text{on} \quad A_\sigma. \tag{3.12}$$

Eq. (2.10), in which we assume $g = f = g' = f' = 0$, $Q' = \partial(x - \xi)\,\delta(t)$, $\theta' = \hat{G}(x, \xi, t)$, will take the form

$$\begin{aligned} \theta(\xi, t) = & \int_0^t d\tau \int_V Q(x, \tau)\, \hat{G}(x, \xi, t-\tau)\, dV(x) + \\ & + \varkappa \int_0^t \left[\int_{A_\sigma} \hat{G}(x, \xi, t-\tau) \frac{k(x, \tau)}{\lambda_0} dA(x) - \int_{A_u} \frac{\partial \hat{G}(x, \xi, t-\tau)}{\partial n} h(x, \tau)\, dA(x)\right]. \end{aligned} \tag{3.13}$$

The function $\hat{G}$ chosen in an appropriate manner enables us to obtain the temperature by integration. It is assumed that the functions $Q(x, \tau)$ $x \in V$ and $k(\boldsymbol{x}, \tau)$, $\boldsymbol{x} \in A_\sigma$, $h(\boldsymbol{x}, \tau)$, $\boldsymbol{x} \in A_u$ are prescribed

If the temperature $\theta(\boldsymbol{x}, t)$ is prescribed over the entire surface A, then

$$(3.14) \quad \theta(\boldsymbol{\xi}, t) = \int_0^t dt \int_V Q(x, \tau) \overset{0}{G}(x, \boldsymbol{\xi}, t-\tau)\, dV(x) + \\ - \varkappa \int_0^t d\tau \int_V \theta(\boldsymbol{x}, \tau) \frac{\partial \overset{0}{G}(\boldsymbol{x}, \boldsymbol{\xi}, t-\tau)}{\partial n} dV(x),$$

where the function $\overset{0}{G}(\boldsymbol{x}, \boldsymbol{\xi}, t)$ has to satisfy Eq. (3.10) with homogeneous initial conditions and homogeneous boundary condition

$$\overset{0}{G}(\boldsymbol{x}, \boldsymbol{\xi}, t) = 0 \quad \text{on} \quad A.$$

4. The analogue of the Kirchhoff theorem

Let us consider an internal region B^+, the function θ and $\dfrac{\partial\theta}{\partial n}$ being prescribed on its boundary A. Our aim is to determine the function θ at the point $\boldsymbol{\xi} \in B^+$ by expressing it as a surface integral over A using the Green's function G for the infinite region and the function $\theta, \dfrac{\partial\theta}{\partial n}$ on A.

It is assumed that there is no heat source in the region B^+ $(Q = 0)$, and that the initial conditions are homogeneous $(f = g = f' = g' = 0)$. By substituting in addition $\theta' = G(\boldsymbol{x}, \boldsymbol{\xi}, t)$, $Q' = \delta(\boldsymbol{x} - \boldsymbol{\xi})\,\delta(t)$ into Eq. (2.9), we have

$$(4.1) \quad \bar{\theta}(\boldsymbol{\xi}, p) = \varkappa \int_A \left[\bar{G}(\boldsymbol{x}, \boldsymbol{\xi}, p) \frac{\partial \bar{\theta}(\boldsymbol{x}, p)}{\partial n} - \bar{\theta}(\boldsymbol{x}, \boldsymbol{\xi}, p) \frac{\partial \bar{G}(\boldsymbol{x}, \boldsymbol{\xi}, p)}{\partial n} \right] dA(\boldsymbol{x}).$$

Let us present $\bar{G}(\boldsymbol{x}, \boldsymbol{\xi}, p)$ of Eq. (3.3) in the form

$$(4.2) \qquad \bar{G}(\boldsymbol{x}, \boldsymbol{\xi}, p) = \frac{1}{4\pi R\varkappa} [e^{-R\beta p} + p\bar{F}(\boldsymbol{x}, \boldsymbol{\xi}, p)],$$

where $\bar{F} = \dfrac{1}{p}\left[\exp\left(-R\beta\sqrt{p\left(\dfrac{1}{\beta^2\varkappa} + p\right)}\right) - e^{-R\beta p}\right]$.

Thus, Eq. (4.1) will take the form

$$(4.3) \quad \bar{\theta}(\boldsymbol{\xi}, p) = \frac{1}{4\pi} \int_A \left[\left(\frac{e^{-R\beta p}}{R} \right) \frac{\partial\bar{\theta}}{\partial n} - \bar{\theta} \frac{\partial}{\partial n} \left(\frac{e^{-R\beta p}}{R} \right) \right] dA(\boldsymbol{x}) + \\ + \frac{1}{4\pi} \int_V \left\{ \left(\frac{p\bar{F}}{R} \right) \frac{\partial\bar{\theta}}{\partial n} - \bar{\theta} \frac{\partial}{\partial n} \left(\frac{p\bar{F}}{R} \right) \right\} dA(\boldsymbol{x}).$$

Performing the inverse Laplace transformation of (4.3) we find, after some simple rearrangements,

$$(4.4)\qquad \theta(\xi, t) = -\frac{1}{4\pi}\int_A \left\{[\theta(x, t)]\frac{\partial}{\partial n}\left(\frac{1}{R}\right) - \frac{\beta}{R}\frac{\partial R}{\partial n}\left[\frac{\partial\theta(x, t)}{\partial t}\right] + \right.$$
$$\left. -\frac{1}{R}\left[\frac{\partial\theta(x, t)}{\partial n}\right]\right\} dA(x) + \frac{1}{4\pi}\int_0^t d\tau \int_A \left\{\frac{1}{R}\frac{\partial F(x, \tau)}{\partial \tau}\frac{\partial\theta(x, t-\tau)}{\partial n} + \right.$$
$$\left. -\theta(x, t-\tau)\frac{\nu}{\partial n}\left[\frac{1}{R}\frac{\partial F(x, \tau)}{\partial\tau}\right]\right\} dA(x), \qquad \xi \in B^+,$$

where [4]:

$$F(x, t) = \begin{cases} 0, & t \leqslant R\beta, \\ -\dfrac{R}{2\beta\varkappa}\displaystyle\int_t^\infty e^{-\frac{t}{\beta^2\varkappa}} \frac{I_1\left(\dfrac{1}{\beta^2\varkappa}\sqrt{t' - R^2\beta^2}\right)}{\sqrt{t'^2 - R^2\beta^2}}\, dt', & t > R\beta \end{cases}$$

and

$$[\theta(x, t)] = \theta(x, t - R\beta), \left[\frac{\partial\theta(x, t)}{\partial t}\right] = \frac{\partial\theta(x, t - R\beta)}{\partial t}, \left[\frac{[\partial\theta(x, t)}{\partial n}\right] = \frac{\partial\theta(x, t - R\beta)}{\partial n}.$$

The first integral in (3.4) has a form analogous to that of the Kirchhoff integral in the classical wave equation, [5]. If $\frac{1}{\varkappa} = 0$ in the left-hand member of Eq. (2.1), that is if we disregard the diffusion term in Eq. (2.1), we have $F = 0$ and what is left is the Kirchhoff integral.

In the particular case of a harmonically varying temperature field, Eq. (4.1) becomes

$$(4.5)\qquad \theta^*(\xi) = \varkappa\int_A \left\{G^*(x, \xi)\frac{\partial\theta^*(x)}{\partial n} - \theta^*(x)\frac{\partial G^*(x, \xi)}{\partial n}\right\} dA(x), \qquad \xi \in B^+,$$

where the function G^* has the form

$$(4.6)\qquad G^* = \frac{1}{4\pi R\varkappa}\exp\left[-R\sqrt{\frac{i\omega}{\varkappa} - \beta\,\omega^2}\right]$$

or

$$(4.7)\qquad G^* = \frac{1}{4\pi\varkappa R}\exp\left\{-\frac{R\omega}{2\varkappa}[1 + i(\beta^2\omega\varkappa + \right.$$
$$\left. + \sqrt{1 + \varkappa^2\beta\omega^2})]\frac{1}{\sqrt{\dfrac{\beta^2\omega^2}{2} + \dfrac{1}{2}\sqrt{\beta^4\omega^4 + \omega^2/\varkappa^2}}}\right\}.$$

Eq. (4.5) is the analogue of the Helmholtz integral for the classical wave equation. Let us observe that if the diffusion term in Eq. (2.1) is dropped, we find from (4.6), $\tilde{G}^* = \frac{1}{4\pi R\varkappa}\exp(-R\beta i\omega)$.

5. Another form of the reciprocity theorem

Let us consider Eq. (2.7) assuming the homogeneous initial conditions. The second set of causes and effects will be that of the wave phenomenon as described by the equation

$$(\nabla^2 - \beta^2\,\partial_t^2)\,\theta'(x, t) = -\frac{Q'(x, t)}{\varkappa}. \tag{5.1}$$

Let us perform on (5.1) the Laplace transformation and assume that the initial conditions are homogeneous. We find

$$(\nabla^2 - p^2\beta^2)\,\bar{\theta}'(x, p) = -\frac{\bar{Q}'(x, p)}{\varkappa}. \tag{5.2}$$

Let us multiply (2.7) by $\bar{\theta}'$ and Eq. (5.2) by $\bar{\theta}$, subtract the two equations from one another and integrate over the body. Applying the Green transformation, we have

$$\int\limits_A \left(\bar{\theta}'\frac{\partial\theta}{\partial n} - \bar{\theta}\,\frac{\partial\bar{\theta}'}{\partial n}\right) dA - \frac{p}{\varkappa}\int\limits_V \bar{\theta}\bar{\theta}'\,dV = -\frac{1}{\varkappa}\int\limits_V (\bar{Q}\bar{\theta}' - \bar{Q}'\bar{\theta})\,dV. \tag{5.3}$$

Let us assume now that $\theta' = F(x, \xi, t)$ the function F satisfying the equation

$$(\nabla^2 - \beta^2\,\partial_t)\,F(x, \xi, t) = -\frac{\delta(x-\xi)\,\delta(t)}{\varkappa} \tag{5.4}$$

with the homogeneous initial conditions and the following boundary conditions:

$$F(x, \xi, t) = 0 \quad \text{on} \quad A_u, \qquad \frac{\partial F(x, \xi, t)}{\partial n} = 0 \quad \text{on} \quad A_\sigma, \quad A = A_u + A_\sigma. \tag{5.5}$$

Eq. (4.3) takes the form

$$\int\limits_{A_\sigma} \bar{F}\frac{\partial\bar{\theta}}{\partial n}\,dA - \int\limits_{A_u}\frac{\partial\bar{F}}{\partial n}\,dA - \frac{p}{\varkappa}\int\limits_V \bar{\theta}\bar{F}dV = -\frac{1}{\varkappa}\int\limits_V \bar{Q}\bar{F}dV + \frac{1}{\varkappa}\bar{\theta}(\xi, p).$$

After performing the inverse Laplace transformation, we obtain

$$\theta(\xi, t) = M(\xi, t) - \int\limits_0^t d\tau \int\limits_V \theta(x, \tau)\frac{\partial F(x, \xi, t-\tau)}{\partial\tau}\,dV(x), \tag{5.6}$$

where

$$M(\xi, t) = \varkappa \int_0^t d\tau \int_{A_\sigma} F(x, \xi, t-\tau) \frac{\partial \theta (x\tau)}{\partial n} dA(x) -$$

$$-\varkappa \int_0^t d\tau \int_{A_u} \theta(x, \tau) \frac{\partial F(x, \xi, t-\tau)}{\partial n} dA(x) + \int_0^t d\tau \int_V Q(x, \tau) F(x, \xi, t-\tau)\, dA(x).$$

We have obtained an integral equation for determining the function $\theta(\xi, t)$. In the case of an infinite body the surface integrals in the expression for $M(\xi, t)$ vanish and the function F takes the form

$$F(x, \xi, t) = \frac{1}{4\pi R \varkappa} \delta(t - R\beta). \tag{5.7}$$

In the case of a temperature field varying harmonically in time we have

$$\theta^*(\xi) = M^*(\xi) + i\omega \int_V \theta^*(x) F^*(x, \xi)\, dV(x), \tag{5.8}$$

where

$$M^*(\xi) = \int_V \theta^* F^* dV + \varkappa \left\{ \int_{A_\sigma} F^* \frac{\partial \theta^*}{\partial n} dA - \int_{A_u} \theta^* \frac{\partial F^*}{\partial n} dA \right\}.$$

We have obtained the Fredholm integral equation for the determination of the temperature amplitude θ^*. For the infinite body the surface integrals vanish in the expression for $M^*(\xi)$ and $F^* = \frac{1}{4\pi R\varkappa} \exp(-R\beta i\omega)$.

The set of integral theorems presented above constitute a generalization of the classical theorems for the wave equation and the parabolic heat equation to the case of the generalized, wave-type heat equation.

DEPARTMENT OF VIBRATIONS, INSTITUTE OF BASIC TECHNICAL PROBLEMS, POLISH ACADEMY OF SCIENCES
(ZAKŁAD BADANIA DRGAŃ, INSTYTUT PODSTAWOWYCH PROBLEMÓW TECHNIKI, PAN)

REFERENCES

[1] S. Kaliski, *Wave equation of heat conduction*, Bull. Acad. Polon. Sci., Sér. sci. techn., **13** (1965), 211 [353].

[2] — , *Čerenkov generation of thermal waves by means of slowed down electromagnetic waves*, Arch. Mech. Stos., **18** (1966), No. 6; *Irreversible aspects of continuum mechanics*, IUTAM Symp. Vienna 1966, Springer Vlg., 1968.

[3] W. Nowacki, *Thermoelastic waves motion*, Proc. Vibr. Probl., **9** (1968), No. 4.

[4] V. A. Ditkin, P. T. Kuznietsov, *Spravochnik po operatsionnomu ischisleniyu* [in Russian], [*Handbook of operational calculus*], Moscow, 1951.

[5] G. Kirchhoff, Berliner Sitzungsberichte (1882), 641, Annalen der Physic, **18** (1883), 663.

С. КАЛИСКИЙ и В. НОВАЦКИЙ, **ИНТЕГРАЛЬНЫЕ ТЕОРЕМЫ ДЛЯ ВОЛНОВОГО УРАВНЕНИЯ ТЕПЛОПРОВОДНОСТИ**

В работе приводятся основные теоремы взаимности и интегральные теоремы для обобщенного волнового уравнения теплопроводности [1]. Упомянутые теоремы в классических случаях переходят в известные интегральные теоремы для волнового уравнения и параболического уравнения теплопроводности.