BULLETIN
DE
L'ACADÉMIE POLONAISE DES SCIENCES

SÉRIE DES SCIENCES TECHNIQUES

Volume XVII, Numéro 4

VARSOVIE 1969

BULLETIN DE L'ACADÉMIE
6. POLONAISE DES SCIENCES
Série des sciences techniques
Volume XVII, No. 4 — 1969

APPLIED MECHANICS
(THERMOELASTICITY)

Formulae for Overall Thermoelastic Deformation in a Micropolar Body

by

W. NOWACKI

Presented on February 25, 1969

1. Introduction

In this paper we shall be concerned with a simply connected body. The body is assumed to be micropolar, isotropic, homogeneous and centrosymmetric. Under the effect of external forces and heating the body becomes deformed. This deformation is characterized by two asymmetric tensors, namely the deformation tensor γ_{ji} and the curvature — twist tensor $\varkappa_{ji}$. The following relations hold [1]—[4]:

$$\gamma_{ji} = u_{i,j} - \varepsilon_{kji}\,\omega_k, \quad \varkappa_{ji} = \omega_{i,j} \tag{1.1}$$

Here the symbol **u** stands for the displacement vector, **φ** is the rotation vector, while ϵ_{kji} is the well known Cartesian ϵ — tensor (its components are $+1$ (-1) is i, j, k is an even (odd) permutation of 1, 2, 3; they are zero, if two subsripts are equal).

The state of stress in the body is defined by two asymmetric tensors: The force-stress-tensor and the couple-stress-tensor. The relations between the state of stress and the state of strain are given by the following formulae:

$$\sigma_{ji} = (\mu+\alpha)\,\gamma_{ji} + (\mu-\alpha)\,\gamma_{ij} + (\lambda\gamma_{kk} - \nu\theta)\,\delta_{ij}, \tag{1.2}$$

$$\mu_{ji} = (\gamma+\varepsilon)\,\varkappa_{ji} + (\gamma-\varepsilon)\,\varkappa_{ij} + \beta\varkappa_{kk}\,\delta_{ij}. \tag{1.3}$$

Here the symbols $\mu, \lambda, \alpha, \beta, \gamma, \varepsilon$ represent the material constants. $\theta = T - T_0$, where T denotes the absolute temperature. The above relations should be supplemented by the equation of equilibrium

$$\sigma_{ji,j} + X_i = 0, \tag{1.4}$$

$$\varepsilon_{ijk}\,\sigma_{jk} + \mu_{ji,j} + Y_i = 0, \quad i, j, k = 1, 2, 3. \tag{1.5}$$

In the above equations the symbol **X** denotes the vector of body forces, while **Y** stands for the vector of body couples. The loadings p_i and the moments m_i appearing at the surface A of the body are connected by the following relations

$$p_i = \sigma_{ji}\,n_j, \quad m_i = \mu_{ji}\,n_j, \tag{1.6}$$

where **n** is the unary vector of the normal to the surface A.

In this paper we are going to determine the integrals

$$I_1 = \int_V \gamma_{kk}\, dV, \qquad I_2 = \int_V \varkappa_{kk}\, dV, \tag{1.7}$$

$$\gamma_{kk} = \operatorname{div} \mathbf{u}, \qquad \varkappa_{kk} = \operatorname{div} \boldsymbol{\omega}.$$

The first integral denotes the increment of the volume of the simply connected body due to its deformation. The second integral defines the mean value of the quantitty div $\boldsymbol{\omega}$ in the simply connected body. Since

$$\int_V \operatorname{div} \boldsymbol{\omega}\, dV = \int_A \boldsymbol{\omega} \cdot \mathbf{n} dA = \int_A \omega_n\, dA, \tag{1.8}$$

where ω_n is the projection of the vector $\boldsymbol{\omega}$ onto the direction of the normal $\mathbf{n}$; the integral $I_2 = \int_V \omega_n\, dA$ represents also the mean value of the quantity ω_n on the surface A bounding the body.

2. Application of the theorem of reciprocity of work

In order to determine the quantities I_1, I_2 we shall make use of the theorem of reciprocity of works for the static problem [5]

$$\int_A (p_i u_i' + m_i \omega_i')\, dA + \int_V (X_i u_i' + Y_i \omega_i')\, dV + \nu \int_V \gamma_{kk}' \theta dV = \tag{2.1}$$

$$= \int_A (p_i' u_i + m_i' \omega_i)\, dA + \int_V (X_i' u_i + Y_i' \omega_i)\, dV + \nu \int_V \gamma_{kk} \theta'\, dV.$$

Here $\nu = 3K\alpha_t$, where $K = \lambda + \frac{2}{3}\mu$ is the module of compressibility and α_t is the coefficient of linear thermal dilatability. In Eq. (2.1) we have two systems of causes and effects. To the causes in the first system we shall assign the external forces p_i, m_i, X_i, Y_i and temperature θ, while to the effects: displacements u_i and rotations ω_i. The causes and effects of the second system will be marked with "primes".

In order to determine the integral $I_1 = \int_V \gamma_{kk}\, dV$ we assume that the second system of causes refers to the overall unary tension in isothermic state. Consequently, we have to assume

$$X_i' = 0, \quad Y_i' = 0, \quad m_i' = 0, \quad \theta' = 0, \quad p_i' = 1 \cdot n_i. \tag{2.2}$$

Since in this case there is

$$\sigma_{ij}' = 1\delta_{ij}, \quad \gamma_{ij}' = \frac{1}{2\mu + 3\lambda}\delta_{ij}, \quad \varkappa_{ij}' = 0, \quad \mu_{ij}' = 0, \tag{2.3}$$

we have

$$u_i' = \frac{x_i}{3K}, \qquad \omega_i' = 0.$$

From Eq. (2.1) we obtain

$$\int_A n_i u_i \, dA = \frac{1}{3K}\left\{\int_A p_i x_i \, dA + \int_V X_i x_i \, dV\right\} + 3a_t \int_V \theta dV. \tag{2.4}$$

As

$$\int_A n_i u_i \, dA = \int_V u_{k,k} \, dV = I_1$$

we get

$$I_1 = \int_V \gamma_{kk} \, dV = \frac{1}{3K}\left\{\int_A p_i x_i \, dA + \int_V X_i x_i \, dV\right\} + 3a_t \int_V \theta dV. \tag{2.4'}$$

The changes in the volume of the body depend here solely on the loadings p_i, body forces X_i and temperature θ. The moments m_i and body couples Y_i do not exert any influence on the changes of the volume of the body.

The changes of the volume of the body may be described in a particularly simple form if no external forces are acting. In this case there is

$$I_1 = 3a_t \int_V \theta dV. \tag{2.5}$$

Let us remark that the increase of the volume of the body depends only on the distribution of heat in the body and on the coefficient of thermal dilatation.

Formulae (2.4′) and (2.5) are identical with those known from the classical theory of thermoelasticity [6], [7]. Let us consider now the integral $\int_V \sigma_{kk} \, dV$. In virtue of Eq. (1.2) we have

$$\int_V \sigma_{kk} \, dV = 3K \int_V \gamma_{kk} \, dV - 9a_t \int_V \theta dV.$$

Taking into account (2.4′), we get finally

$$\int_V \sigma_{kk} \, dV = \int_A p_i x_i \, dA + \int_V X_i x_i \, dV. \tag{2.6}$$

If no external loadings are acting (the body, being, however, heated), there is

$$\int_V \sigma_{kk} \, dV = 0. \tag{2.6'}$$

Eq. (2.6′) is identical with Hieke's formula [8] in classical thermoelasticity.

Let us return to the theorem on reciprocity (2.1), assuming that the body was subjected to overall unary twisting. We assume:

$$X_i' = 0, \quad Y_i' = 0, \quad p_i' = 0, \quad \theta' = 0, \tag{2.7}$$

and

$$m_i' = 1 \cdot n_i, \quad \mu_{ji}' = 1 \cdot \delta_{ij}.$$

Substituting the above expressions into Eq. (2.1), we get

$$\int_V \varkappa_{kk}\,dV = \int_A (p_i u_i' + m_i \omega_i')\,dA + \int_V (X_i u_i' + Y_i \omega_i')\,dV + \nu \int_V \theta \gamma_{kk}'\,dV. \tag{2.8}$$

Here use was made of Gauss' transformation

$$\int_A \omega_i n_i\,dA = \int_V \omega_{k,k}\,dV = \int_V \varkappa_{kk}\,dV.$$

Solving Eqs. (1.3) with respect to $\varkappa_{ij}'$, we obtain

$$\varkappa_{ij}' = \frac{1}{3\Omega}\delta_{ij}, \qquad \Omega = \beta + \frac{2}{3}\gamma.$$

Hence,

$$\omega_i' = \frac{x_i}{3\Omega}. \tag{2.9}$$

Since $\sigma_{ij}' = 0$, there is

$$\gamma_{(ij)}' = \frac{1}{2}(u_{i,j}' + u_{j,i}') = 0, \qquad \gamma_{\langle ij\rangle}' = \frac{1}{2}(u_{i,j}' - u_{j,i}') - \varepsilon_{kji}\omega_k' = 0.$$

From the above formula we have

$$u_{i,j}' = \epsilon_{kji}\varphi_k', \qquad u_{i,j}' = \frac{1}{3\Omega}\epsilon_{kj\cdot}x_k, \qquad u_{j,j}' = \gamma_{kk}' = 0. \tag{2.10}$$

Introducing these values into (2.8), we obtain

$$\int_V \varkappa_{kk}\,dV = \frac{1}{3\Omega}\left\{\int_V Y_i x_i\,dV + \int_A m_i x_i\,dA\right\} + \int_V X_i u_i'\,dV + \int_A p_i u_i'\,dA. \tag{2.11}$$

Making use of the equation of equilibrium (1.4), of the relations

$$\gamma_{j,i}' = u_{ij}' - \varepsilon_{kji}\omega_k'$$

and of the constitutive equation for σ_{ji} and σ_{ji}', we shall transform the expression

$$\int_A p_i u_i'\,dA + \int_V X_i u_i'\,dV + \nu \int_V \theta \gamma_{kk}'\,dV = P.$$

As a result we obtain

$$P = \int_V [\sigma_{ji} u_{i,j}' + \nu\theta\gamma_{kk}']\,dV = \int_V [\sigma_{ji}(\gamma_{ji}' + \varepsilon_{kji}\omega_k') + \nu\theta\gamma_{kk}']\,dV =$$

$$= \int_V \sigma_{ji}'\gamma_{ji}\,dV + \int_V \sigma_{ji}\varepsilon_{kji}\omega_k'\,dV = \int_V \sigma_{ji}'\gamma_{ji}\,dV + \frac{1}{3\Omega}\int_V \varepsilon_{kji}x_k\sigma_{ji}\,dV.$$

Since $\sigma_{ji}' = 0$. Eq. (2.8) — wherein the term P appears — will take the form

$$I_2 = \int_V \varkappa_{kk}\,dV = \frac{1}{3\Omega}\left\{\int_A m_i x_i\,dA + \int_V Y_i x_i\,dV + \int_V \varepsilon_{kji}x_k\sigma_{ji}\,dV\right\}. \tag{2.12}$$

Let us consider now the integral $\int_V \mu_{kk}\, dV$. Making use of the relations (1.3) we get

$$\mu_{kk} = 3\Omega\varkappa_{kk}.$$

Introducing the above expression into the formula (2.12) we obtain

$$\int_V \mu_{kk}\, dV = \int_A m_i\, x_i\, dA + \int_V Y_i\, x_i\, dV + \int_V \varepsilon_{kji}\, x_k\, \sigma_{ji}\, dV. \tag{2.13}$$

The mean value of the quantity μ_{kk} depends on external loadings.

In a particular case — for $m_i = 0$, $Y_i = 0$ — we have

$$\int_V \mu_{kk}\, dV = \int_V \varepsilon_{kji}\, x_k\, \sigma_{ji}\, dV. \tag{2.14}$$

This integral becomes equal to zero if the stress tensor σ_{ji} is symmetric.

Besides, we may derive the formulae (2.4′) and (2.12) in another way also. Let us namely multiply the equations of equilibrium (1.4) and (1.5) by x_i and integrate them over the region of the body. We obtain

$$\int_V (\sigma_{ji,j} + X_i)\, x_i\, dV = 0, \tag{2.15}$$

$$\int_V (\varepsilon_{ijk}\, \sigma_{jk} + \mu_{ji,j} + Y_i)\, x_i\, dV = 0. \tag{2.16}$$

We shall transform the above equations making use of the theorem on divergence and of the relations (1.6)

$$\int_A p_i\, x_i\, dA + \int_V X_i\, x_i\, dV = \int_V \sigma_{kk}\, dV, \tag{2.17}$$

$$\int_A m_i\, x_i\, dA + \int_V Y_i\, x_i\, dV + \int_V \varepsilon_{ijk}\, x_i\, \sigma_{jk}\, dV = \int_V \mu_{kk}\, dV. \tag{2.18}$$

It is seen at once that the formula (2.18) is identical with (2.13) and (2.17) with (2.6). Taking advantage of the expressions

$$\sigma_{kk} = 3(K\gamma_{kk} - \nu\theta), \quad \mu_{kk} = 3\Omega\varkappa_{kk}, \tag{2.19}$$

derived from the relations (1.2) and (1.3), we obtain successively

$$I_1 = \frac{1}{3K}\left\{\int_A p_i\, x_i\, dA + \int_V X_i\, x_i\, dV\right\} + 3\alpha_t \int_V \theta dV, \tag{2.20}$$

$$I_2 = \frac{1}{3\Omega}\left\{\int_A m_i\, x_i\, dA + \int_V Y_i\, x_i\, dV + \int_V \varepsilon_{kji}\, x_k\, \sigma_{ji}\, dV\right\}. \tag{2.21}$$

As this way to the final result proved to be quite simple, it may be readily used when considering a more complex structure of the body.

Let us consider an isotropic and homogeneous — although not centrosymmetric body. The relations between the state of stress and state of strain will be given by the formulae [9]:

$$\sigma_{ji} = (\mu+\alpha)\gamma_{ji}+(\mu-\alpha)\gamma_{ij}+\chi\varkappa_{ji}+\sigma\varkappa_{ij}+(\lambda\gamma_{kk}+\varrho\varkappa_{kk}-\eta\theta)\delta_{ij}, \tag{2.22}$$

$$\mu_{ji} = (\gamma+\varepsilon)\varkappa_{ji}+(\gamma-\varepsilon)\varkappa_{ij}+\chi\gamma_{ji}+\sigma\gamma_{ij}+(\beta\varkappa_{kk}+\varrho\gamma_{kk}-\zeta\theta)\delta_{ij}. \tag{2.23}$$

Here $\mu, \lambda, \alpha, \beta, \gamma, \varepsilon, \chi, \sigma, \varrho,$ are material constants. Contracting the tensors σ_{ji} and μ_{ji}, we obtain

$$\sigma_{kk} = 3(K\gamma_{kk}+\Gamma\varkappa_{kk}-\eta\theta), \tag{2.24}$$

$$\mu_{kk} = 3(\Omega\varkappa_{kk}+\Gamma\gamma_{kk}-\zeta\theta), \quad 3\Gamma = \chi+\sigma+3\varrho. \tag{2.25}$$

Introducing the above expressions into the formulae (2.17) and (2.18), we obtain

$$I_1+\frac{\Gamma}{K}I_2 = \frac{1}{3K}\left\{\int_A p_i x_i\,dA+\int_V X_i x_i\,dV\right\}+3\eta\int_V \theta dV, \tag{2.26}$$

$$I_2+\frac{\Gamma}{\Omega}I_1 = \frac{1}{3\Omega}\left\{\int_A m_i x_i\,dA+\int_V Y_i x_i\,dV+\int_V \varepsilon_{kji}x_k\sigma_{ji}\,dV\right\}+3\zeta\int_V \theta dV. \tag{2.27}$$

We have to solve these equations with respect I_1 and I_2. It is obvious that the increment of the volume of the body $I_1 = \Delta V$ as well as the mean value of the integral $I_2 = \int_V \varkappa_{kk}\,dV$ depend on all the component of external forces and the temperature θ.

DEPARTMENT OF MECHANICS OF CONTINUOUS MEDIA, INSTITUTE OF BASIC TECHNICAL PROBLEMS, POLISH ACADEMY OF SCIENCES
(ZAKŁAD MECHANIKI OŚRODKÓW CIĄGŁYCH, INSTYTUT PODSTAWOWYCH PROBLEMÓW TECHNIKI, PAN)

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В. НОВАЦКИЙ, **ФОРМУЛЫ СПЛОШНОЙ ТЕРМОУПРУГОЙ ДЕФОРМАЦИИ В МИКРОПОЛЯРНОМ ТЕЛЕ**

В настоящей работе выведены формулы на изменение объема односвязного, упругого, микрополярного тела, а также на среднее значение дивергенции оборота в таком теле. Упомянутые формулы получены по двум методам: путем применения теоремы о взаимности работ, а затем непосредственно, исходя из уравнений равновесия тела.