BULLETIN
DE
L'ACADÉMIE POLONAISE DES SCIENCES

SÉRIE DES SCIENCES TECHNIQUES

Volume XVII, Numéro 4

VARSOVIE 1969

BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Série des sciences techniques
Volume XVII, No. 4 — 1969

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APPLIED MECHANICS

(ELASTICITY)

Generalized Love's Functions in Micropolar Elasticity

by

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Presented on February 25, 1969

1. Introduction

Within the framework of the classical theory of elasticity, the Love's function [1] verifying the biharmonic equations plays an important role in solving axi-symmetric problems.

In this Note we shall attempt to obtain an analogous function (strictly speaking, two analogous functions) for axi-symmetric problems in micropolar elasticity.

The state of strain is described in micropolar elasticity by the following system of equations of equilibrium in displacements and rotations [2]:

$$(\mu+\alpha)\,\nabla^2\mathbf{u}+(\lambda+\mu-\alpha)\,\mathrm{grad}\,\mathrm{div}\,\mathbf{u}+2\alpha\,\mathrm{rot}\,\boldsymbol{\varphi}+\mathbf{X}=0, \tag{1.1}$$

$$(\gamma+\varepsilon)\,\nabla^2\boldsymbol{\varphi}+(\gamma+\beta-\varepsilon)\,\mathrm{grad}\,\mathrm{div}\,\boldsymbol{\varphi}-4\alpha\boldsymbol{\varphi}+2\alpha\,\mathrm{rot}\,\mathbf{u}+\mathbf{Y}=0. \tag{1.2}$$

Here, $\mathbf{X}$ denotes the body force, $\mathbf{Y}$ the body couple, $\mathbf{u}(\mathbf{x}, t)$ the displacement and $\boldsymbol{\varphi}(\mathbf{x}, t)$ the rotation. The quantities $\alpha, \beta, \gamma, \varepsilon, \mu, \lambda$ are material constants. The functions $\mathbf{X}, \mathbf{Y}, \mathbf{u}, \boldsymbol{\varphi}$ are functions of the position $\mathbf{x}$.

Passing with Eqs. (1.1) and (1.2) to the (r, θ, z) coordinate system and assuming that both the causes and effects are independent of the angle θ, we obtain from Eqs. (1.1) and (1.2) two independent of each other systems of equations:

$$\begin{aligned}
&(\mu+\alpha)\left(\nabla^2-\frac{1}{r^2}\right)u_r+(\lambda+\mu-\alpha)\frac{\partial e}{\partial r}-2\alpha\frac{\partial\varphi_\theta}{\partial z}+X_r=0,\\
&(\gamma+\varepsilon)\left(\nabla^2-\frac{1}{r^2}\right)\varphi_\theta+2\alpha\left(\frac{\partial u_r}{\partial z}-\frac{\partial u_z}{\partial r}\right)-4\alpha\varphi_\theta+Y_\theta=0,\\
&(\mu+\alpha)\,\nabla^2u_z+(\lambda+\mu-\alpha)\frac{\partial e}{\partial z}+2\alpha\frac{1}{r}\frac{\partial}{\partial r}(r\varphi_\theta)+X_z=0,
\end{aligned} \tag{1.3}$$

and

$$(\gamma+\varepsilon)\left(\nabla^2-\frac{1}{r^2}\right)\varphi_r-4\alpha\varphi_r+(\beta+\gamma-\varepsilon)\frac{\partial\varkappa}{\partial r}-2\alpha\frac{\partial u_\theta}{\partial z}+Y_r=0,$$

$$(\mu+\alpha)\left(\nabla^2-\frac{1}{r^2}\right)u_\theta+2\alpha\left(\frac{\partial\varphi_r}{\partial z}-\frac{\partial\varphi_z}{\partial r}\right)+X_\theta=0, \tag{1.4}$$

$$(\gamma+\varepsilon)\nabla^2\varphi_z-4\alpha\varphi_z+(\beta+\gamma-\varepsilon)\frac{\partial\varkappa}{\partial z}+2\alpha\frac{\partial}{\partial r}(ru_\theta)+Y_z=0.$$

In the above equations the following notations have been introduced:

$$\mathbf{u}\equiv(u_r,u_\theta,u_z),\quad \boldsymbol{\varphi}\equiv(\varphi_r,\varphi_\theta,\varphi_z),\quad \mathbf{X}\equiv(X_r,X_\theta,X_z),$$

$$\mathbf{Y}\equiv(Y_r,Y_\theta,Y_z),\quad e=\frac{1}{r}\frac{\partial}{\partial r}(u_r r)+\frac{\partial u_z}{\partial z},$$

$$\varkappa=\frac{1}{r}\frac{\partial}{\partial r}(\varphi_r r)+\frac{\partial\varphi_z}{\partial z}.$$

It is seen that the state of displacements and rotations $\mathbf{u}\equiv(u_r,0,u_z)$, $\boldsymbol{\varphi}\equiv(0,\varphi_\theta,0)$ appearing in the system of Eqs. (1.3) may be induced by the action of body forces and body couples in the form:

$$\mathbf{X}\equiv(X_r,0,X_z),\quad \mathbf{Y}\equiv(0,Y_\theta,0),$$

with boundary conditions chosen appropriately. Let us quote, as an example, that such a state may be induced by the action of a load parallel to the z-axis (e.g. Boussinesq problem for an elastic half-space).

The state of displacements and rotations

$$\mathbf{u}\equiv(0,u_\theta,0),\quad \boldsymbol{\varphi}\equiv(\varphi_r,0,\varphi_z),$$

appearing in the system of Eqs. (1.4) may be due to the action of body forces and body couples

$$\mathbf{X}\equiv(0,X_\theta,0),\quad \mathbf{Y}\equiv(Y_r,0,Y_z),$$

the boundary conditions being chosen appropriately. E.g., such a state of displacements and rotations will appear in the elastic half-space $z\geqslant 0$ loaded on the surface $z=0$, by the twist moments with vectors directed along the z-axis.

2. Generalized Love's function for the system of Eqs. (1.3)

Let us consider the system of Eqs. (1.3) under the assumption that $\mathbf{X}\equiv(0,0,X_z)$. Thereafter we put into Eqs. (1.3) the quantities

$$u_r=\frac{\partial v}{\partial r},\quad \varphi_\theta=\frac{\partial\omega}{\partial r},\quad u_z=w. \tag{2.1}$$

We take into account that

$$\left(\nabla^2-\frac{1}{r^2}\right)\frac{\partial v}{\partial r}=\frac{\partial}{\partial r}\nabla^2 v, \qquad \left(\nabla^2-\frac{1}{r^2}\right)\frac{\partial w}{\partial r}=\frac{\partial}{\partial r}\nabla^2 w.$$

Introducing the above expressions into Eqs. (1.3) and $(1.3)_2$, we obtain a system of equations, wherein only the operators $\nabla^2 w$, $\nabla^2 v$, $\nabla^2 \omega$ appear. Integrating these equations with respect to r and taking into consideration Eq. $(1.3)_3$ we get the following system of equations:

$$\begin{aligned}
&\left[(\lambda+2\mu)\nabla^2-(\lambda+\mu-\alpha)\frac{\partial^2}{\partial z^2}\right]v-2\alpha\frac{\partial\omega}{\partial z}+(\lambda+\mu-\alpha)\frac{\partial w}{\partial z}=0,\\
&2\alpha\frac{\partial v}{\partial z}+[(\gamma+\varepsilon)\nabla^2-4\alpha]\,\omega-2\alpha w=0,\\
&(\lambda+\mu-\alpha)\frac{\partial}{\partial z}\left(\nabla^2-\frac{\partial^2}{\partial z^2}\right)v+2\alpha\left(\nabla^2-\frac{\partial^2}{\partial z^2}\right)\omega+\\
&\qquad+\left[(\mu+\alpha)\nabla^2+(\lambda+\mu-\alpha)\frac{\partial^2}{\partial z^2}\right]w+X_z=0.
\end{aligned}\tag{2.2}$$

This system of equation may be written also in the form

$$\begin{aligned}
&L_{vv}\,v+L_{v\omega}\,\omega+L_{vw}\,w=0,\\
&L_{\omega v}\,v+L_{\omega\omega}\,\omega+L_{\omega w}\,w=0,\\
&L_{wv}\,v+L_{w\omega}\,\omega+L_{\omega\omega}\,w=0.
\end{aligned}\tag{2.3}$$

Let us now introduce the stress function $\chi(r,z)$ connected with the function v, ω, w by the following relations

$$v=\begin{vmatrix}0 & L_{v\omega} & L_{vw}\\ 0 & L_{\omega\omega} & L_{\omega w}\\ \chi & L_{w\omega} & L_{ww}\end{vmatrix},\qquad \omega=\begin{vmatrix}L_{vv} & 0 & L_{vw}\\ L_{\omega v} & 0 & L_{\omega w}\\ L_{wv} & \chi & L_{ww}\end{vmatrix},\qquad w=\begin{vmatrix}L_{vv} & L_{v\omega} & 0\\ L_{\omega v} & L_{\omega\omega} & 0\\ L_{wv} & L_{w\omega} & \chi\end{vmatrix},\tag{2.4}$$

wherefrom we find that

$$u_r=\frac{\partial v}{\partial r}=-\frac{\partial^2}{\partial r\,\partial z}(\Gamma\chi),\tag{2.5}$$

$$\varphi_0=\frac{\partial\omega}{\partial r}=2\alpha(\lambda+2\mu)\frac{\partial}{\partial r}(\nabla^2\chi),\tag{2.6}$$

$$u_z=w=\Theta\chi-\frac{\partial^2}{\partial z^2}(\Gamma\chi),\tag{2.7}$$

where

$$\Gamma=(\gamma+\varepsilon)(\lambda+\mu-\alpha)\nabla^2-4\alpha(\lambda+\mu),$$

$$\Theta=(\lambda+2\mu)\nabla^2[(\gamma+\varepsilon)\nabla^2-4\alpha].$$

Substituting (2.5)–(2.7) into the last of Eqs. (2.3), we obtain the equation

$$(\lambda+2\mu)\nabla^2\nabla^2[(\mu+\alpha)(\gamma+\varepsilon)\nabla^2-4\alpha\mu]\chi(r,z)+X_z(r,z)=0. \tag{2.8}$$

This is the sought for generalization of Love's equation known from the classical theory of elasticity on the micropolar elasticity.

The state of displacements $\mathbf{u}\equiv(u_r,0,u_z)$ and rotations $\boldsymbol{\varphi}\equiv(0,\varphi_\theta,0)$ here considered is connected with the state of stress by the following relations

$$\boldsymbol{\sigma}=\begin{vmatrix}\sigma_{rr} & 0 & \sigma_{rz}\\ 0 & \sigma_{\theta\theta} & 0\\ \sigma_{zr} & 0 & \sigma_{zz}\end{vmatrix},\qquad \boldsymbol{\mu}=\begin{vmatrix}0 & \mu_{r\theta} & 0\\ \mu_{\theta r} & 0 & \mu_{\theta z}\\ 0 & \mu_{z\theta} & 0\end{vmatrix}, \tag{2.9}$$

where

$$\begin{aligned}
&\sigma_{rr}=2\mu\frac{\partial u_r}{\partial r}+\lambda e,\qquad \sigma_{\theta\theta}=2\mu\frac{u_r}{r}+\lambda e,\\
&\sigma_{zz}=2\mu\frac{\partial u_z}{\partial z}+\lambda e,\\
&\sigma_{rz}=\mu\left(\frac{\partial u_z}{\partial r}+\frac{\partial u_r}{\partial z}\right)-\alpha\left(\frac{\partial u_r}{\partial z}-\frac{\partial u_z}{\partial r}\right)+2\alpha\varphi_\theta,\\
&\sigma_{zr}=\mu\left(\frac{\partial u_z}{\partial r}+\frac{\partial u_r}{\partial z}\right)+\alpha\left(\frac{\partial u_r}{\partial z}-\frac{\partial u_z}{\partial r}\right)-2\alpha\varphi_\theta,
\end{aligned} \tag{2.10}$$

and

$$\begin{aligned}
&\mu_{r\theta}=\gamma\left(\frac{\partial\varphi_\theta}{\partial r}-\frac{\varphi_\theta}{r}\right)+\varepsilon\left(\frac{\partial\varphi_\theta}{\partial r}+\frac{\varphi_\theta}{r}\right),\\
&\mu_{\theta r}=\gamma\left(\frac{\partial\varphi_\theta}{\partial r}-\frac{\varphi_\theta}{r}\right)-\varepsilon\left(\frac{\partial\varphi_\theta}{\partial r}+\frac{\varphi_\theta}{r}\right),\\
&\mu_{\theta z}=(\gamma-\varepsilon)\frac{\partial\varphi_\theta}{\partial z},\qquad \mu_{z\theta}=(\gamma+\varepsilon)\frac{\partial\varphi_\theta}{\partial z}.
\end{aligned} \tag{2.11}$$

Expressing the above stresses by u_r, φ_θ, u_z from the formulae (2.5)–(2.7), we get finally

$$\begin{aligned}
&\sigma_{rr}=-\frac{\partial}{\partial z}\left[2\mu\frac{\partial^2}{\partial r^2}(\Gamma\chi)-\lambda(\Psi\chi)\right],\\
&\sigma_{\varphi\varphi}=-\frac{\partial}{\partial z}\left[2\mu\frac{1}{r}\frac{\partial}{\partial r}(\Gamma\chi)-\lambda(\Psi\chi)\right],\\
&\sigma_{zz}=\frac{\partial}{\partial z}\left[2\mu\left(\frac{\partial^2\Gamma}{\partial z^2}-\Theta\right)\chi+\lambda(\Psi\chi)\right],
\end{aligned} \tag{2.12}$$

where

$$\Psi=\nabla^2[(\mu+\alpha)(\gamma+\varepsilon)\nabla^2-4\alpha\mu].$$

$$\sigma_{rz} = \frac{\partial}{\partial r}\left[(\mu+\alpha)\,\Theta\chi - 2\mu\frac{\partial^2}{\partial z^2}(\Gamma\chi) + 4\alpha^2(\lambda+2\mu)\,\nabla^2\chi\right],$$

(2.13)

$$\sigma_{zr} = \frac{\partial}{\partial r}\left[(\mu-\alpha)\,\Theta\chi - 2\mu\frac{\partial^2}{\partial z^2}(\Gamma\chi) - 4\alpha^2(\lambda+2\mu)\,\nabla^2\chi\right],$$

and

$$\mu_{r\theta} = 2\alpha(\lambda+2\mu)\left[(\gamma+\varepsilon)\frac{\partial^2}{\partial r^2} - (\gamma-\varepsilon)\frac{1}{r}\frac{\partial}{\partial r}\right]\nabla^2\chi,$$

$$\mu_{\theta r} = 2\alpha(\lambda+2\mu)\left[(\gamma-\varepsilon)\frac{\partial^2}{\partial r^2} - (\gamma+\varepsilon)\frac{1}{r}\frac{\partial}{\partial r}\right]\nabla^2\chi,$$

(2.14)

$$\mu_{\theta z} = 2\alpha(\lambda+2\mu)(\gamma-\varepsilon)\frac{\partial^2}{\partial r\,\partial z}(\nabla^2\chi),$$

$$\mu_{z\theta} = 2\alpha(\lambda+2\mu)(\gamma+\varepsilon)\frac{\partial^2}{\partial r\,\partial z}(\nabla^2\chi).$$

Now, let us pass to the classical theory of elasticity putting $\alpha = 0$ in (2.5)–(2.14). Introducing, moreover, a new function

$$L(r, z) = (\gamma+\varepsilon)\,\nabla^2\chi(r, z), \tag{2.15}$$

we obtain from (2.5)–(2.7):

$$u_r = -(\lambda+\mu)\frac{\partial^2 L}{\partial r\,\partial z}, \qquad \varphi_\theta = 0, \qquad u_z = (\lambda+2\mu)\,\nabla^2 L - (\lambda+\mu)\frac{\partial^2 L}{\partial z^2}. \tag{2.16}$$

Eq. (2.8) transforms into

$$\nabla^2\nabla^2 L(r, z) + \frac{1}{\lambda+2\mu}X_z(r, z) = 0, \tag{2.17}$$

and – for $X_z = 0$ – becomes a biharmonic equation.
The following formulae will describe the stresses [3]:

$$\sigma_{rr}/\mu = \frac{\partial}{\partial z}\left[\lambda\nabla^2 L - 2(\lambda+\mu)\frac{\partial^2 L}{\partial r^2}\right],$$

$$\sigma_{\theta\theta}/\mu = \frac{\partial}{\partial z}\left[\lambda\nabla^2 L - \frac{2}{r}(\lambda+\mu)\frac{\partial L}{\partial r}\right],$$

(2.18)

$$\sigma_{zz}/\mu = \frac{\partial}{\partial z}\left[(3\lambda+4\mu)\,\nabla^2 L - 2(\lambda+\mu)\frac{\partial^2 L}{\partial z^2}\right],$$

$$\sigma_{rz}/\mu = \frac{\partial}{\partial r}\left[(\lambda+2\mu)\,\nabla^2 L - 2(\lambda+\mu)\frac{\partial^2 L}{\partial z^2}\right],$$

$$\mu_{r\theta} = \mu_{\theta r} = 0, \qquad \mu_{zr} = \mu_{rz} = 0.$$

In this way we arrived at the Love's stress function $L(r, z)$ known from the classical theory of elasticity [1].

Returning now to the micropolar elasticity we shall present, *exempli modo*, the application of the stress function χ.

Consider an elastic half-space $z \geqslant 0$ subjected to the action of a loading $p(r)$ in parallel to the z-axis in the plane $z = 0$. The solution of Eq. (2.8) may be written in the form of Hankel's integral

$$\chi(r, z) = \int_0^\infty Z(z)\, J_0(\zeta r)\, d\zeta, \tag{2.19}$$

where

$$Z(z) = (A + B\zeta)\, e^{-\zeta z} + Ce^{-\nu z}, \qquad \nu = (\zeta^2 + \sigma^2)^{1/2}, \qquad \sigma^2 = \frac{4\alpha\mu}{(\mu+\alpha)(\gamma+\varepsilon)}.$$

The function $Z(z)$ is chosen so as to have $(r, z) \to 0$ for $z \to \infty$. The quantities A, B, C, functions of the parameter ζ, will be determined from the following three boundary conditions

$$\sigma_{zz}(r, 0) = -p(r), \qquad \sigma_{zr}(r, 0) = 0, \qquad \mu_{r\theta}(r, 0) = 0. \tag{2.20}$$

Here we make use of the formula (2.12)–(2.14), wherein the stresses $\sigma_{zz}, \sigma_{zr}, \mu_{r\theta}$ are expressed by the function χ.

3. Functions of Love's type for the system of Eqs. (1.4)

We shall now consider the system of Eqs. (1.4) putting $Y \equiv (0, 0, Y_z)$. Substituting into these equations the following quantities

$$\varphi_r = \frac{\partial \eta}{\partial r}, \qquad u_\theta = \frac{\partial \vartheta}{\partial r}, \qquad \varphi_z = \varrho,$$

and bearing in mind

$$\left(\nabla^2 - \frac{1}{r^2}\right)\frac{\partial \eta}{\partial r} = \frac{\partial}{\partial r}\nabla^2 \eta, \qquad \left(\nabla^2 - \frac{1}{r^2}\right)\frac{\partial \vartheta}{\partial r} = \frac{\partial}{\partial r}\nabla^2 \vartheta,$$

we obtain a system of equations, wherein only the Laplacians $\nabla^2\eta$, $\nabla^2\vartheta$ and $\nabla^2\varrho$ occur. Integrating Eqs. $(1.4)_1$ and $(1.4)_2$ with respect to r, we arrive at the following system of equations

$$\begin{aligned}
&\left[(\beta+2\gamma)\nabla^2 - 4\alpha - (\beta+\gamma-\varepsilon)\frac{\partial^2}{\partial z^2}\right]\eta - 2\alpha\frac{\partial \vartheta}{\partial z} + (\beta+\gamma-\varepsilon)\frac{\partial \varrho}{\partial z} = 0,\\
&2\alpha\frac{\partial \eta}{\partial z} + (\mu+\alpha)\nabla^2\vartheta - 2\alpha\varrho = 0,\\
&(\beta+\gamma-\alpha)\left(\nabla^2 - \frac{\partial^2}{\partial z^2}\right)\frac{\partial \eta}{\partial z} + 2\alpha\left(\nabla^2 - \frac{\partial^2}{\partial z^2}\right)\vartheta +\\
&\qquad + \left[(\gamma+\alpha)\nabla^2 - 4\alpha + (\gamma+\beta-\varepsilon)\frac{\partial^2}{\partial z^2}\right]\varrho + Y_z = 0.
\end{aligned} \tag{3.1}$$

We may write this system in an abbreviated form

$$(3.2)\qquad \begin{aligned} &L_{\eta\eta}\,\eta+L_{\eta\vartheta}\vartheta+L_{\eta\varrho}\,\varrho=0,\\ &L_{\vartheta\eta}\,\eta+L_{\vartheta\vartheta}\vartheta+L_{\vartheta\varrho}\,\varrho=0,\\ &L_{\varrho\eta}\,\eta+L_{\varrho\vartheta}\vartheta+L_{\varrho\varrho}\,\varrho+Y_z=0. \end{aligned}$$

Let us introduce the stress function $\varphi(r,z)$ connected with the functions η,ϑ,ϱ by the relations

$$(3.3)\qquad \eta=\begin{vmatrix} 0 & L_{\eta\vartheta} & L_{\eta\varrho}\\ 0 & L_{\vartheta\vartheta} & L_{\vartheta\varrho}\\ \psi & L_{\varrho\vartheta} & L_{\varrho\varrho}\end{vmatrix},\qquad \vartheta=\begin{vmatrix} L_{\eta\eta} & 0 & L_{\eta\varrho}\\ L_{\vartheta\eta} & 0 & L_{\vartheta\varrho}\\ L_{\varrho\eta} & \psi & L_{\varrho\varrho}\end{vmatrix},\qquad \varrho=\begin{vmatrix} L_{\eta\eta} & L_{\eta\vartheta} & 0\\ L_{\vartheta\eta} & L_{\vartheta\vartheta} & 0\\ L_{\varrho\eta} & L_{\varrho\vartheta} & \psi\end{vmatrix}.$$

From the above equations we get

$$(3.4)\qquad \begin{aligned} &\varphi_r=\frac{\partial\eta}{\partial r}=-\frac{\partial^2}{\partial r\,\partial z}(\Omega\psi),\\ &u_0=\frac{\partial\vartheta}{\partial r}=2\alpha\frac{\partial}{\partial r}(\Xi\psi),\\ &\varphi_z=\varrho=\Phi\psi-\frac{\partial^2}{\partial z^2}(\Omega\psi), \end{aligned}$$

where

$$\Omega=(\mu+\alpha)(\beta+\gamma-\varepsilon)\nabla^2-4\alpha^2,\qquad \Phi=(\mu+\alpha)\nabla^2[(\beta+2\gamma)\nabla^2-4\alpha],$$
$$\Xi=(\beta+2\gamma)\nabla^2-4\alpha.$$

Substituting (3.4) into Eq. $(1.4)_3$, we obtain the equation

$$(3.5)\qquad [(\beta+2\gamma)\nabla^2-4\alpha]\,\nabla^2[(\mu+\alpha)(\gamma+\varepsilon)\nabla^2-4\alpha\mu]\,\psi(r,z)+Y_z(r,z)=0.$$

Eq. (3.5) is an analogon with Eq. (2.8). There is no counterpart of this equation in the classical theory of elasticity. Putting $\alpha=0$ in the relations (3.4) and (3.5) we obtain the equation

$$(3.6)\qquad (\beta+2\gamma)(\gamma+\varepsilon)\nabla^2\nabla^2\varphi+Y_z=0,\qquad \varphi=\mu\nabla^2\psi,$$

and the relations

$$(3.7)\qquad \begin{aligned} &\varphi_r=-(\beta+\gamma-\varepsilon)\frac{\partial^2\varphi}{\partial r\,\partial z},\\ &\mu_0=0,\\ &\varphi_z=(\beta+2\gamma)\nabla^2\varphi-(\beta+\gamma-\varepsilon)\frac{\partial^2\varphi}{\partial z^2}. \end{aligned}$$

Eq. (3.6) and the relations (3.7) refer to a quasi-elastic body in which the rotation $\boldsymbol{\varphi}$ is the only possible. The state of displacement $\mathbf{u}\equiv(0,u_0,0)$ and rotations $\boldsymbol{\varphi}\equiv(\varphi_r,0,\varphi_z)$ is connected with the state of stress

$$(3.8)\qquad \boldsymbol{\sigma}=\begin{vmatrix} 0 & \sigma_{r\theta} & 0\\ \sigma_{\theta r} & 0 & \sigma_{\theta z}\\ 0 & \sigma_{z\theta} & 0\end{vmatrix},\qquad \boldsymbol{\mu}=\begin{vmatrix} \mu_{rr} & 0 & \mu_{rz}\\ 0 & \mu_{\theta\theta} & 0\\ \mu_{zr} & 0 & \mu_{zz}\end{vmatrix},$$

where

$$
(3.9)\quad
\begin{aligned}
\sigma_{r\theta} &= \mu\left(\frac{\partial u_\theta}{\partial r}-\frac{u_\theta}{r}\right)+\alpha\frac{1}{r}\frac{\partial}{\partial r}(ru_\theta)-2\alpha\varphi_z,\\
\sigma_{\theta r} &= \mu\left(\frac{\partial u_\theta}{\partial r}-\frac{u_\theta}{r}\right)-\alpha\frac{1}{r}\frac{\partial}{\partial r}(ru_\theta)+2\alpha\varphi_z,\\
\sigma_{\theta z} &= \mu\frac{\partial u_\theta}{\partial z}-\frac{\alpha}{r}\frac{\partial}{\partial z}(ru_\theta)-2\alpha\varphi_r,\\
\sigma_{z\theta} &= \mu\frac{\partial u_\theta}{\partial z}+\frac{\alpha}{r}\frac{\partial}{\partial z}(ru_\theta)+2\alpha\varphi_r,
\end{aligned}
$$

and

$$
(3.10)\quad
\begin{aligned}
&\mu_{rr}=2\gamma\frac{\partial\varphi_r}{\partial r}+\beta\varkappa,\qquad \mu_{\theta\theta}=2\gamma\frac{\varphi_r}{r}+\beta\varkappa,\qquad \mu_{zz}=2\gamma\frac{\partial\varphi_z}{\partial z}+\beta\varkappa,\\
&\mu_{rz}=\gamma\left(\frac{\partial\varphi_z}{\partial r}+\frac{\partial\varphi_r}{\partial z}\right)+\varepsilon\left(\frac{\partial\varphi_r}{\partial z}-\frac{\partial\varphi_z}{\partial r}\right),\\
&\mu_{zr}=\gamma\left(\frac{\partial\varphi_z}{\partial r}+\frac{\partial\varphi_r}{\partial z}\right)-\varepsilon\left(\frac{\partial\varphi_r}{\partial z}-\frac{\partial\varphi_z}{\partial r}\right).
\end{aligned}
$$

Expressing the above stresses in function ψ, we have

$$
(3.11)\quad
\begin{aligned}
\sigma_{r\theta} &= 2\alpha\left[(\mu+\alpha)\frac{\partial^2}{\partial r^2}-(\mu-\alpha)\frac{1}{r}\frac{\partial}{\partial r}\right](\Xi\psi)-2\alpha\left[\Phi\psi-\frac{\partial^2}{\partial z^2}(\Omega\psi)\right],\\
\sigma_{\theta r} &= 2\alpha\left[(\mu+\alpha)\frac{\partial^2}{\partial r^2}-(\mu+\alpha)\frac{1}{r}\frac{\partial}{\partial r}\right](\Xi\psi)+2\alpha\left[\Phi\psi-\frac{\partial^2}{\partial z^2}(\Omega\psi)\right],\\
\sigma_{z\theta} &= 2\alpha\frac{\partial^2}{\partial r\,\partial z}\left[(\mu+\alpha)(\Xi\psi)-\Omega\psi\right],\\
\sigma_{\theta z} &= 2\alpha\frac{\partial^2}{\partial r\,\partial z}\left[(\mu-\alpha)(\Xi\psi)+\Omega\psi\right],
\end{aligned}
$$

$$
(3.12)\quad
\begin{aligned}
\mu_{rr} &= -\frac{\partial}{\partial z}\left[2\gamma\frac{\partial^2}{\partial r^2}(\Omega\psi)-\beta(\Phi-\nabla^2\Omega)\psi\right],\\
\mu_{\theta\theta} &= -\frac{\partial}{\partial z}\left[2\gamma\frac{1}{r}\frac{\partial}{\partial r}(\Omega\psi)-\beta(\Phi-\nabla^2\Omega)\psi\right],\\
\mu_{zz} &= -\frac{\partial}{\partial z}\left[2\gamma\left(\frac{\partial^2\Omega}{\partial z^2}-\Phi\right)\psi-\beta(\Phi-\nabla^2\Omega)\psi\right],\\
\mu_{rz} &= (\gamma-\varepsilon)\frac{\partial}{\partial r}\left[\Phi-\frac{\partial^2\Omega}{\partial z^2}\right]\psi-(\gamma+\varepsilon)\frac{\partial^3}{\partial r\,\partial z^2}(\Omega\psi),\\
\mu_{zr} &= (\gamma+\varepsilon)\frac{\partial}{\partial r}\left[\Phi-\frac{\partial^2\Omega}{\partial z^2}\right]\psi-(\gamma-\varepsilon)\frac{\partial^3}{\partial r\,\partial z^2}(\Omega\psi).
\end{aligned}
$$

Let us consider an example of the application of the function ψ. Suppose a half-space $z \geqslant 0$ loaded in the plane $z = 0$ by the twisting moments $m(r)$ with vectors directed in parallel to the z-axis.

The function ψ will be expressed in the form of Hankel's integral

$$\psi(r, z) = \int_0^\infty Z(z)\, J_0(r\zeta)\, d\zeta,$$

where

$$Z(z) = Ae^{-\zeta z} + Be^{-\tau z} + Ce^{-\nu z}.$$

Here

$$\tau = (\zeta^2 + a^2)^{1/2}, \quad \nu = (\zeta^2 + \sigma^2)^{1/2}$$

and

$$a^2 = \frac{4\alpha}{\beta + 2\gamma}, \quad \sigma = \frac{4\alpha\mu}{(\gamma + \varepsilon)(\alpha + \mu)}.$$

The quantities A, B, C, functions of parameter ζ, will be determined from the boundary conditions

$$\mu_{rr}(r, 0) = -m(r), \quad \mu_{rz}(r, 0) = 0, \quad \sigma_{r\theta}(r, 0) = 0.$$

The stresses $\mu_{rr}, \mu_{rz}, \sigma_{r\theta}$ are expressed in function ψ by the formulae (3.11) and (3.12).

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В. НОВАЦКИЙ, **ОБОБЩЕННЫЕ ФУНКЦИИ ЛОВЭ В МИКРОПОЛЯРНОЙ ТЕОРИИ УПРУГОСТИ**

В настоящем сообщении автор ограничивается осе-симметрическими вопросами микрополярной теории упругости. Для упомянутых проблем выведены две функции напряжений χ и ψ, которые играют аналогичную роль как функции Ловэ в классической теории упругости. Пользуясь этими функциями можно выразить все составляющие состояния напряжения. Функции χ и ψ удовлетворяют уравнениям (2.8) и (3.5).