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Green Functions for Micropolar Thermoelasticity

by

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1. Introduction

The aim of this paper is to give basic solutions of wave equations in an unlimited medium for micropolar thermoelasticity and, in particular, to present in a closed form the wave functions as well as the displacement, rotation and temperature fields formed in an unlimited space under the action of a concentrated force, of a couple or else a concentrated heat source changing harmonically in time.

Let us consider, first, the system of linearized equations of micropolar thermoelasticity [1]:

$$(\mu+\alpha)\,\nabla^2\mathbf{u}+(\lambda+\mu-\alpha)\,\mathrm{grad}\,\mathrm{div}\,\mathbf{u}+2\alpha\,\mathrm{rot}\,\boldsymbol{\omega}+\mathbf{X}=\varrho\ddot{\mathbf{u}}+\nu\,\mathrm{grad}\,\theta, \tag{1.1}$$

$$(\gamma+\varepsilon)\,\nabla^2\boldsymbol{\omega}+(\beta+\gamma-\varepsilon)\,\mathrm{grad}\,\mathrm{div}\,\boldsymbol{\omega}-4\alpha\boldsymbol{\omega}+2\alpha\,\mathrm{rot}\,\mathbf{u}+\mathbf{Y}=J\ddot{\boldsymbol{\omega}}, \tag{1.2}$$

$$\nabla^2\theta-\frac{1}{\varkappa}\dot{\theta}-\eta\,\mathrm{div}\,\dot{\mathbf{u}}=-\frac{Q}{\varkappa}. \tag{1.3}$$

Eqs. (1.1) and (1.2) are the equations of motion, while Eq. (1.3) is an expanded equation of heat conductivity. These equations are coupled. In the above equations the following notations are used: $\mathbf{u}$ denotes the displacement vector, $\boldsymbol{\omega}$ = the rotation vector, $\mathbf{X}$ = the vector of body-forces, while $\mathbf{Y}$ stands for the vector of body-couples. The symbols $\mu, \lambda, \alpha, \beta, \gamma, \varepsilon$ denote the material constants. $\theta = T - T_0$ is the difference between the absolute temperature T and the temperature of the body T_0 in natural state. Q stands for the function describing the intensity of heat sources. $\varkappa = \lambda_0/\varrho c$ is a coefficient, wherein λ_0 denotes the heat conductivity, ϱ = density and c_ε = specific heat, the deformation being assumed constant. $\eta = \nu T_0/\lambda_0$, where $\nu = (3\lambda+2\mu)\,\alpha_t$, α_t denoting the coefficient of linear heat dilatation. Finally, $Q = W/\varrho c_\varepsilon$, where W stands for the quantity of heat generated in a volume unit of the body and within a time unit.

The quantities $\mathbf{u}, \boldsymbol{\omega}, \theta, \mathbf{X}, \mathbf{Y}, Q$ are functions of the position $\mathbf{x}$ and time t.

The system of Eqs. (1.1)—(1.3) may be separated by decomposing the vectors $\mathbf{u}, \boldsymbol{\omega}$ and also $\mathbf{X}, \mathbf{Y}$ into their potential and solenoidal parts introducing to this end into the system of Eqs. (1.1)—(1.3) the following Helmholtz representations

$$\mathbf{u} = \operatorname{grad}\Phi + \operatorname{rot}\mathbf{\Psi}, \quad \operatorname{div}\mathbf{\Psi} = 0, \tag{1.4}$$

$$\boldsymbol{\omega} = \operatorname{grad}\Sigma + \operatorname{rot}\mathbf{H}, \quad \operatorname{div}\mathbf{H} = 0, \tag{1.5}$$

and

$$\mathbf{X} = \varrho(\operatorname{grad}\vartheta + \operatorname{rot}\boldsymbol{\chi}), \tag{1.6}$$

$$\mathbf{Y} = J(\operatorname{grad}\sigma + \operatorname{rot}\boldsymbol{\eta}), \tag{1.7}$$

We obtain in this way the following system of wave equations

$$(\Box_1 D - \nu\eta\partial_t \nabla^2)\Phi = -\varrho D\vartheta - \frac{\nu}{\varkappa}Q, \tag{1.8}$$

$$(\Box_1 D - \nu\eta\partial_t \nabla^2)\theta = -\varrho\eta\partial_t \nabla^2\vartheta - \frac{1}{\varkappa}\Box_1 Q, \tag{1.9}$$

$$\Box_3\Sigma + J\sigma = 0, \tag{1.10}$$

$$(\Box_2\Box_4 + 4\alpha^2\nabla^2)\mathbf{\Psi} = 2\alpha J \operatorname{rot}\boldsymbol{\eta} - \varrho\Box_4\boldsymbol{\chi}, \tag{1.11}$$

$$(\Box_2\Box_4 + 4\alpha^2\nabla^2)\mathbf{H} = 2\alpha\varrho \operatorname{rot}\boldsymbol{\chi} - J\Box_2\boldsymbol{\eta}. \tag{1.12}$$

In the above formulae the following notations have been introduced

$$D = \nabla^2 - \frac{1}{\varkappa}\partial_t, \quad \Box_1 = (\lambda+2\mu)\nabla^2 - \varrho\partial_t^2, \quad \Box_2 = (\mu+\alpha)\nabla^2 - \varrho\partial_t^2,$$

$$\Box_3 = (\beta+2\gamma)\nabla^2 - 4\alpha - J\partial_t^2, \quad \Box_4 = (\gamma+\varepsilon)\nabla^2 - 4\alpha - J\partial_t^2, \tag{1.13}$$

$$\nabla^2 = \frac{\partial}{\partial x_l}\frac{\partial}{\partial x_l}, \quad \partial_t^2 = \partial^2/\partial t^2.$$

Eq. (1.8) represents the longitudinal wave, Eq. (1.9) describes the thermal wave, Eq. (1.10) is the expression for the rotation wave, while Eqs. (1.11) and (1.12) correspond to transverse waves. Let us remark that the heat source Q and the body forces $\mathbf{X} = \varrho \operatorname{grad}\vartheta$ may induce in the infinite space only a longitudinal wave and a temperature field.

Body forces $\mathbf{X} = \varrho \operatorname{rot}\boldsymbol{\chi}$ and body couples $\mathbf{Y} = J\operatorname{rot}\boldsymbol{\eta}$ induce only transverse waves not accompanied by the temperature field. Eq. (1.10) does not depend on the remaining ones. The rotation wave may be provoked by the body couple $\mathbf{Y} = J\operatorname{grad}\sigma$. We assume that the action of causes inducing the wave disturbances changes harmonically in time

$$\mathbf{X}(\mathbf{x},t) = \mathbf{X}^*(\mathbf{x})e^{-i\omega t}, \quad \mathbf{Y}(\mathbf{x},t) = \mathbf{Y}^*(\mathbf{x})e^{-i\omega t}, \quad Q(\mathbf{x},t) = Q^*(\mathbf{x})e^{-i\omega t}. \tag{1.14}$$

Thus, the effects provoked, namely the displacements $\mathbf{u}$, rotations $\boldsymbol{\omega}$ and the functions $\Phi, \Sigma, \theta, \mathbf{\Psi}, \mathbf{H}$ change harmonically in time also. Marking with an asterisk the amplitudes of these functions, we reduce Eqs. (1.8)—(1.12) to the form

$$(1.15) \qquad (\nabla^2+\mu_1^2)(\nabla^2+\mu_2^2)\Phi^* = -\frac{1}{c_1^2}(\nabla^2+q)\vartheta^* - \frac{m}{\varkappa}Q^*,$$

$$(1.16) \qquad (\nabla^2+\mu_1^2)(\nabla^2+\mu_2^2)\theta^* = -\frac{1}{\varkappa}(\nabla^2+\sigma^2)Q^* + \frac{q\eta\varkappa}{c_1^2}\nabla^2\vartheta^*,$$

$$(1.17) \qquad (\nabla^2+\sigma_3^2)\Sigma^* = -\frac{1}{c_3^2}\sigma^*,$$

$$(1.18) \qquad (\nabla^2+k_1^2)(\nabla^2+k_2^2)\mathbf{\Psi}^* = \frac{r}{c_4^2}\operatorname{rot}\boldsymbol{\eta}^* - \frac{1}{c_2^2}D_2\boldsymbol{\chi}^*,$$

$$(1.19) \qquad (\nabla^2+k_1^2)(\nabla^2+k_2^2)\mathbf{H}^* = \frac{p}{c_2^2}\operatorname{rot}\boldsymbol{\chi}^* - \frac{1}{c_4^2}D_1\boldsymbol{\eta}^*.$$

The following notations have been introduced in the above equations

$$\sigma = \frac{\omega}{c_1}, \quad c_1 = \left(\frac{\lambda+2\mu}{\varrho}\right)^{1/2}, \quad \sigma_3 = \left(\frac{\omega^2-\omega_0^2}{c_3}\right)^{1/2}, \quad c_3 = \left(\frac{\beta+2\gamma}{J}\right)^{1/2}, \quad \omega_0^2 = \frac{4\alpha}{J},$$

$$r = \frac{2\alpha}{\varrho c_2^2}, \quad p = \frac{2\alpha}{Jc_4^2}, \quad c_2 = \left(\frac{\mu+\alpha}{\varrho}\right)^{1/2}, \quad c_4 = \left(\frac{\gamma+\varepsilon}{J}\right)^{1/2}, \quad D_1 = \nabla^2+\sigma_2^2,$$

$$D_2 = \nabla^2+\sigma_4^2-2p, \quad \sigma_2 = \frac{\omega}{c_2}, \quad \sigma_4 = \frac{\omega}{c_4}, \quad q = \frac{i\omega}{\varkappa}.$$

The quantities μ_1^2, μ_2^2 are the roots of the equation

$$(1.20) \qquad \mu^4 - \mu^2[\sigma^2+q(1+\varepsilon)]+\sigma^2 q = 0, \quad \varepsilon = \eta\varkappa m.$$

The roots k_1, k_2 are coupled quantities with real and imaginary parts. A more detailed discussion of these roots was given in [2]. The quantities k_1^2, k_2^2 are the roots of equation

$$(1.21) \qquad k^4 - k^2\left(\sigma_2^2+\sigma_4^2+p(r-2)\right)+\sigma_2^2(\sigma_4^2-2p) = 0.$$

Let us now consider the homogeneous equation (1.15). The solution of this equation has the form of the integrals $\frac{1}{R}e^{\pm i\mu_\alpha R}$, $\alpha = 1, 2, 3$. We shall take into consideration only the integral $\frac{1}{R}e^{i\mu_\alpha R}$, as only the waves

$$(1.22) \qquad \operatorname{Re}\left[e^{-i\omega t}\frac{1}{R}e^{i\mu_\alpha R}\right] = \frac{e^{-\vartheta_\alpha R}}{R}\cos\omega\left(t-\frac{R}{v_\alpha}\right),$$

$$\vartheta_\alpha = \operatorname{Im}(\mu_\alpha), \quad v_\alpha = \frac{\omega}{\operatorname{Re}(\mu_\alpha)}, \quad \alpha = 1, 2$$

are the divergent waves propagating from the origin of the system $R = 0$ to infinity. Thus

$$(1.23) \qquad \Phi^* = \frac{1}{R}(N_1 e^{i\mu_1 R}+N_2 e^{i\mu_2 R}), \quad N_1, N_2 = \text{const}$$

and similarly

$$\theta^* = \frac{1}{R}(M_1 e^{i\mu_1 R} + M_2 e^{i\mu_2 R}), \qquad M_1, M_2 = \text{const}. \tag{1.24}$$

It is easily seen that these waves are dispersed, as the phase velocities v_a $(a = 1, 2)$ depend on the frequency ω. These waves are damped, as evidenced by the appearance of the damping coefficient ϑ_a.

The discussion of Eq. (1.21) leads to the conclusion that the quantities k_1, k_2 are real for $\sigma_4 > 2p$ or $\omega^2 > \frac{4a}{J}$. The function (1.25) is the solution of the homogeneous equation (1.18)

$$\mathbf{\Psi}^* = \mathbf{A}\frac{e^{ik_1 R}}{R} + \mathbf{B}\frac{e^{ik_2 R}}{R}. \tag{1.25}$$

In the above formula two waves appear. They are both divergent, non-damped but undergoing dispersion.

2. Effect of the concentrated force

Let us first consider the action of body forces. Since $\mathbf{Y} = 0$, there is also $\sigma = 0$ $\boldsymbol{\eta} = 0$. In an infinite space the rotation wave will not appear. Thus, we have to solve the following system of equations:

$$(\nabla^2 + \mu_1^2)(\nabla^2 + \mu_2^2)\,\Phi^* = -\frac{1}{c_1^2}(\nabla^2 + q)\,\vartheta^*, \tag{2.1}$$

$$(\nabla^2 + \mu_1^2)(\nabla^2 + \mu_2^2)\,\theta^* = \frac{q\eta\varkappa}{c_1^2}\nabla^2\vartheta^*, \tag{2.2}$$

$$(\nabla^2 + k_1^2)(\nabla^2 + k_2^2)\,\mathbf{\Psi}^* = -\frac{1}{c_2^2}D_2\,\boldsymbol{\chi}^*, \tag{2.3}$$

$$(\nabla^2 + k_1^2)(\nabla^2 + k_2^2)\,\mathbf{H}^* = \frac{p}{c_2^2}\operatorname{rot}\boldsymbol{\chi}. \tag{2.4}$$

In a general approach, the functions ϑ^* and χ^* for any arbitrary vector of body forces will be determined from the formulae [4]

$$\vartheta^*(\mathbf{x}) = -\frac{1}{4\pi\varrho}\int_V X_j^*(\boldsymbol{\xi})\frac{\partial}{\partial x_j}\left(\frac{1}{R(\boldsymbol{\xi}, \mathbf{x})}\right)dV(\boldsymbol{\xi}), \tag{2.5}$$

$$X_l^*(\mathbf{x}) = -\frac{1}{4\pi\varrho}\,\epsilon_{ljk}\int_V X_j^*(\boldsymbol{\xi})\frac{\partial}{\partial x_k}\left(\frac{1}{R(\boldsymbol{\xi}, \mathbf{x})}\right)dV(\boldsymbol{\xi}), \qquad l, j, k = 1, 2, 3. \tag{2.6}$$

We introduce into Eqs. (2.5) and (2.6) the concentrated force $X_j^* = \delta(x_1)\,\delta(x_2)\,\delta(x_3)\,\delta_{jl}$ acting at the origin of the coordinate system and oriented along the x_l-axis. In this way we get

$$\vartheta^{(l)*} = -\frac{1}{4\pi\varrho}\,\frac{\partial}{\partial x_l}\left(\frac{1}{R}\right), \qquad \chi_j^{(l)*} = -\frac{1}{4\pi\varrho}\,\epsilon_{jlk}\frac{\partial}{\partial x_k}\left(\frac{1}{R}\right), \tag{2.7}$$

where

$$R = (x_1^2+x_2^2+x_3^2)^{1/2}, \quad j,k,l = 1,2,3.$$

We have to solve the following system of equations

$$\begin{aligned} (\nabla^2+\mu_1^2)(\nabla^2+\mu_2^2)\,\Phi^{*(l)} &= \frac{1}{4\pi\varrho c_1^2}(\nabla^2+q)\frac{\partial}{\partial x_l}\left(\frac{1}{R}\right),\\ (\nabla^2+\mu_1^2)(\nabla^2+\mu_2^2)\,\theta^{*(l)} &= -\frac{1}{4\pi\varrho c_1^2}\,q\eta\varkappa\nabla^2\frac{\partial}{\partial x_2}\left(\frac{1}{R}\right), \end{aligned} \tag{2.8}$$

$$(\nabla^2+k_1^2)(\nabla^2+k_2^2)\,\Phi_j^{*(l)} = \frac{1}{4\pi\varrho c_2^2}\,\epsilon_{jlk}(\nabla^2+\sigma_4^2-2p)\frac{\partial}{\partial x_k}\left(\frac{1}{R}\right), \tag{2.9}$$

(2.10)

$$(\nabla^2+k_1^2)(\nabla^2+k_2^2)\,H_j^{*(l)} = -\frac{p}{4\pi\varrho c_2^2}\left(\nabla^2\,\delta_{jl}-\frac{\partial}{\partial x_j}\,\frac{\partial}{\partial x_l}\right)\left(\frac{1}{R}\right), \quad j,k,l = 1,2,3.$$

The solution of the system of Eqs. (2.8) is known; it was given by the present author in a previous work [5]

$$\Phi^{(l)*} = -\frac{1}{4\pi\varrho\omega^2}\,\frac{\partial}{\partial x_l}F(R,\omega), \tag{2.11}$$

where

$$F(R,\omega) = \frac{1}{R}\left[A_1 e^{i\mu_1 R}+A_2 e^{i\mu_2 R}+A_3\right].$$

The following notations have been introduced

$$A_1 = \frac{(\mu_1^2-q)\,\sigma^2}{\mu_1^2(\mu_1^2-\mu_2^2)}, \qquad A_2 = \frac{(\mu_2^2-q)\,\sigma^2}{\mu_2^2(\mu_2^2-\mu_1^2)}, \qquad A_3 = 1.$$

For the temperature $\theta^{*(l)}$ the following formula is obtained

$$\theta^{*(l)} = \frac{q\varepsilon}{4\pi\varrho m c_1^2(\mu_1^2-\mu_2^2)}\,\frac{\partial}{\partial x_l}\left(\frac{e^{i\mu_1 R}-e^{i\mu_2 R}}{R}\right). \tag{2.12}$$

The waves appearing in the formulae (2.11) and (2.12), i.e. $\frac{1}{R}e^{i\mu_a R}$, $(a=1,2)$ are damped and dispersed.

The solutions of Eqs. (2.9) and (2.10) have been given by the present author in [3]:

$$(2.13)\quad \Psi^{*(l)} = \frac{1}{4\pi\varrho\omega^2}\in_{ljk}\frac{\partial}{\partial x_k}\left(B_1\frac{e^{ik_1R}}{R}+B_2\frac{e^{ik_2R}}{R}+B_3\frac{e^{ik_3R}}{R}\right), \quad j,l,k=1,2,3,$$

where

$$B_1=\frac{\sigma_2^2-k_1^2}{k_1^2-k_2^2}, \qquad B_2=\frac{\sigma_2^2-k_1^2}{k_2^2-k_1^2}, \qquad B_3=1,$$

and

$$(2.14)\quad H_j^{*(l)}=\frac{p}{4\pi\varrho c_2^2}\left[\left(\frac{e^{ik_1R}-e^{ik_2R}}{R(k_1^2-k_2^2)}\right)\delta_{jl}+\right.$$
$$\left.+\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_l}\left(C_1\frac{e^{ik_1R}}{R}+C_2\frac{e^{ik_2R}}{R}+C_3\frac{1}{R}\right)\right],$$
$$j,l=1,2,3,$$

where

$$C_1=\frac{1}{k_1^2(k_1^2-k_2^2)}, \qquad C_2=\frac{1}{k_2^2(k_1^2-k_2^2)}, \qquad C_3=\frac{1}{k_1^2k_2^2}.$$

We derive formulae describing the displacements and rotations from Eqs. (1.4) and (1.5). Since $\boldsymbol{\eta}=0$, we have

$$(2.15)\qquad \mathbf{u}=\operatorname{grad}\Phi+\operatorname{rot}\mathbf{\Psi}, \quad \boldsymbol{\omega}=\operatorname{rot}\mathbf{H}.$$

After some simple calculations we get

$$(2.16)\quad u_j^*=U_j^{*(l)}=\frac{1}{4\pi\varrho\omega^2}\left\{\delta_{jl}\left(\frac{k_1^2B_1e^{ik_1R}+k_2^2B_2e^{ik_2R}}{R}\right)+\right.$$
$$\left.+\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_l}\left(B_1\frac{e^{ik_1R}}{R}+B_2\frac{e^{ik_2R}}{R}-A_1\frac{e^{i\mu_1R}}{R}-A_2\frac{e^{i\mu_2R}}{R}\right)\right\}$$

$$(2.17)\quad \omega_j^*=\Omega_j^{*(l)}=\frac{p}{4\pi\varrho c_2^2(k_1^2-k_2^2)}\in_{ljk}\frac{\partial}{\partial x_k}\left(\frac{e^{ik_1R}-e^{ik_2R}}{R}\right), \quad j,k,l=1,2,3.$$

We have further to shift the concentrated force from the origin of the coordinate system to point $\boldsymbol{\xi}$. In this way we obtain the Green functions $U_j^{*(l)}(\mathbf{x},\boldsymbol{\xi})$, $\Omega_j^{*(l)}(\mathbf{x},\boldsymbol{\xi})$ and $\theta^{*(l)}(\mathbf{x},\boldsymbol{\xi})$. Eqs. (2.16)–(2.18) hold true, R denoting the distance between the points $\mathbf{x}$ and $\boldsymbol{\xi}$.

$$R=[(x_1-\xi_1)^2+(x_2-\xi_2)^2+(x_3-\xi_3)^2]^{1/2}.$$

The functions $U_j^{*(l)}(\mathbf{x},\boldsymbol{\xi})$ and $\Omega_j^{*(l)}(\mathbf{x},\boldsymbol{\xi})$ describe the displacement and rotation tensors, respectively. Together with $\theta^{*(l)}(\mathbf{x},\boldsymbol{\xi})$ they form the group of Green functions for an unlimited medium — as a result of the action of the unitary and concentrated force. Let us remark that for the rotation ω_l^* parallel to the x_l-axis we obtain the zero-value ($\in_{llk}=0$).

We shall now consider the case of the uncoupled problem. In the engineering theory of stresses the term $-\eta \operatorname{div} \mathbf{u}$ in Eq. (1.3) is disregarded. Thus Eqs. (1.1) and (1.3) become uncoupled. Putting $\eta = 0$ (or $\varepsilon = 0$*), which is equivalent, into the formulae (2.16) and (2.17) we get

$$(2.18) \quad u_j^* = U_j^{*(l)}(\mathbf{x}, \boldsymbol{\xi})|_{\varepsilon=0} = \frac{1}{4\pi\varrho\omega^2}\left\{\delta_{jl}\left(\frac{k_1^2 B_1 e^{ik_1 R} + k_2^2 B_2 e^{ik_2 R}}{R}\right) + \right.$$
$$\left. + \frac{\partial}{\partial x_j}\frac{\partial}{\partial x_l}\left[\frac{1}{R}(B_1 e^{ik_1 R} + B_2 e^{ik_2 R} + B_3 e^{i\gamma R})\right]\right\},$$

$$(2.19) \quad \omega_j^* = \Omega_j^{*(l)}(\mathbf{x}, \boldsymbol{\xi})|_{\varepsilon=0} = \frac{p}{4\pi\varrho c_2^2 (k_1^2 - k_2^2)} \in_{ljk} \frac{\partial}{\partial x_k}\left(\frac{e^{ik_1 R} - e^{ik_2 R}}{R}\right).$$

As may be easily seen, only the formula (2.16) is changed.

Passing from the micropolar to the classical elastic medium, we have to assume in (2.18) and (2.19) $\alpha = 0$. In this way we get the known formulae [4]

$$(2.20) \quad U_j^{*(l)} = \frac{e^{i\tau R}}{4\pi\mu R}\delta_{jl} - \frac{1}{4\pi\varrho\omega^2}\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_l}\left(\frac{e^{i\sigma R} - e^{i\tau R}}{R}\right),$$

$$(2.21) \quad \Omega_j^{*(l)} = 0, \quad \tau = \frac{\omega}{c_2^2}, \quad c_2^0 = \left(\frac{\mu}{\varrho}\right)^{1/2}, \quad \sigma = \frac{\omega}{c_1}, \quad c_1 = \left(\frac{\lambda + 2\mu}{\varrho}\right)^{1/2}.$$

3. Effect of concentrated couples

Since $\mathbf{X}^* = 0$, there is $\vartheta^* = 0$ and $\chi^* = 0$, too. Assuming, moreover, $Q^* = 0$ we make sure that the functions Φ^* and θ^* are also equal to zero. It means that the action of concentrated couples does not induce the appearance of the longitudinal wave. Thus we have to solve the following system of equations:

$$(3.1) \quad (\nabla^2 + \sigma_3^2)\Sigma^* = -\frac{1}{c_3^2}\sigma^*,$$

$$(3.2) \quad (\nabla^2 + k_1^2)(\nabla^2 + k_2^2)\boldsymbol{\Psi}^* = \frac{r}{c_4}\operatorname{rot}\boldsymbol{\eta}^*,$$

$$(3.3) \quad (\nabla^2 + k_1^2)(\nabla^2 + a_2^2)\mathbf{H}^* = -\frac{1}{c_4^2}D_1\boldsymbol{\eta}^*.$$

The action of body couples results only in the appearance of the longitudinal wave — described by Eq. (3.1), and of transverse waves — described by Eqs. (3.2) and (3.3). They are not accompanied by the formation of a temperature field. Assume that at point $\boldsymbol{\xi}$ a unitary concentrated moment is acting oriented along the x_l-axis, i.e. $Y_j^* = \delta(\mathbf{x} - \boldsymbol{\xi})\delta_{jl}$. The quantities θ^* and $\boldsymbol{\eta}^*$ will be determined from the formulae resembling Eqs. (2.5) and (2.6). Denoting by $V_j^{*(l)}(\mathbf{x}, \boldsymbol{\xi})$ the displacement tensor and by $W_j^{*(l)}$ the rotation tensor we derive from the solutions of Eqs. (3.1)—(3.3) the following formulae [3]:

*) For $\varepsilon = 0$ there is $\mu_1 = \sigma$, $\mu_2 = \sqrt{q}$.

$$(3.4)\qquad u_j^* = V_j^{*(l)} = \frac{r}{4\pi J c_4^2 (k_1^2 - k_2^2)} \epsilon_{ljk} \frac{\partial}{\partial x_k}\left(\frac{e^{ik_1 R} - e^{ik_2 R}}{R}\right),$$

$$(3.5)\qquad \omega_j^* = W_j^{*(l)} = -\frac{1}{4\pi J c_4^2}\left\{\frac{1}{R}(k_1^2 G_1 + k_2^2 G_2)\,\delta_{lj} + \right.$$

$$\left. + \frac{\partial}{\partial x_l}\frac{\partial}{\partial x_j}\left[\frac{1}{R}(G_1 e^{ik_1 R} + G_2 e^{ik_2 R} + G_3 e^{ik_3 R})\right]\right\},$$

$$l, j, k = 1, 2, 3.$$

Here:

$$G_1 = \frac{k_1^2 - \sigma_2^2}{k_1^2 (k_1^2 - k_2^2)}, \qquad G_2 = \frac{k_2^2 - \sigma_2^2}{k_2^2 (k_2^2 - k_1^2)}, \qquad G_3 = -\frac{\sigma_2^2}{k_1^2 k_2^2}.$$

The function $V_j^{*(l)}(\mathbf{x}, \boldsymbol{\xi})$ and $W_j^{*(l)}(\mathbf{x}, \boldsymbol{\xi})$ represent, respectively, the displacement tensor and the rotation tensor which appears in an unlimited medium under the action of a unitary concentrated couple Y_j^* parallel to the x_l-axis. Let us remark that the displacement $u_l^* = V_l^{*(l)} = 0$, since $\epsilon_{llk} = 0$.

4. Effect of concentrated heat sources

Assuming $\mathbf{X} = 0$, $\mathbf{Y} = 0$, $Q = 0$ we see that what remains from the system of wave equations (1.15)—(1.19) is the system of the following two equations

$$(4.1)\qquad (\nabla^2 + \mu_1^2)(\nabla^2 + \mu_2^2)\,\Phi^* = -\frac{m}{\varkappa} Q^*,$$

$$(4.2)\qquad (\nabla^2 + \mu_1^2)(\nabla^2 + \mu_1^2)\,\theta^* = -\frac{1}{\varkappa}(\nabla^2 + \sigma^2)\, Q^*.$$

The above system of equations is identical with that describing the phenomena occurring in the classical elastic medium. The solutions of these equations are known [5].

$$(4.3)\qquad \Phi^* = \frac{m}{4\pi\varkappa(\mu_1^2 - \mu_2^2) R}(e^{i\mu_2 R} - e^{i\mu_1 R}),$$

$$(4.4)\qquad \theta^* = \frac{1}{4\pi\varkappa(\mu_1^2 - \mu_2^2) R}\left[(\mu_1^2 - \sigma^2)\, e^{i\mu_1 R} - (\mu_2^2 - \sigma^2)\, e^{i\mu_2 R}\right].$$

Since $\mathbf{u}^* = \operatorname{grad}\Phi^*$, $\boldsymbol{\omega}^* = 0$ we get

$$(4.5)\qquad u_j^* = \Gamma_j^*(\mathbf{x}, \boldsymbol{\xi}) = \frac{\partial}{\partial x_j}\Phi^* = \frac{m}{4\pi\varkappa(\mu_1^2 - \mu_2^2)}\frac{\partial}{\partial x_j}\left(\frac{e^{i\mu_2 R} - e^{i\mu_1 R}}{R}\right),$$

$$(4.6)\qquad \omega_j^* = 0, \quad j = 1, 2, 3.$$

In this way we obtained all the Green functions for the micropolar thermoelasticity. The functions $U_j^{*(l)}$, $\Omega_j^{*(l)}$, $\theta^{*(l)}$, $V_j^{*(l)}$, $W_j^{*(l)}$ and $\Gamma_j^{*(l)}$ allow to determine — by the method of superposition — the functions $\mathbf{u}^*(\mathbf{x})$, $\boldsymbol{\omega}^*(\mathbf{x})$, $\theta^*(\mathbf{x})$ for any arbitrary distribution of body forces, body couples as well as heat sources within a limited region.

Let us consider two particular cases.

a) Assume that a concentrated force $X_j^* = \delta(\mathbf{x}-\boldsymbol{\xi})\,\delta_{jr}$ is acting at point $\boldsymbol{\xi}$. The action of this force will induce the appearance of the displacement field $U_j^{*(r)}(\mathbf{x},\boldsymbol{\xi})$, the rotation field $\Omega_j^{*(r)}(\mathbf{x},\boldsymbol{\xi})$ and the temperature field $\theta^{*(r)}(\mathbf{x},\boldsymbol{\xi})$.

Now, let us assume that at point $\boldsymbol{\eta}$ the body couple $Y_j^* = \delta(\mathbf{x}-\boldsymbol{\eta})\,\delta_{jl}$ acts inducing the displacement field $V_j^{*(l)}(\mathbf{x},\boldsymbol{\eta})$ and the rotation field $W_j^{*(l)}(\mathbf{x},\boldsymbol{\eta})$.

To the above-mentioned causes we apply the reciprocity theorem, taken from [6] and adapted for the causes varying harmonically in time.

$$(4.7)\qquad \frac{\eta\varkappa i\omega}{\nu}\left[\int_V (X_i^* u_i^{*\prime} + Y_i^* \omega_i^{*\prime} - X_i^{*\prime} u_i^* - Y_i^{*\prime} \omega_i^*)\,dV\right] = \int_V (Q^*\theta^{*\prime} - Q^{*\prime}\theta^*)\,dV.$$

In view of the causes listed above under a) we have

$$\int_V \delta(\mathbf{x}-\boldsymbol{\xi})\,\delta_{jr}\,V_j^{*(l)}(\mathbf{x},\boldsymbol{\eta})\,dV(\mathbf{x}) = \int_V \delta(\mathbf{x}-\boldsymbol{\eta})\,\delta_{jl}\,\Omega_j^{*(r)}(\mathbf{x},\boldsymbol{\xi})\,dV(\mathbf{x}).$$

Hence, we obtain the most interesting relation

$$(4.8)\qquad V_r^{*(l)}(\boldsymbol{\xi},\boldsymbol{\eta}) = \Omega_l^{*(r)}(\boldsymbol{\eta},\boldsymbol{\xi}).$$

Bearing in mind Eqs. (2.19) and (3.4) we may assert, that the above relation is verified.

b) Let a concentrated force $X_j^* = \delta(\mathbf{x}-\boldsymbol{\xi})\,\delta_{jr}$ act at point $\boldsymbol{\xi}$. This action will induce the appearance of the fields $U_j^{*(l)}$, $\Omega_j^{*(l)}$, $\theta^{*(l)}$. Assume that a heat source with amplitude $Q^* = \delta(\mathbf{x}-\boldsymbol{\eta})$ acts at the point $\boldsymbol{\eta}$. The action of this heat source will provoke the appearance of the displacement field $\Gamma_j^*(\mathbf{x},\boldsymbol{\eta})$ and the temperature field $\theta^*(\mathbf{x},\eta)$. Making use of Eqs. (4.7) we get

$$\frac{\eta\varkappa i\omega}{\nu}\int_V \delta(\mathbf{x}-\boldsymbol{\xi})\,\delta_{jr}\,\Gamma_j^*(\mathbf{x},\boldsymbol{\eta})\,dV(\mathbf{x}) = -\int_V \delta(\mathbf{x}-\boldsymbol{\eta})\,\theta^{*(r)}(\mathbf{x},\boldsymbol{\xi})\,dV(\mathbf{x}),$$

hence,

$$(4.9)\qquad \frac{\eta\varkappa i\omega}{\nu}\,\Gamma_r^*(\boldsymbol{\xi},\boldsymbol{\eta}) = -\theta^{*(r)}(\boldsymbol{\eta},\boldsymbol{\xi}).$$

As may be easily seen — bearing in mind Eqs. (2.12) and (4.5) — Eq. (4.9) is, obviously, fulfilled.

Some attention should be paid to the two-dimensional problem.

If in an unlimited elastic medium the concentrated force $X_j = \delta(x_1)\,\delta(x_2)\,e^{-i\omega t}\,\delta_{j1}$ oriented along the x_1-axis and distributed uniformly along the x_3-axis is acting, then the displacements and rotations will be independent of the variable x_3.

Green functions appropriate for the two-dimensional problem may be obtained by superposition starting with Green functions for the three-dimensional problem. We take advantage of the formula (2.16) assuming that the concentrated force oriented along the x_1-axis is acting at the point $(0, 0, \xi_3)$. Integ-ating the $U_j^{*(1)}$ function along the x_3-axis within the interval $-\infty - +\infty$ we obtain the corresponding Green functions for the two-dimensional problem.

Taking into account that

$$\int_{-\infty}^{\infty} \frac{\exp - ik_a \sqrt{r^2+\xi_3^2}}{\sqrt{r^2+\xi_3^2}} d\xi_3 = 2K_0(-ik_a r) = \pi i H_0^{(1)}(k_a r), \quad a = 1, 2,$$

where $r = (x_1^2+x_2^2)^{1/2}$, we obtain from Eq. (2.16) the following formula:

$$(4.10) \quad U_j^{*(1)}(\mathbf{x}, 0) = \frac{i}{4\varrho\omega^2} \{\delta_{j1}(k_1^2 H_0^{(1)}(k_1 r)+k_2^2 H_0^{(1)}(k_2 r)+ + \frac{\partial}{\partial x_j}\frac{\partial}{\partial x_j}(B_1 H_0^{(1)}(k_1 r)+B_2 H_0^{(1)}(k_2 r) - A_1 H_0^{(1)}(\mu_1 r) - - A_2 H_0^{(1)}(\mu_2 r))\}, \quad j = 1, 2.$$

We have now to direct the concentrated force $X_j^* = \delta(x_1)\delta(x_2)\delta_{jl}$ along the x_l-axis $(l = 1, 2)$ and to shift it from the origin of the coordinate system to point $\boldsymbol{\xi} \equiv (\xi_1, \xi_2, 0)$. In this way we get

$$(4.11) \quad U_j^{*(l)}(x_1, x_2; \xi_1, \xi_2) = \frac{i}{4\varrho\omega^2}\{\delta_{jl}(k_1^2 H_0^{(1)}(k_1 r)+k_2^2 H_0^{(1)}(k_2 r))+ + \frac{\partial}{\partial x_j}\frac{\partial}{\partial x_l}(B_1 H_0^{(1)}(k_1 r) + B_2 H_0^{(1)}(k_2 r) - A_1 H_0^{(1)}(k_1 r) - A_2 H_0^{(1)}(k_2 r))\},$$

where $r = [(x_1-\xi_1)^2+(x_2-\xi_2)^2]^{1/2}$.

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В. НОВАЦКИЙ, **ФУНКЦИИ ГРИНА ДЛЯ МИКРОПОЛЯРНОЙ ТЕРМОУПРУГОСТИ**

Приводятся основные решения дифференциальных уравнений для сопряженной микрополярной термоупругости. Даются функции Грина для перемещений, оборотов и температуры, вызванных действием поочередных источников: сосредоточенной единичной силы, сосредоточенного единичного момента и сосредоточенного единичного источника тепла. Функции Грина относятся к бесконечной упругой среде.