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Green Functions for Micropolar Elasticity

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1. Introduction

The aim of this paper is to give basic solutions of wave equations in an unlimited medium for micropolar elasticity and, in particular, to present in a closed form wave functions as well as the displacement and rotation field formed in such a medium under the action of a concentrated force or a couple changing harmonically in time.

Let us consider first the system of linearized equations for micropolar elasticity [1]–[4]:

$$(1.1)\qquad (\mu+\alpha)\,\nabla^2\mathbf{u}+(\lambda+\mu-\alpha)\,\mathrm{grad\,div}\,\mathbf{u}+2\alpha\,\mathrm{rot}\,\boldsymbol{\omega}+\mathbf{X}=\varrho\ddot{\mathbf{u}},$$

$$(1.2)\qquad (\gamma+\varepsilon)\,\nabla^2\boldsymbol{\omega}+(\beta+\gamma-\varepsilon)\,\mathrm{grad\,div}\,\boldsymbol{\omega}-4\alpha\boldsymbol{\omega}+2\alpha\,\mathrm{rot}\,\mathbf{u}+\mathbf{Y}=J\ddot{\boldsymbol{\omega}}.$$

The following notations are adopted throughout the present paper: $\mathbf{u}$ denotes the displacement vector, $\boldsymbol{\omega}$ — the rotation vector, $\mathbf{X}$ — the vector of body-forces, $\mathbf{Y}$ — the vector of body-couples, the symbols $\mu, \lambda, \alpha, \beta, \gamma, \varepsilon$ stand for material constants, ϱ — for density and J — for rotational inertia. The quantities $\mathbf{u}, \boldsymbol{\omega}, \mathbf{X}, \mathbf{Y}$ are functions of the position $\mathbf{x}$ and time t.

Eqs. (1.1) and (1.2) are coupled.

Decomposing the displacements and rotation vectors into their potential and solenoidal parts, we get

$$(1.3)\qquad \mathbf{u}=\mathrm{grad}\,\Phi+\mathrm{rot}\,\boldsymbol{\Psi},\quad \mathrm{div}\,\boldsymbol{\Psi}=0,$$

$$(1.4)\qquad \boldsymbol{\omega}=\mathrm{grad}\,\Sigma+\mathrm{rot}\,\mathbf{H},\quad \mathrm{div}\,\mathbf{H}=0,$$

and, similarly, decomposing the body-force and body-couple vectors into two terms each we obtain

$$(1.5)\qquad \mathbf{X}=\varrho\,(\mathrm{grad}\,\vartheta+\mathrm{rot}\,\boldsymbol{\chi}),$$

$$(1.6)\qquad \mathbf{Y}=J\,(\mathrm{grad}\,\sigma+\mathrm{rot}\,\boldsymbol{\eta}),$$

Thus we may transform the system of Eqs. (1.1) and (1.2) into the system of the following four equations

$$\Box_1 \Phi + \varrho\vartheta = 0, \tag{1.7}$$

$$\Box_3 \Sigma + J\sigma = 0, \tag{1.8}$$

$$(\Box_2 \Box_4 + 4\alpha^2 \nabla^2)\mathbf{\Psi} = 2\alpha J \operatorname{rot}\boldsymbol{\eta} - \varrho\Box_4 \boldsymbol{\chi}, \tag{1.9}$$

$$(\Box_2 \Box_4 + 4\alpha^2 \nabla^2)\mathbf{H} = 2\alpha\varrho \operatorname{rot}\boldsymbol{\chi} - J\Box_2 \boldsymbol{\eta}. \tag{1.10}$$

with the following notations

$$\Box_1 = (\lambda+2\mu)\nabla^2 - \varrho\partial_t^2, \qquad \Box_2 = (\mu+\alpha)\nabla^2 - \varrho\partial_t^2,$$
$$\Box_3 = (\beta+2\gamma)\nabla^2 - 4\alpha - J\partial_t^2, \quad \Box_4 = (\gamma+\varepsilon)\nabla^2 - 4\alpha - J\partial_t^2,$$
$$\nabla^2 = \partial_i\partial_i, \quad \partial_t^2 = \partial^2/\partial t^2.$$

Eq. (1.7) describes the longitudinal, while Eq. (1.8) the rotational wave. Let us remark that in an infinite elastic space the body-force $\mathbf{X}' = \varrho \operatorname{grad}\vartheta$ generates only the longitudinal, whereas the body-couple $Y' = J \operatorname{grad}\sigma$ only the rotational waves.

Eqs. (1.9) and (1.10) represent the modified transverse waves. We assume that the body-forces and-body couples responsible for the wave disturbances change harmonically in time. This may be noted in the form

$$\mathbf{X}(\mathbf{x}, t) = \mathbf{X}^*(\mathbf{x})\, e^{-i\omega t}, \quad \mathbf{Y}(\mathbf{x}, t) = \mathbf{Y}^*(\mathbf{x})\, e^{-i\omega t}. \tag{1.11}$$

Consequently, the displacements $\mathbf{u}$, the rotations $\boldsymbol{\omega}$ and also the functions $\Phi, \Sigma, \mathbf{\Psi}, \mathbf{H}$ change harmonically in time, too.

Marking with an asterisk the amplitudes of these functions, we reduce Eqs. (1.7)—(1.10) to the forms

$$(\nabla^2 + \sigma_1^2)\Phi^* = -\frac{1}{c_1^2}\vartheta^*, \tag{1.12}$$

$$(\nabla^2 + \sigma_3^2)\Sigma^* = -\frac{1}{c_3^2}\sigma^*, \tag{1.13}$$

$$(\nabla^2 + k_1^2)(\nabla^2 + k_2^2)\mathbf{\Psi}^* = \frac{r}{c_4^2}\operatorname{rot}\boldsymbol{\eta}^* - \frac{1}{c_2^2}D_2\boldsymbol{\chi}^*; \tag{1.14}$$

$$(\nabla^2 + k_1^2)(\nabla^2 + k_2^2)\mathbf{H}^* = \frac{p}{c_2^2}\operatorname{rot}\boldsymbol{\chi}^* - \frac{1}{c_4^2}D_1\boldsymbol{\eta}^*. \tag{1.15}$$

wherein the following notations have been introduced

$$\sigma_1 = \frac{\omega}{c_1}, \quad c_1 = \left(\frac{\lambda+2\mu}{\varrho}\right)^{1/2}, \quad \sigma_3 = \left(\frac{\omega^2 - \omega_0^2}{c_3}\right)^{1/2}, \quad c_3 = \left(\frac{\beta+2\gamma}{J}\right)^{1/2},$$
$$\omega_0^2 = \frac{4\alpha}{J}, \quad r = \frac{2\alpha}{\varrho c_2^2}, \quad \sigma_2 = \frac{\omega}{c_2}, \quad c_2 = \left(\frac{\mu+\alpha}{\varrho}\right)^{1/2}, \quad p = \frac{2\alpha}{Jc_4^2},$$
$$\sigma_4 = \frac{\omega}{c_4}, \quad c_4 = \left(\frac{\gamma+\varepsilon}{J}\right)^{1/2}, \quad D_1 = \nabla^2 + \sigma_2^2, \quad D_2 = \nabla^2 + \sigma_4^2 - 2p.$$

The quantities k_1^2, k_2^2 stand for the roots of the following biquadratic equation

$$k^4 - k^2 (\sigma_2^2+\sigma_4^2+p (r-2))+\sigma_2^2 (\sigma_4^2-2p) = 0. \tag{1.16}$$

Thus we have

$$k_{1,2}^2 = \frac{1}{2} [\sigma_2^2+\sigma_4^2+p (r-2)\pm\sqrt{[\sigma_4^2-\sigma_2^2+p (r-2)]^2+4pr\sigma_2^2}]. \tag{1.17}$$

The discriminant appearing in (1.17) is, obviously, positive. Let us now consider the homogeneous Eq. (1.14). The solution of this equation may be presented — according to a theorem due to Boggio [5] — in the form of a sum of two partial solutions $\mathbf{\Psi}'^*$ and $\mathbf{\Psi}''^*$:

$$\mathbf{\Psi}^* = \mathbf{\Psi}'^*+\mathbf{\Psi}''^*, \tag{1.18}$$

satisfying the Helmholtz vector equations

$$(\nabla^2+k_1^2)\,\mathbf{\Psi}'^* = 0, \quad (\nabla^2+k_2^2)\,\mathbf{\Psi}''^* = 0. \tag{1.19}$$

The singular integrals of Eqs. (1.19) are the functions $\frac{e^{\pm ik_a R}}{R}$, $a=1,2$. However, only the solution $\frac{1}{R}\,e^{ik_a R}$ have a physical meaning as only the expressions

$$\mathrm{Re}\left[e^{-i\omega t}\frac{1}{R}\,e^{ik_a R}\right] = \frac{1}{R}\cos\omega\left(t-\frac{R}{v_a}\right), \quad v_a = \frac{\omega}{k_a}, \quad a = 1, 2,$$

represent the divergent waves propagating from the point of disturbance towards infinity. Thus, the solution of the homogeneous Eq. (1.14) will take the form

$$\mathbf{\Psi}^* = \mathbf{A}\frac{e^{ik_1 R}}{R}+\mathbf{B}\,\frac{e^{ik_2 R}}{R}. \tag{1.20}$$

Similarly, the solution of the homogeneous Eq. (1.15) will be given in the form of the following function

$$\mathbf{H}^* = \mathbf{C}\,\frac{e^{ik_1 R}}{R}+\mathbf{D}\,\frac{e^{ik_2 R}}{R}. \tag{1.21}$$

Only real phase velocities may appear in terms representing the functions $\mathbf{\Psi}$ and $\mathbf{H}$. Thus, we should have $k_1^2 > 0$, $k_2^2 > 0$. The first condition is already satisfied. The second one will be satisfied if $\sigma_4 > 2p$ or $\omega^2 > \frac{4\alpha}{J}$, what results from the relation: $k_1^2\, k_2^2 = \sigma_2^2 (\sigma_4^2 - 2p) > 0$. In expressions (1.20) and (1.21) two waves appear undergoing dispersion (since k_1 and k_2 are the functions of frequency ω).

2. Effect of the concentrated force

Let us first consider the action of body forces. Since $\mathbf{Y} = 0$, there is also $\sigma = 0$ and $\boldsymbol{\eta} = 0$. In an infinite elastic space rotation waves will not appear ($\Sigma^* = 0$). Thus we have to solve the system of equations

$$(2.1)\qquad (\nabla^2+\sigma_1^2)\,\Phi^* = -\frac{1}{c_1^2}\,\vartheta^*,$$

$$(2.2)\qquad (\nabla^2+k_1^2)(\nabla^2+k_2^2)\,\boldsymbol{\Psi}^* = -\frac{1}{c_2^2}\,D_2\,\boldsymbol{\chi}^*,$$

$$(2.3)\qquad (\nabla^2+k_1^2)(\nabla^2+k_2^2)\,\mathbf{H}^* = \frac{p}{c_2^2}\operatorname{rot}\boldsymbol{\chi}^*.$$

In a general approach, we determine the function ϑ^* and χ^* for an arbitrary vector of body forces from the following formulae [6]

$$(2.4)\qquad \vartheta^*(\mathbf{x}) = -\frac{1}{4\pi\varrho}\int_V X_j^*(\boldsymbol{\xi})\,\frac{\partial}{\partial x_j}\left(\frac{1}{R(\boldsymbol{\xi},\mathbf{x})}\right)dV(\boldsymbol{\xi}),$$

$$(2.5)\qquad \chi_i^*(\mathbf{x}) = -\frac{1}{4\pi\varrho}\int_V \varepsilon_{ijk}\,X_j^*(\boldsymbol{\xi})\,\frac{\partial}{\partial x_k}\left(\frac{1}{R(\boldsymbol{\xi},\mathbf{x})}\right)dV(\boldsymbol{\xi}),\qquad i,j,k=1,2,3.$$

Now, introducing into Eqs. (2.4) and (2.5) the expression

$$X_j(\mathbf{x}) = \delta(x_1)\,\delta(x_2)\,\delta(x_3)\,\delta_{1j},\qquad j=1,2,3,$$

which describes the action of the concentrated force starting with the origin of the coordinate system and acting along the x_1-axis we obtain successively

$$(2.6)\qquad \begin{aligned} &\vartheta^*(\mathbf{x}) = -\frac{1}{4\pi\varrho}\,\frac{\partial}{\partial x_1}\left(\frac{1}{R}\right),\qquad \chi_1^*=0,\qquad \chi_2^* = \frac{1}{4\pi\varrho}\,\frac{\partial}{\partial x_3}\left(\frac{1}{R}\right),\\ &\chi_3^* = -\frac{1}{4\pi\varrho}\,\frac{\partial}{\partial x_2}\left(\frac{1}{R}\right),\qquad R=(x_1^2+x_2^2+x_3^2)^{1/2}.\end{aligned}$$

Thus, we have to solve the following equations

$$(2.7)\qquad (\nabla^2+\sigma_1^2)\,\Phi^* = \frac{1}{4\pi\varrho c_1^2}\,\frac{\partial}{\partial x_1}\left(\frac{1}{R}\right),$$

$$(2.8)\qquad \begin{aligned} &(\nabla^2+k_1^2)(\nabla^2+k_2^2)\,\Psi_2^* = -\frac{1}{4\pi\varrho c_2^2}(\nabla^2+\sigma_4^2-2p)\,\frac{\partial}{\partial x_3}\left(\frac{1}{R}\right),\\ &(\nabla^2+k_1^2)(\nabla^2+k_2^2)\,\Psi_3^* = \frac{1}{4\pi\varrho c_2^2}(\nabla^2+\sigma_4^2-2p)\,\frac{\partial}{\partial x_2}\left(\frac{1}{R}\right),\qquad \Psi_1^*=0,\end{aligned}$$

$$(2.9)\qquad \begin{aligned} &(\nabla^2+k_1^2)(\nabla^2+k_2^2)\,H_1^* = -\frac{p}{4\pi\varrho c_2^2}(\nabla^2-\partial_1^2)\left(\frac{1}{R}\right),\\ &(\nabla^2+k_1^2)(\nabla^2+k_2^2)\,H_2^* = \frac{p}{4\pi\varrho c_2^2}\,\frac{\partial}{\partial x_1}\,\frac{\partial}{\partial x_2}\left(\frac{1}{R}\right),\\ &(\nabla^2+k_1^2)(\nabla^2+k_2^2)\,H_3^* = \frac{p}{4\pi\varrho c_2^2}\,\frac{\partial}{\partial x_1}\,\frac{\partial}{\partial x_3}\left(\frac{1}{R}\right).\end{aligned}$$

The solution of Eq. (2.7) is known from classical elastokinetics [6]. It reads as follows

$$\Phi^*(\mathbf{x}) = -\frac{1}{4\pi\varrho\omega^2}\frac{\partial}{\partial x_1}\left(\frac{e^{i\sigma_1 R}-1}{R}\right). \tag{2.10}$$

We shall solve Eqs. (2.8) and (2.9) applying the exponential Fourier integral transformation. Thus, the solution for Ψ_2^*, e.g., will have the form

$$\Psi_2^*(\mathbf{x}) = \frac{1}{8\varrho c_2^2\pi^3}\frac{\partial}{\partial x_3}\underset{-\infty}{\overset{\infty}{\iiint}}\frac{(\alpha^2-\sigma_4^2+2p)\,e^{i\alpha_k x_k}\,d\alpha_1\,d\alpha_2\,d\alpha_3}{\alpha^2(\alpha^2-k_1^2)(\alpha^2-k_2^2)}, \tag{2.11}$$

$$\alpha^2 = \alpha_1^2+\alpha_2^2+\alpha_3^2.$$

Taking into consideration that

$$\underset{-\infty}{\overset{\infty}{\iiint}}\frac{e^{i\alpha_k x_k}\,d\alpha_1\,d\alpha_2\,d\alpha_3}{\alpha^2-k_a^2} = 2\pi^2\frac{e^{ik_j}}{R},$$

we obtain from (2.11)

$$\Psi_2^* = \frac{1}{4\pi\varrho\omega^2}\frac{\partial}{\partial x_3}\left(A_1\frac{e^{ik_1R}}{R}+A_2\frac{e^{ik_2R}}{R}+A_3\frac{1}{R}\right), \tag{2.12}$$

where

$$A_1 = \frac{\sigma_2^2-k_2^2}{k_1^2-k_2^2},\qquad A_2 = \frac{\sigma_2^2-k_1^2}{k_2^2-k_1^2},\qquad A_3 = -1.$$

Solving the equation for Ψ_2^*, we get

$$\Psi_3^* = -\frac{1}{4\pi\varrho\omega^2}\frac{\partial}{\partial x_2}\left(A_1\frac{e^{ik_1R}}{R}+A_2\frac{e^{ik_2R}}{R}+A_3\frac{1}{R}\right). \tag{2.13}$$

The application of the exponential Fourier integral transformation to the system of Eqs. (2.9) affords

$$H_1^* = \frac{p}{4\pi\varrho c_2^2}\left\{\frac{e^{ik_1R}-e^{ik_2R}}{R(k_1^2-k_2^2)}+\partial_1^2\left(B_1\frac{e^{ik_1R}}{R}+B_2\frac{e^{ik_2R}}{R}+B_3\frac{1}{R}\right)\right\}, \tag{2.14}$$

$$H_2^* = \frac{p}{4\pi\varrho c_2^2}\partial_1\partial_2\left(B_1\frac{e^{ik_1R}}{R}+B_2\frac{e^{ik_2R}}{R}+B_3\frac{1}{R}\right), \tag{2.15}$$

$$H_3^* = \frac{p}{4\pi\varrho c_2^2}\partial_1\partial_3\left(B_1\frac{e^{ik_1R}}{R}+B_2\frac{e^{ik_2R}}{R}+B_3\frac{1}{R}\right), \tag{2.16}$$

where

$$B_1 = \frac{1}{k_1^2(k_1^2-k_2^2)},\qquad B_2 = \frac{1}{k_2^2(k_2^2-k_1^2)},\qquad B_3 = \frac{1}{k_1^2k_2^2}.$$

We obtain the displacements $\mathbf{u}$ and the rotations $\boldsymbol{\omega}$ from the formulae (1.3) and (1.4). Since $\boldsymbol{\eta}^* = 0$, there is

$$(2.17)\quad \begin{aligned} u_1^* &= \partial_1 \Phi^* + \partial_2 \Psi_3^* - \partial_3 \Psi_2^*, \quad u_2^* = \partial_2 \Phi^* - \partial_1 \Psi_3^*, \quad u_3^* = \partial_3 \Phi^* + \partial_1 \Psi_2^*, \\ \omega_1^* &= \partial_2 H_3^* - \partial_3 H_2^*, \quad \omega_2^* = \partial_3 H_1^* - \partial_1 H_3^*, \quad \omega_3^* = \partial_1 H_2^* - \partial_2 H_1^*. \end{aligned}$$

In this way, we arrive at the following formulae for the displacement $\mathbf{u}^*$ and rotation $\boldsymbol{\omega}^*$ amplitudes

$$(2.18)\quad u_j^* = U_j^{*(1)} = \frac{1}{4\pi\varrho\omega^2}\left(A_1 k_1^2 \frac{e^{ik_1 R}}{R} + A_2 k_2^2 \frac{e^{ik_2 R}}{R}\right)\delta_{1j} + \\ + \frac{1}{4\pi\varrho\omega^2}\partial_1\partial_j\left(A_1\frac{e^{ik_1 R}}{R} + A_2\frac{e^{ik_2 R}}{R} + A_3\frac{e^{i\sigma_1 R}}{R}\right).$$

$$(2.19)\quad \omega_j^* = \Omega_j^{*(1)} = \frac{p\epsilon_{1jk}}{4\pi\varrho c_2^2 (k_1^2 - k_2^2)}\frac{\partial}{\partial x_k}\left(\frac{e^{ik_1 R} - e^{ik_2 R}}{R}\right).$$

Thus we get three components of the displacement vector $U_j^{*(1)}$ and three components of the rotation vector $\Omega_j^{*(1)}$ as well. Let us now shift the concentrated force to the point $\boldsymbol{\xi}$ and direct it parallelly to the x_l-axis. The following formulae may serve as an example:

$$(2.20)\quad \omega_j^* = \Omega_j^{*(l)} = \frac{p\epsilon_{ljk}}{4\pi\varrho c_2^2 (k_1^2 - k_2^2)}\frac{\partial}{\partial x_k}\left(\frac{e^{ik_1 R} - e^{ik_2 R}}{R}\right),$$

where

$$R = [(x_i - \xi_i)(x_i - \xi_i)]^{1/2}.$$

By this method we obtain the rotation tensor $\Omega_j^{(l)}(\mathbf{x}, \boldsymbol{\xi})$, $j, l = 1, 2, 3$ and, similarly, the displacement tensor $U_j^{(l)}(\mathbf{x}, \boldsymbol{\xi})$, $j, l = 1, 2, 3$.

Putting into Eqs. (2.18) and (2.19) $\alpha = 0$ we pass to the classical elastokinetics. We obtain thus [6]

$$(2.21)\quad U_j^{*(l)} = \frac{\delta_{jl}}{4\pi\mu}\frac{e^{i\tau R}}{R} - \frac{1}{4\pi\varrho\omega^2}\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_l}\left(\frac{e^{i\sigma R} - e^{i\tau R}}{R}\right), \\ \Omega^{*(l)} = 0, \quad j, l = 1, 2, 3,$$

where

$$\tau = \frac{\omega}{c_2^0}, \quad c_2^0 = \left(\frac{\mu}{\varrho}\right)^{1/2}, \quad \sigma = \frac{\omega}{c_1}, \quad c_1 = \left(\frac{\lambda + 2\mu}{\varrho}\right)^{1/2}.$$

We return now once more to the formulae (2.18) and (2.19). Observe that the concentrated force acting along the x_1-axis effectuates the rotation $\omega_1^* = \Omega_1^{*(1)} = 0$. Thus it results that the components $\varkappa_{j1}$ $j = 1, 2, 3$ of the curvature-twist tensor $\varkappa_{ji} = u_{i,j}$ are equal to zero. The components of the strain tensor $\gamma_{ji} = u_{i,j} - \varepsilon_{kji}\omega_k$ are less than zero. In (2.18) and (2.19) three kinds of waves appear. Those connected with the quantities k_1, k_2 undergo dispersion.

3. Effect of the body couples

Let us now consider the effect of body couples. Since $\mathbf{X} = 0$, ϑ and $\boldsymbol{\chi}$ are also equal to zero. In an infinite space the longitudinal wave will not occur ($\boldsymbol{\Phi}^* = 0$). Thus, we have to solve only the following system of equations

$$(\nabla^2 + k_3^2)\,\Sigma^* = -\frac{1}{c_3^2}\,\sigma^*, \tag{3.1}$$

$$\begin{cases} (\nabla^2 + k_1^2)(\nabla^2 + k_2^2)\,\boldsymbol{\Psi}^* = \dfrac{r}{c_4^2}\,\mathrm{rot}\,\boldsymbol{\eta}^*, \\ (\nabla^2 + k_1^2)(\nabla^2 + k_2^2)\,\mathbf{H}^* = -\dfrac{1}{c_4^2}\,D_1\boldsymbol{\eta}^*. \end{cases} \tag{3.2}$$

Let us now assume that at the origin of the coordinate system the concentrated couple $Y_j^* = \delta(x_1)\,\delta(x_2)\,\delta(x_3)\,\delta_{1j}$, $j = 1, 2, 3$, is acting. The components σ^* and $\boldsymbol{\eta}^*$ will be obtained from formulae resembling (2.4) and (2.5), namely

$$\sigma^*(\mathbf{x}) = -\frac{1}{4\pi J}\,\frac{\partial}{\partial x_1}\left(\frac{1}{R}\right), \quad \eta_1^* = 0, \quad \eta_2^* = \frac{1}{4\pi J}\,\frac{\partial}{\partial x_3}\left(\frac{1}{R}\right), \tag{3.3}$$

$$\eta_3^* = \frac{1}{4\pi J}\,\frac{\partial}{\partial x_2}\left(\frac{1}{R}\right).$$

Applying a procedure similar to that used in the preceding section, we get the following solutions of Eqs. (3.1) and (3.2)

$$\Sigma^* = -\frac{1}{4\pi J c_3^2 k_3^2}\,\frac{\partial}{\partial x_1}\left(\frac{e^{ik_3R} - 1}{R}\right) \tag{3.4}$$

$$\Psi_j^* = \frac{r}{4\pi J c_4^2 (k_1^2 - k_2^2)}\left(\frac{e^{ik_1R} - e^{ik_2R}}{R}\right)\delta_{j1} + \tag{3.5}$$

$$+\frac{r}{4\pi J c_4^2}\,\partial_1\,\partial_j\left(B_1\frac{e^{ik_1R}}{R} + B_2\frac{e^{ik_2R}}{R} + B_3\frac{1}{R}\right).$$

$$H_j^* = \frac{1}{4\pi J c_4^2}\,\epsilon_{1jk}\,\frac{\partial}{\partial x_k}\left(C_1\frac{e^{ik_1R}}{R} + C_2\frac{e^{ik_2R}}{R} + C_3\frac{1}{R}\right), \quad j = 1, 2, 3, \tag{3.6}$$

$$C_1 = \frac{k_1^2 - \sigma_2^2}{k_1^2(k_1^2 - k_2^2)}, \qquad C_2 = \frac{k_2^2 - \sigma_2^2}{k_1^2(k_2^2 - k_1^2)}, \qquad C_3 = -\frac{\sigma_2^2}{k_1^2 k_2^2}.$$

The displacements and rotations will be obtained from the formulae below

$$\begin{aligned} &u_1^* = \partial_2\,\Psi_3^* - \partial_3\,\Psi_2^*, \quad u_2^* = \partial_3\,\Psi_1^* - \partial_1\,\Psi_3^*, \quad u_3^* = \partial_1\,\Psi_2^* - \partial_2\,\Psi_1^*, \\ &\omega_1^* = \partial_1\,\Sigma^* + \partial_2\,H_3^* - \partial_3\,H_2^*, \quad \omega_2^* = \partial_2\,\Sigma^* - \partial_1\,H_3^*, \\ &\omega_3^* = \partial_3\,\Sigma^* + \partial_1\,H_2^*. \end{aligned} \tag{3.7}$$

Introducing Eqs. (3.4)—(3.6) into Eqs. (3.7) we have

$$(3.8)\qquad u_j^* = V_j^{*(1)} = \frac{r}{4\pi J c_4^2 (k_1^2-k_2^2)}, \quad \varepsilon_{1jk} \frac{\partial}{\partial x_k}\left(\frac{e^{ik_1R}-e^{ik_2R}}{R}\right),$$

$$(3.9)\qquad \omega_j^* = W_j^{*(1)} = -\frac{1}{4\pi J c_4^2}\left(k_1^2 C_1 \frac{e^{ik_1R}}{R} + k_2^2 C_2 \frac{e^{ik_2R}}{R}\right)\delta_{1j} + \\ + \frac{\partial_1 \partial_j}{4\pi J c_4^2}\left(C_1 \frac{e^{ik_1R}}{R} + C_2 \frac{e^{ik_2R}}{R} + C_3 \frac{e^{ik_3R}}{R}\right), \quad k,j=1,2,3.$$

Transferring the concentrated moment to point $\boldsymbol{\xi}$ and directing the moment vector parallelly to the x_l-axis we obtain Green's displacement tensor $V_j^{*(l)}(\mathbf{x},\boldsymbol{\xi})$ and the rotation tensor $W_j^{*(l)}(\mathbf{x},\boldsymbol{\xi})$. To quote an example, we obtain

$$(3.10)\quad V_j^{*(l)}(\mathbf{x},\boldsymbol{\xi}) = \frac{r\epsilon_{ljk}}{4\pi J c_4^2 (k_1^2-k_2^2)} \frac{\partial}{\partial x_k}\left(\frac{e^{ik_1R}-e^{ik_2R}}{R}\right), \quad l,j,k=1,2,3.$$

where $R = [(x_i-\xi_i)(x_i-\xi_i)]^{1/2}$.

Returning to Eqs. (3.8) and (3.9) let us remark that the action of the concentrated moment $Y^* = \delta(x_1)\,\delta(x_2)\,\delta(x_3)\,\delta_{1j}$ leads to the zero-value of the displacement along the x_1-axis ($u_1^*=0$). Thus, also $\gamma_{11}=0$. Since k_1, k_2, k_3 are functions of the frequency ω, all kinds of waves appearing in (3.8) and (3.9) undergo dispersion.

Let us consider a particular case. Assume the concentrated force $X_j^* = \delta(\mathbf{x}-\boldsymbol{\xi})\,\delta_{jr}$ acting at point $\boldsymbol{\xi}$ and oriented parallelly to the x_r-axis. This force will induce a displacement field $U_j^{(r)}(\mathbf{x},\boldsymbol{\xi})$ and a rotation field $\Omega_j^{(r)}(\mathbf{x},\boldsymbol{\xi})$. Now, let the concentrated moment $Y_j^* = \delta(\mathbf{x}-\boldsymbol{\eta})\,\delta_{jl}$ oriented parallelly to the x_l-axis act at point $\boldsymbol{\eta}$. It will induce the displacement $V_j^{(l)}(\mathbf{x},\boldsymbol{\eta})$ and the rotations $W_j^{(l)}(\mathbf{x},\boldsymbol{\eta})$. We apply the theorem on reciprocity [7] to the causes and effects mentioned above

$$(3.11)\qquad \int_V (X_i^* u_i'^* + Y_i^* \omega_i'^*)\,dV = \int_V (X_i'^* u_i^* + Y_i'^* \omega_i^*)\,dV.$$

Eq. (3.11) affords

$$\int_V \delta(\mathbf{x}-\boldsymbol{\xi})\,\delta_{jr} V_j^{(l)}(\mathbf{x},\boldsymbol{\eta})\,dV(\mathbf{x}) = \int_V \delta(\mathbf{x}-\boldsymbol{\eta})\,\delta_{jl}\,\Omega_j^{(r)}(\mathbf{x},\boldsymbol{\xi})\,dV(\mathbf{x})$$

whence

$$(3.12)\qquad V_r^{(l)}(\boldsymbol{\xi},\boldsymbol{\eta}) = \Omega_l^{(r)}(\boldsymbol{\eta},\boldsymbol{\xi}).$$

Making use of Eqs. (2.20) and (3.10), we arrive at

$$V_r^{(l)}(\boldsymbol{\xi},\boldsymbol{\eta}) = \frac{r}{4\pi J c_4^2 (k_1^2-k_2^3)} \epsilon_{lrk} \left|\frac{\partial}{\partial x_k}\left(\frac{e^{ik_1R}-e^{ik_2R}}{R(\mathbf{x},\boldsymbol{\eta})}\right)\right|_{\mathbf{x}=\xi},$$

$$\Omega_l^{(r)}(\boldsymbol{\eta},\boldsymbol{\xi}) = \frac{p}{4\pi\varrho c_2^2 (k_1^2-k_2^2)} \epsilon_{rlk} \left|\frac{\partial}{\partial x_k}\left(\frac{e^{ik_1R}-e^{ik_2R}}{R(\mathbf{x},\boldsymbol{\xi})}\right)\right|_{\mathbf{x}=\eta}.$$

As $r = \dfrac{2\alpha}{\varrho c_2^2}$ and $p = \dfrac{2\alpha}{J c_4^2}$ the relation (3.12) is obviously, verified.

Eq. (3.12) may be considered as an expansion of the J. C. Maxwell theorem on the reciprocity of works known from classical elastokinetics.

A more ample discussion of the problem of solutions of basic equations (1.1) and (1.2) will appear in a separate paper to be published in Proc. of Vibrations Problems.

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В. НОВАЦКИЙ, ФУНКЦИИ ГРИНА ДЛЯ МИКРОПОЛЯРНОЙ УПРУГОСТИ

В настоящей работе дается основное решение дифференциальных уравнений для микрополярной упругости. Приводятся функции Грина (тензор перемещения и тензор оборота) для сосредоточенной силы и для сосредоточенного момента, действующих в бесконечной упругой среде.