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5.

APPLIED MECHANICS

Propagation of Rotation Waves in Asymmetric Elasticity

by

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1. Introduction

In this Note we shall consider an elastic, homogeneous, isotropic and centro-symmetric body filling up the region B bounded by the surface A. Under the effect of external loadings the body will get deformed: A displacement field $\mathbf{u}(\mathbf{x}, t)$ and a rotation field $\boldsymbol{\omega}(\mathbf{x}, t)$ will appear changing with $\mathbf{x}$ (position) and t (time).

The state of strain will be characterized by two asymmetric tensors, namely by the strain tensor γ_{ji} and the curvature -twist tensor $\varkappa_{ji}$. They are connected by the relations [1]—[4]:

$$\gamma_{ji} = u_{i,j} - \varepsilon_{kji}\,\omega_k, \qquad \varkappa_{ji} = \omega_{i,j}. \tag{1.1}$$

The state of stress is described also by two asymmetric tensors, namely by the force-stress tensor σ_{ji} and the couple-stress tensor μ_{ji}. The relation between the state of stress and that of strain is described by the following constitutive equations

$$\sigma_{ji} = (\mu+\alpha)\,\gamma_{ji} + (\mu-\alpha)\,\gamma_{ij} + \lambda\gamma_{kk}\,\delta_{ij}, \tag{1.2}$$

$$\mu_{ji} = (\gamma+\varepsilon)\,\varkappa_{ji} + (\gamma-\varepsilon)\,\varkappa_{ij} + \beta\varkappa_{kk}\,\delta_{ij}. \tag{1.3}$$

The symbols $\alpha, \beta, \gamma, \varepsilon, \mu, \lambda$ stand for the material constants.

Substituting Eqs. (1.2) and (1.3) into the equations of motion we get

$$\sigma_{ji,j} + X_i - \varrho\ddot{u}_i = 0 \tag{1.4}$$

$$\varepsilon_{ijk}\,\sigma_{jk} + \mu_{ji,j} + Y_i - J\ddot{\omega}_i = 0, \qquad i, j, k = 1, 2, 3. \tag{1.5}$$

Next, expressing the quantities γ_{ji} and $\varkappa_{ji}$ in terms of u_i and ω_i, respectively, we obtain a system of equations in vector form

$$\Box_2\,\mathbf{u} + (\lambda+\mu-\alpha)\,\mathrm{grad\,div}\,\mathbf{u} + 2\alpha\,\mathrm{rot}\,\boldsymbol{\omega} + \mathbf{X} = 0, \tag{1.6}$$

$$\Box_4\,\boldsymbol{\omega} + (\beta+\gamma-\varepsilon)\,\mathrm{grad\,div}\,\boldsymbol{\omega} + 2\alpha\,\mathrm{rot}\,\mathbf{u} + \mathbf{Y} = 0, \tag{1.7}$$

where

$$\Box_2 = (\mu+\alpha)\,\nabla^2 - \varrho\partial_t^2, \qquad \Box_4 = (\gamma+\varepsilon)\,\nabla^2 - 4\alpha - J\partial_t^2,$$
$$\nabla^2 = \frac{\partial}{\partial x_i}\,\frac{\partial}{\partial x_i}, \qquad \ddot{\boldsymbol{\omega}} = \frac{\partial^2\boldsymbol{\omega}}{\partial t^2} = \partial_t^2\,\boldsymbol{\omega}.$$

Here $\mathbf{X}$ denotes the body force vector and $\mathbf{Y}$ the body couple vector, while ε_{ijk} is the well-known Cartesian ε-tensor, the components of which are $+1$ (-1) if i, j, k is an even permutation of 1, 2, 3; they are 0 if two subscripts are equal. Eqs. (1.6) and (1.7) can be separated by decomposing the vectors $\mathbf{u}$ and $\boldsymbol{\omega}$ into potential and solenoidal parts

$$\mathbf{u} = \operatorname{grad} \vartheta + \operatorname{rot} \boldsymbol{\varphi}, \quad \boldsymbol{\omega} = \operatorname{grad} \varphi + \operatorname{rot} \boldsymbol{\Phi}, \quad \operatorname{div} \boldsymbol{\psi} = 0, \quad \operatorname{div} \boldsymbol{\Phi} = 0. \tag{1.8}$$

Decomposing in a similar way the expressions for the body forces and couple-body forces

$$\mathbf{X} = \varrho (\operatorname{grad} \lambda + \operatorname{rot} \boldsymbol{\chi}), \quad \mathbf{Y} = J (\operatorname{grad} \sigma + \operatorname{rot} \boldsymbol{\eta}) \tag{1.9}$$

we obtain from (1.6) and (1.7) the following system of wave equations

$$\Box_1 \vartheta + \varrho \lambda = 0, \tag{1.10}$$

$$\Box_3 \varphi + J\sigma = 0, \tag{1.11}$$

$$(\Box_2 \Box_4 + 4\alpha^2 \nabla^2) \boldsymbol{\Psi} = 2\alpha J \operatorname{rot} \boldsymbol{\eta} - \varrho \Box_4 \boldsymbol{\chi} \tag{1.12}$$

$$(\Box_2 \Box_4 + 4\alpha^2 \nabla^2) \boldsymbol{\Phi} = 2\alpha\varrho \operatorname{rot} \boldsymbol{\chi} - J \Box_2 \boldsymbol{\eta}, \tag{1.13}$$

In the above equations there is

$$\Box_1 = (\lambda + 2\mu) \nabla^2 - \varrho \partial_t^2, \quad \Box_3 = (\beta + 2\gamma) \nabla^2 - 4\alpha - J\partial_t^2.$$

Eq. (1.10) describes the longitudinal wave, Eq. (1.11) corresponds to the rotation wave, while Eqs. (1.12) and (1.13) represent the modified transverse waves.

In what follows we shall be concerned solely with the wave equation (1.11). We write it in the form

$$\left(\nabla^2 - \frac{1}{c^2} \partial_t^2 - \frac{\nu^2}{c^2}\right) \varphi(\mathbf{x}, t) + \frac{J}{c^2} \sigma(\mathbf{x}, t) = 0, \tag{1.14}$$

where $\nu^2 = \dfrac{4\alpha}{J}$, $c = \left(\dfrac{\beta + 2\gamma}{J}\right)^{1/2}$.

The differential equation (1.14) is known in mathematical physics as Klein—Gordon equation. It is of use in quantum electrodynamics (interaction of electrons with the radiation field) [5], [6]. For $\nu = 0$, Eq. (1.14) reduces to the classical scalar wave equation. In this communication we shall attempt to generalize the known theorems on the longitudinal wave of classical elastikinetics on the problem of propagation of rotation wave in the theory of asymmetric elasticity.

2. Solution of non homogeneous wave equation

Let us consider the wave equation (1.14) in an infinite region. The propagation of the rotation wave is brought about by the action of sources $\sigma(\mathbf{x}, t)$ distributed within a finite region. The initial conditions for the φ-function are assumed to be homogeneous.

In order to solve Eq. (1.14) we recur to Green's function $G(\mathbf{x}, \boldsymbol{\xi}, t)$ verifying the following differential equation

$$\left(\nabla^2 - \frac{1}{c^2}\partial_t^2 - \frac{\nu^2}{c^2}\right)\varphi(\mathbf{x}, t) = -4\pi\delta(\mathbf{x}-\boldsymbol{\xi})\,\delta(t) \tag{2.1}$$

with homogeneous initial conditions

$$G(\mathbf{x}, \boldsymbol{\xi}, 0) = 0, \quad \dot{G}(\mathbf{x}, \boldsymbol{\xi}, 0) = 0, \tag{2.2}$$

under the assumption that $G \to 0$ for $|x_1^2+x_2^2+x_3^3| \to \infty$.

The right hand side of Eq. (2.1) represents a concentrated instantaneous disturbance, acting at the point $\boldsymbol{\xi}$. Performing on Eq. (2.1) the Laplace integral transformation — the initial conditions as described by Eq. (2.2) being preserved — we get

$$\left(\nabla^2 - \frac{1}{c^2}(p^2+\nu^2)\right)\tilde{G}(\mathbf{x}, \boldsymbol{\xi}, p) = -4\pi\delta(\mathbf{x}-\boldsymbol{\xi}), \tag{2.3}$$

where

$$\tilde{G}(\mathbf{x}, \boldsymbol{\xi}, p) = \int_0^\infty G(\mathbf{x}, \boldsymbol{\xi}, t)\, e^{-pt}\, dt.$$

The solution of Eq. (2.3) satisfying the condition $\tilde{G} \to 0$ at infinity is given by the function

$$\tilde{G}(\mathbf{x}, \boldsymbol{\xi}, p) = \frac{1}{R}\exp\left(-\frac{R}{c}\sqrt{p^2+\nu^2}\right), \tag{2.4}$$

where

$$R = [(x_i - \xi_i)(x_i - \xi_i)]^{1/2}.$$

Let us perform the integral transformation on Eq. (1.14) bearing in mind homogeneous initial conditions for the φ-function. We obtain the following equation

$$\left(\nabla^2 - \frac{1}{c^2}(p^2+\nu^2)\right)\tilde{\varphi}(\mathbf{x}, p) + \frac{1}{c^2}\tilde{\sigma}(\mathbf{x}, p) = 0. \tag{2.5}$$

It may be easily seen that by an appropriate combination of Eqs. (2.3) and (2.5) we arrive at the equation

$$\int_V (\tilde{G}\nabla^2\tilde{\varphi} - \tilde{\varphi}\,\nabla^2\tilde{G})\,dV = -\frac{1}{c^2}\int_V \tilde{\sigma}\tilde{G}\,dV + 4\pi\int_V \delta(\mathbf{x}-\boldsymbol{\xi})\,\tilde{\varphi}(\mathbf{x}, p)\,dV(\mathbf{x})$$

whence, performing Green's transformation, we get

$$\tilde{\varphi}(\boldsymbol{\xi}, p) = \frac{1}{4\pi c^2}\int_V \tilde{\sigma}(\mathbf{x}, p)\,\tilde{G}(\mathbf{x}, \boldsymbol{\xi}, p)\,dV(\mathbf{x}) + \frac{1}{4\pi}\int_A\left(\tilde{G}\frac{\partial\tilde{\varphi}}{\partial n} - \tilde{\varphi}\frac{\partial\tilde{G}}{\partial n}\right)dA(\mathbf{x}). \tag{2.6}$$

The term $\partial/\partial n$ represents the differentiation with respect to the external normal. As regards the propagation of the rotation wave in an infinite region we may drop the surface integral in (2.6). Taking into account (2.4) we have

$$\tag{2.7} \tilde{\varphi}(\boldsymbol{\xi}, p) = \frac{1}{4\pi c^2} \int_v \frac{\tilde{\sigma}(\mathbf{x}, p)}{R(\mathbf{x}, \boldsymbol{\xi})} \exp\left(-\frac{R}{c} p\right) dV(\mathbf{x}) + \frac{1}{4\pi c^2} \int_v \frac{\tilde{\sigma}(\mathbf{x}, p)}{R(\mathbf{x}, \boldsymbol{\xi})} \tilde{F}(\mathbf{x}, \boldsymbol{\xi}, p)\, dV(\mathbf{x}),$$

where

$$\tag{2.7'} \tilde{F}(\mathbf{x}, \boldsymbol{\xi}, p) = \exp\left(-\frac{R}{c}\sqrt{p^2+\nu^2}\right) - \exp\left(-\frac{Rp}{c}\right).$$

Bearing in mind the reciprocals of Laplace transformation [7]

$$\tag{2.8} \mathcal{L}^{-1}\left(\frac{1}{R}\exp\left(-\frac{Rp}{c}\right)\right) = \frac{1}{R}\,\delta\left(\frac{R}{c} - t\right),$$

$$\tag{2.9} \mathcal{L}^{-1}\left(\tilde{F}(R, p)\right) = -\frac{R\nu J_1\left(\nu\sqrt{t^2 - R^2/c^2}\right)}{c\sqrt{t^2 - R^2/c^2}}\, H\left(t - \frac{R}{c}\right) = F(R, t),$$

where

$$H\left(t - \frac{R}{c}\right) = \begin{cases} 0 & \text{for} \quad t < \dfrac{R}{c}, \\ 1 & \text{for} \quad t > \dfrac{R}{c} \end{cases}$$

is the Heaviside function, and bearing in mind also the formula for the convolution

$$\tag{2.10} \mathcal{L}^{-1}(\tilde{\sigma}\cdot\tilde{f}) = \int_0^t \sigma(\mathbf{x}, t-\tau) f(\mathbf{x}, \tau)\, d\tau = \int_0^t \sigma(\mathbf{x}, \tau) f(\mathbf{x}, t-\tau)\, d\tau$$

we can write the function $\varphi(\boldsymbol{\xi}, t)$ in the form

$$\tag{2.11} \varphi(\boldsymbol{\xi}, t) = \frac{1}{4\pi c^2} \int_v \frac{\sigma\left(\mathbf{x}, t - \frac{R}{c}\right)}{R(\mathbf{x}, \boldsymbol{\xi})}\, dV(\mathbf{x}) + \frac{1}{4\pi c^2} \int_v \frac{dV(\mathbf{x})}{R(\mathbf{x}, \boldsymbol{\xi})} \int_0^t \sigma(\mathbf{x}, t-\tau)\, F(\mathbf{x}, \boldsymbol{\xi}, \tau)\, d\tau.$$

The function $J_1(z)$ appearing in (2.9) is Bessel's function of the first kind and first order.

In the first integral at the right-hand side of Eq. (2.11) we have in the term describing the function σ the argument $t - \dfrac{R}{c}$. This is to denote the moment prior to the moment t for which the integral is calculated. The difference $\dfrac{R}{c}$ separating these two moments corresponds to the time interval necessary for the wave to cover the distance between the point $\mathbf{x}$ and the point $\boldsymbol{\xi}$, for which the integral is calculated. The first integral in the right-hand part of the relation (2.11) represents the retarded potential. In particular, for $\sigma = f(t)\,\delta(\mathbf{x})$, i.e. when the disturbance is concentrated at the origin of the coordinate system and changes with time starting with the moment $t = 0$ in accordance with the function $f(t)$, we obtain from (2.11)

$$\varphi(\mathbf{x}, t) = \frac{1}{4\pi c^2 R_0} f\left(t - \frac{R_0}{c}\right) + \frac{1}{4\pi c^2 R_0} \int_0^t f(t-\tau)\, F(R_0, \tau)\, d\tau, \tag{2.12}$$

where $R_0 = (x_1^2 + x_2^2 + x_3^2)^{1/2}$.

Let us consider, in turn, the two-dimensional problem, wherein the wave function does not depend on the variable x_3. Green's function for the two-dimensional. problem has to verify the following equation

$$\left(\partial_1^2 + \partial_2^2 - \frac{1}{c^2}\partial_t^2 - \frac{\nu^2}{c^2}\right) G(x_1, x_2; \xi_1, \xi_2, t) = -2\pi\delta(t)\,\delta(x_1 - \xi_1)\,\delta(x_2 - \xi_2) \tag{2.13}$$

with homogeneous initial conditions and with the condition $G \to 0$ for $|x_1^2 + x_2^2| \to \infty$

Performing on the relation (2.13) the integral Laplace transformation we obtain

$$\left(\partial_1^2 + \partial_2^2 - \frac{1}{c^2}(p^2 + \nu^2)\right) \bar{G}(x_1, x_2; \xi_1, \xi_2, p) = -4\pi\delta(x_1 - \xi_1)\,\delta(x_2 - \xi_2). \tag{2.14}$$

The solution of this equation reads as follows [8]

$$\bar{G}(r, p) = K_0\left(r\sqrt{p^2 + \nu^2}\right), \tag{2.15}$$

where $r = ((x_1 - \xi_1)^2 + (x_2 - \xi_2)^2]^{1/2}$. Here, the symbol $K_0(z)$ stands for the modified Bessel's function of the third kind and zero order. After performing on Eq. (2.15) the inverse Laplace transformation we have

$$G(r, t) = \frac{c \cos\left(\nu\sqrt{t^2 - r^2/c^2}\right)}{\sqrt{(ct)^2 - r^2}} H\left(t - \frac{r}{c}\right). \tag{2.16}$$

The function $\bar{G}$ being thus known we can determine the function $\tilde{\varphi}$ from the following formula similar to (2.6)

$$\tilde{\varphi}(\boldsymbol{\xi}, p) = \frac{1}{2\pi c^2} \int_A \tilde{\sigma}(\mathbf{x}, p)\, \bar{G}(\mathbf{x}. \boldsymbol{\xi}, p)\, dA(\mathbf{x}) + \frac{1}{2\pi c^2} \int_l \left(\bar{G}\frac{\partial\tilde{\varphi}}{\partial n} - \tilde{\varphi}\frac{\partial\bar{G}}{\partial n}\right) dl.$$

If we consider the infinite region, the curvilinear integral, wanishes and what remains reads as below

$$\tilde{\varphi}(\xi_1, \xi_2, p) = \frac{1}{2\pi c^2} \int_A \tilde{\sigma}(x_1, x_2, p)\, \bar{G}(x_1, x_2; \xi_1, \xi_2, p)\, dx_1\, dx_2. \tag{2.17}$$

After performing on (2.17) the inverse Laplace transformation we arrive at

(2.18)

$$\varphi(\xi_1, \xi_2, t) = \frac{1}{2\pi c} \int_A dx_1, dx_2 \int_0^t \frac{\sigma(x_1, x_2, t-\tau)}{\sqrt{(\tau c)^2 - r^2}} \cos\left(\nu\sqrt{\tau^2 - r^2/c^2}\right) H\left(\tau - \frac{r}{c}\right) d\tau.$$

In the particular case of the disturbance $\sigma = \delta(x_1)\,\delta(x_2) f(t)$ we obtain from (2.18)

$$\varphi(x_1, x_2, t) = \frac{1}{2\pi c} \int_0^t \frac{f(t-\tau)}{\sqrt{(\tau c)^2 - r_0^2}} \cos\left(\nu\sqrt{\tau^2 - r_0^2/c^2}\right) H(\tau - r_0/c)\, d\tau \tag{2.19}$$

where $r_0 = (x_1^2 + x_2^2)^{1/2}$.

Let us remark that for $\nu = 0$ ($F = 0$) the formulae (2.11) and (2.18) give the solution of the classical scalar wave equation [9].

3. Solution of homogeneous wave equation

We shall give this solution taking into account the initial conditions.

Let us consider the homogeneous equation of the rotation wave

$$\left(\nabla^2 - \frac{1}{c^2}\partial_t^2 - \frac{\nu^2}{c^2}\right)\varphi(\mathbf{x}, t) = 0 \tag{3.1}$$

with non-homogeneous initial conditions

$$\varphi(\mathbf{x}, 0) = g(\mathbf{x}), \quad \dot{\varphi}(\mathbf{x}, 0) = h(\mathbf{x}). \tag{3.2}$$

We perform on Eq. (3.1) the integral transformation taking into account (3.2). We get then the equation

$$\left(\nabla^2 - \frac{1}{c^2}(p^2 + \nu^2)\right)\bar{\varphi}(\mathbf{x}, p) = -\frac{1}{c^2}(pg(\mathbf{x}) + h(\mathbf{x})). \tag{3.3}$$

Combining in an appropriate way Eqs. (3.3) and (2.3) and integrating them over the infinite region we obtain the following expression for the function φ.

$$\tilde{\varphi}(\boldsymbol{\xi}, p) = \frac{1}{4\pi c^2} \int_v (pg(\mathbf{x}) + h(\mathbf{x}))\, \tilde{G}(\mathbf{x}, \boldsymbol{\xi}, p)\, dV(\mathbf{x}) \tag{3.4}$$

or else

$$\bar{\varphi}(\boldsymbol{\xi}, p) = \frac{1}{4\pi c^2} \int_v (pg(\mathbf{x}) + h(\mathbf{x})) \frac{\exp\left(-\frac{R}{c}p\right)}{R(\mathbf{x}, \boldsymbol{\xi})} dV(\mathbf{x}) + \frac{1}{4\pi c^2} \int_v (pg(\mathbf{x}) + h(\mathbf{x}))\, \tilde{F}(\mathbf{x}, \boldsymbol{\xi}, p)\, dV(\mathbf{x}). \tag{3.5}$$

The function F appearing in (3.5) is determined by formula (2.7′) while its inverse transform by formula (2.9). The inverse transformation of (3.5) reads as follows

$$(3.6)\qquad \varphi(\mathbf{x}, t) = \frac{1}{4\pi c^2}\int_V \left(h(\boldsymbol{\xi}) + \frac{\partial}{\partial t} g(\boldsymbol{\xi})\right) \frac{\delta\left(\frac{R}{c} - t\right)}{R(\mathbf{x}, \boldsymbol{\xi})} dV(\boldsymbol{\xi}) + \frac{1}{4\pi c^2}\int_V \left(h(\boldsymbol{\xi}) + g(\boldsymbol{\xi})\frac{\partial}{\partial t}\right) F(\boldsymbol{\xi}, \mathbf{x}, t)\, dV(\boldsymbol{\xi}).$$

Let us now consider the integral

$$(3.7)\qquad \frac{1}{4\pi c^2}\int_V \frac{h(\boldsymbol{\xi})}{R(\mathbf{x}, \boldsymbol{\xi})} \delta\left(\frac{R}{c} - t\right) dV(\boldsymbol{\xi}).$$

Introducing spherical coordinates (R, θ, ψ) we can express the current coordinates of the sphere ξ_i in terms of coordinates of the centre of the sphere x_i. Thus

$$\xi_i = x_i + n_i R_i,$$

where

$$n_1 = \sin\theta\cos\psi, \quad n_2 = \sin\theta\sin\psi, \quad n_3 = \cos\theta,$$
$$0 \leqslant \theta \leqslant \pi, \quad 0 \leqslant \psi < 2\pi.$$

The introduction of spherical coordinates gives $dV = R^2 \sin\theta\, d\theta\, dR\, d\psi$. Bearing this in mind and taking advantage of the following property of the Dirac function

$$\int_V f(\eta)\, \delta(\eta - t)\, d\eta = f(t)$$

we reduce the integral (3.7) to the form

$$\frac{1}{4\pi c^2}\int_V h(\boldsymbol{\xi}) \frac{\delta\left(\frac{R}{c} - t\right)}{R(\mathbf{x}, \boldsymbol{\xi})} dV(\boldsymbol{\xi}) = \frac{t}{4\pi}\int_0^{2\pi} d\psi \int_0^{\pi} h_i(x_i + n_i ct)\sin\theta\, d\theta.$$

This integral is to be considered as arithmetical mean of the function h on the surface of the sphere with unitary radius. Introducing the notation

$$M_{ct}\{h(x_i, t)\} = \frac{1}{4\pi}\int_0^{2\pi} d\psi \int_0^{\pi} h(x_i + n_i ct)\sin\theta\, d\theta$$

we rewrite the formula (3.6) in the form

$$(3.8)\qquad \varphi(\mathbf{x}, t) = t\, M_{ct}\{h(\mathbf{x}_i, t)\} + \frac{\partial}{\partial t}[t\, M_{ct}\{hg(\mathbf{x}_i, t)\}] + \frac{1}{4\pi c^2}\int_V \left(h(\boldsymbol{\xi}) + g(\boldsymbol{\xi})\frac{\partial}{\partial t}\right) F(\boldsymbol{\xi}, \mathbf{x}, t)\, dV(\boldsymbol{\xi}).$$

Let us remark that for $\nu = 0$ $(F = 0)$ only the two first terms remain at the right hand side of the formula (3.8). These are the Poisson's integrals for the classical scalar wave equation [10].

For the two-dimensional problem we obtain — following a similar trend of considerations and taking into account the function $G(r, t)$ from the formula (2.16) — the following expression for the function φ:

$$\varphi(x_1, x_2, t) = \frac{1}{2\pi c}\int_0^{ct}\int_0^{2\pi}\Big[h(x_1+r\cos\vartheta, x_2+r\sin\vartheta)+ \\ +g(x_1+r\cos\vartheta, x_2+r\sin\vartheta)\frac{\partial}{\partial t}\Big]\frac{\cos\left(\nu\sqrt{t^2-r^2/c^2}\right)}{\sqrt{(ct)^2-r^2}}\, r\,dr\,d\vartheta, \tag{3.9}$$

where $r = ((x_1-\xi_1)^2+(x_2-\xi_2)^2)^{1/2}$.

Here also for $\nu = 0$ one obtains the known solution of the scalar wave equation. The solution of the one-dimensional problem is known [11]. It reads as follows

$$\varphi(x_1, t) = \frac{1}{2} g(x_1-ct)+g(x_1+ct)- \\ -\frac{\nu t}{2c}\int_{x_1-ct}^{x_1+ct}\frac{J_1\left(\nu\sqrt{t^2-\left(\frac{x_1-\xi_1}{c}\right)^2}\right)}{\sqrt{t^2-\left(\frac{x_1-\xi_1}{c}\right)^2}}\, g(\xi_1)\,d\xi_1+ \\ +\frac{1}{2c}\int_{x_1-ct}^{x_1+ct} J_0\left(\nu\sqrt{t^2-\left(\frac{x_1-\xi_1}{c}\right)^2}\right) h(\xi_1)\,d\xi_1. \tag{3.10}$$

For the particular case $\nu = 0$ we obtain the known solution of the scalar wave equation

$$\varphi(x_1, t) = \frac{1}{2}\left(g(x_1-ct)+g(x_1+ct)\right)+\frac{1}{2c}\int_{x_1-ct}^{x_1+ct} h(\xi_1)\,d\xi_1. \tag{3.11}$$

4. The generalized Kirchhoff formula

Let $\varphi(x, t)$ be the solution of the homogeneous wave

$$\left(\nabla^2-\frac{1}{c^2}\partial_t^2-\frac{\nu^2}{c^2}\right)\varphi(\mathbf{x}, t) = 0. \tag{4.1}$$

Its partial derivatives of the first and second order are continuous on and within the closed surface A. Let $\boldsymbol{\xi}$ be a point within the surface A. We shall demonstrate that the value of the function φ at the point $\boldsymbol{\xi}$ will assume the form

$$\varphi(\boldsymbol{\xi}, t) = -\frac{1}{4\pi}\int_A\left\{[\varphi]\frac{\partial}{\partial n}\left(\frac{1}{R}\right)-\frac{1}{cR}\frac{\partial R}{\partial n}\left[\frac{\partial\varphi}{\partial t}\right]-\frac{1}{R}\left[\frac{\partial\varphi}{\partial n}\right]\right\}dA(\mathbf{x})- \\ -\frac{1}{4\pi}\int_A dA(\mathbf{x})\int_0^t\left\{\left[F(R, t)\frac{\partial}{\partial n}\left(\frac{1}{R}\right)+\frac{1}{R}\frac{\partial R}{\partial n}\frac{\partial F}{\partial R}\right]\varphi(\mathbf{x}, t-\tau)- \\ -\frac{1}{R}F(R, t)\frac{\partial\varphi(\mathbf{x}, t-\tau)}{\partial n}\right\}dA(\mathbf{x}), \tag{4.2}$$

where R denotes the distance between the point $\boldsymbol{\xi}$ and a typical point $\mathbf{x}$ of A, $\partial/\partial n$ indicates the differentiation along the external normal to A and square brackets signify the retarded value. The function $F(R, t)$ is given by formula (2.4).

If, however, the point $\boldsymbol{\xi}$ is situated outside A, the value of the integral in (4.2) is zero.

To prove this assertion we shall take advantage of the formula (2.6) assuming $\tilde{\sigma} = 0$. Introducing into this expression $\tilde{G}$ from the formula (2.4) we get after some simple transformations

$$(4.3) \qquad \tilde{\varphi}(\boldsymbol{\xi}, p) = -\frac{1}{4\pi} \int_A \left\{ \tilde{\varphi} \frac{\partial}{\partial n}\left(\frac{1}{R}\right) \exp\left(-\frac{R}{c}\sqrt{p^2+\nu^2}\right) - \right.$$
$$\left. - \frac{1}{Rc}\frac{\partial R}{\partial n}\left(\tilde{\varphi}\sqrt{p^2+\nu^2}\exp\left(-\frac{R}{c}\sqrt{p^2+\nu^2}\right) - \frac{\partial \tilde{\varphi}}{\partial n}\exp\left(-\frac{R}{c}\sqrt{p^2+\nu^2}\right)\right)\right\} dA(\mathbf{x})$$

Introducing the notations

$$[\tilde{\varphi}] = \tilde{\varphi}\exp\left(-\frac{R}{c}p\right), \quad \left[\frac{\partial \tilde{\varphi}}{\partial n}\right] = \frac{\partial \tilde{\varphi}}{\partial n}\exp\left(-\frac{R}{c}p\right),$$
$$\tilde{F} = \exp\left(-\frac{R}{c}\sqrt{p^2+\nu^2}\right) - \exp\left(-\frac{R}{c}p\right)$$

we reduce (4.3) to the form

$$(4.4) \qquad \tilde{\varphi}(\boldsymbol{\xi}, p) = -\frac{1}{4\pi}\int_A \left\{[\tilde{\varphi}]\frac{\partial}{\partial n}\left(\frac{1}{R}\right) - \frac{1}{cR}\frac{\partial R}{\partial n}[\tilde{\varphi}p] - \frac{1}{R}\left[\frac{\partial \tilde{\varphi}}{\partial n}\right]\right\} dA(\mathbf{x}) +$$
$$-\frac{1}{4\pi}\int_A \left\{\tilde{\varphi}\,\tilde{F}\frac{\partial}{\partial n}\left(\frac{1}{R}\right) + \frac{1}{R}\frac{\partial R}{\partial n}\left(\tilde{\varphi}\frac{\partial \tilde{F}}{\partial R}\right) - \frac{1}{R}\left(\tilde{F}\frac{\partial \tilde{\varphi}}{\partial n}\right)\right\} dA(\mathbf{x}).$$

The inverse Laplace transformation performed on (4.4) leads to Eq. (4.2). We take in this operation the following formulae into account

$$(4.5) \qquad \begin{aligned} \mathcal{L}^{-1}[\tilde{\varphi}] &= \int_0^t \varphi(\mathbf{x}, t-\tau)\,\delta\left(\frac{R}{c}-\tau\right)d\tau = \varphi\left(\mathbf{x}, t-\frac{R}{c}\right) = [\varphi(\mathbf{x}, t)], \\ \mathcal{L}^{-1}[p\tilde{\varphi}] &= \int_0^t \frac{\partial \varphi(\mathbf{x}, t-\tau)}{\partial \tau}\,\delta\left(\frac{R}{c}-\tau\right)d\tau = \frac{\partial \varphi\left(\mathbf{x}, t-\frac{R}{c}\right)}{\partial t} = \left[\frac{\partial \varphi}{\partial t}\right], \\ \mathcal{L}^{-1}\left[\frac{\partial \tilde{\varphi}}{\partial n}\right] &= \int_0^t \frac{\partial \varphi(\mathbf{x}, t-\tau)}{\partial n}\,\delta\left(\frac{R}{c}-\tau\right)d\tau = \frac{\partial \varphi\left(\mathbf{x}, t-\frac{R}{c}\right)}{\partial n} = \left[\frac{\partial \varphi}{\partial n}\right]. \end{aligned}$$

The generalized Kirchhoff formula (4.2) affords the integral formulation of Huygen's principle for the rotation wave.

In the particular case, for $\nu = 0$ ($F = 0$) Eq. (4.2) transforms into the known Kirchhoff formula for the classical wave equation [11].

In the case of monochromatic vibrations, i.e. if

$$\varphi(\mathbf{x}, t) = \varphi^*(\mathbf{x})\, e^{-i\omega t}, \qquad G(\mathbf{x}, \boldsymbol{\xi}, t) = G(\mathbf{x}, \boldsymbol{\xi})\, e^{-i\omega t} \tag{4.6}$$

we obtain from (4.2) the following expression for the function φ^*:

$$\varphi^*(\boldsymbol{\xi}) = \frac{1}{4\pi} \int_A \left[\frac{\partial \varphi^*}{\partial n} \frac{\exp(-ikR)}{R} - \varphi^* \frac{\partial}{\partial n} \left(\frac{\exp(-ikR)}{R} \right) \right] dA(\mathbf{x}), \tag{4.7}$$

where $R = \frac{1}{c}(\omega^2 - \nu^2)^{1/2}$.

Eq. (4.7) holds true for $\omega > \nu$ since only in this case the rotation wave with real phase velocity is possible. The appearance of the term ω corresponding to the vibration frequency in the formula describing the phase velocity indicates that the wave undergoes dispersion. For $\nu = 0$ Eq. (4.7) transforms into the known Helmholtz formula [11].

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REFERENCES

[1] A. C. Eringen, E. S. Suhubi, Int. J. Engin., **2** (1964), 189.

[2] — , — , ibid., **2** (1964), 389.

[3] V. A. Palmov, Prikl. Mat. Mech., **28** (1964), 401.

[4] W. Nowacki, *Couple-stresses in the theory of thermoelasticity*. III, Bull. Acad. Polon. Sci., Sér. sci. techn., **14** (1966), 505 [801].

[5] P. Jordan, W. Pauli, Z. Physik, **47** (1928) 151.

[6] L. J. Schiff, *Quantum Mechanics*, Mc. Graw Hill New York, 1949, p. 371.

[7] A. Erdelyi [ed], *Tables of integral transforms*, vol. II, Mc. Graw Hill, New York, 1954.

[8] V. A. Ditkin, N. J. Kuznietsov, *Spravochnik po operatzionnomy ischisleniyu* [in Russian], [*Handbook of operational calculus*], GITTL, Moscow, 1951, p. 225.

[9] W. J. Smirnov, *Matematyka wyższa* [in Polish, transl. from Russian], [*Higher Mathematics*], PWN, Warsaw, 1963, p. 504.

[10] B. B. Baker, E. T. Copson, *The mathematical theory of Huygen's principle*, Oxford, Clasendon Press, 1953, p. 37.

[11] P. M. Morse, H. Feshbach, *Methods of theoretical physics*, vol. I, Mc. Graw Hill, 1953, p. 803.

В. НОВАЦКИЙ, **РАСПРОСТРАНЕНИЕ РОТАЦИОННЫХ ВОЛН В АССИМЕТРИЧЕСКОЙ УПРУГОСТИ**

В работе рассматривается распространение ротационной волны в бесконечной упругой среде. Даются поочередно: решение неоднородного волнового уравнения, обобщенные интегралы Пуассона, а также обобщение формул Кирхгоффа и Гельмгольца на ассиметрическую упругость. Констатировано, что для $\nu = 0$ получаются известные результаты, относящиеся к классическому скалярному волновому уравнению.