BULLETIN
DE
L'ACADÉMIE POLONAISE DES SCIENCES

SÉRIE DES SCIENCES TECHNIQUES

Volume XV, Numéro 9

VARSOVIE 1967

BULLETIN DE L'ACADÉMIE
8. POLONAISE DES SCIENCES
Série des sciences techniques
Volume XV, No. 9 — 1967

APPLIED MECHANICS
(THERMOELASTICITY)

On the Completeness of Stress Functions in Thermoelasticity

by

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Presented on July 1, 1967

1. Introduction

In this Note we shall be concerned with an elastic, isotropic and homogeneous body, filling the region B bounded by the surface A. The body, when acted upon by external forces and heated, will undergo deformation. A displacement field $\mathbf{u}(x, t)$ and a temperature field $\theta(x, t)$ will form in the body, depending on the position of the point x and the time t. These fields are described by the following system of differential equations [1]

(1.1) $$\mu\nabla^2 \mathbf{u}+(\lambda+\mu)\,\mathrm{grad}\,\mathrm{div}\,\mathbf{u}+\mathbf{X} = \varrho\ddot{\mathbf{u}}+\gamma\,\mathrm{grad}\,\theta\,,$$

(1.2) $$\nabla^2 \theta - \frac{1}{\varkappa}\dot{\theta} - \eta\,\mathrm{div}\,\dot{\mathbf{u}} = -\frac{Q}{\varkappa}.$$

The notations used in the above formulae read as follows: $\theta = T - T_0$ stands for the increment of temperature, T denotes the absolute temperature, T_0 — the temperature of the body in natural state, the strains and stresses being equal to 0. $\mathbf{X}$ means the vector of body forces and ϱ — density. The quantities μ and λ are Lamé constants for the isothermal state; $\gamma = 3K\alpha_t$, where $K = \lambda + \frac{2}{3}\mu$ is the bulk modulus and α_t — the coefficient of linear thermal dilatability. $\varkappa = \lambda_0/c_\varepsilon$, where λ_0 is the coefficient of thermal conductivity and c_ε — specific heat at a constant strain. Finally, $\eta = \frac{\gamma T_0}{\lambda_0}$ and $Q = \frac{W}{\lambda_0}$, where W denotes the quantity of heat produced per volume and time unit. A dot above the symbol of function denotes its time derivative: $\dot{\theta} = \frac{\partial\theta}{\partial t}$, $\ddot{\mathbf{u}} = \frac{\partial^2 \mathbf{u}}{\partial t^2}$. Decomposing the displacement vector into its potential and solenoidal parts

(1.3) $$\mathbf{u} = \mathrm{grad}\,\Phi + \mathrm{rot}\,\boldsymbol{\chi}, \quad \mathrm{div}\,\boldsymbol{\chi} = 0\,,$$

and decomposing the vector of body forces in a similar way

(1.4) $$\mathbf{X} = \varrho\,(\mathrm{grad}\,\vartheta + \mathrm{rot}\,\boldsymbol{\sigma})$$

we reduce the system of Eqs. (1.1) and (1.2) to the form [2]

$$\square_1^2 \Phi = m\theta - \frac{1}{c_1^2}\vartheta, \tag{1.5}$$

$$\square_2^2 \boldsymbol{\chi} = -\frac{1}{c_2^2}\boldsymbol{\sigma}, \tag{1.6}$$

$$D\theta - \eta\nabla^2\dot{\Phi} = -\frac{Q}{\varkappa}. \tag{1.7}$$

The following notations were introduced in Eqs. (1.5)—(1.7):

$$\begin{gathered} \square_\alpha^2 = \nabla^2 - \frac{1}{c_\alpha^2}\frac{\partial^2}{\partial t^2}, \quad \alpha = 1, 2, \quad D = \nabla^2 - \frac{1}{\varkappa}\frac{\partial}{\partial t}, \\ c_1 = \left(\frac{\lambda+2\mu}{\varrho}\right)^{1/2}, \quad c_2 = \left(\frac{\mu}{\varrho}\right)^{1/2}, \quad m = \frac{\gamma}{\lambda+2\mu}. \end{gathered} \tag{1.8}$$

Eliminating the temperature from Eqs. (1.5) and (1.7) we obtain the equations

$$\Omega\Phi = -\frac{m}{\varkappa}Q - \frac{1}{c_2^1}D\vartheta, \tag{1.9}$$

$$\square_2^2 \boldsymbol{\chi} = -\frac{1}{c_2^2}\boldsymbol{\sigma}, \tag{1.10}$$

where

$$\Omega = \square_1^2 D - m\eta\,\partial_t\nabla^2, \qquad \partial_t = \frac{\partial}{\partial t}.$$

Eq. (1.9) describes the longitudinal thermoelastic wave, while Eq. (1.10) — the transverse one.

In classical elastokinetics, the adiabatic process being assumed, we have to deal with the system of displacement equations

$$\begin{gathered} \mu_s\nabla^2\mathbf{u} + (\lambda_s+\mu_s)\operatorname{grad}\operatorname{div}\mathbf{u} + \mathbf{X} = \varrho\ddot{\mathbf{u}}, \\ \theta = -\eta_T m_T \varkappa \operatorname{div}\mathbf{u}. \end{gathered} \tag{1.11}$$

The Lamé constants μ_s and λ_s appearing in these equations refer to the adiabatic state and $\eta_T = \dfrac{\gamma_T T_0}{\lambda_0}$, $m_T = \dfrac{\gamma_T}{\lambda_T+2\mu_T}$, $\gamma_T = (3\lambda_T+2\mu_T)\alpha_t$ where μ_T and λ_T are Lamé constants in isothermal conditions.

Representing the displacement $\mathbf{u}$ by Eq. (1.3) we obtain the following wave equation

$$\square_1^2\Phi = -\frac{1}{a_1^2}\vartheta, \quad \square_2^2\boldsymbol{\chi} = -\frac{1}{a_2^2}\boldsymbol{\sigma}, \tag{1.12}$$

where

$$a_1 = \left(\frac{\lambda_s+2\mu_s}{\varrho}\right)^{1/2}, \quad a_2 = \left(\frac{\mu_s}{\varrho}\right)^{1/2}, \quad \mu_s = \mu_T, \quad \lambda_s = \lambda_T + \gamma_T\eta_T\varkappa.$$

Somigliana [3] and Duhem [4] have shown that representing of (1.3) in terms of the potentials Φ and χ affords the complete solution of Eqs. (1.11). Sternberg in [5] gave conclusive arguments for the completeness of the general solution of coupled equations of thermoelasticity (1.1) and (1.2) for the (1.3) representation under the assumption that the functions Φ and χ verify the system of Eqs. (1.9) and (1.10). We shall deal with this problem in par. 3 of this paper.

The system of Eqs. (1.1) and (1.2) may be also dissociated in another way, namely by introducing the vector function $\boldsymbol{\varphi}$ and the scalar function ψ connected with the displacement $\mathbf{u}$ and the temperature θ by the following representation

$$\mathbf{u} = \frac{\lambda+2\mu}{\mu}\,\Omega\boldsymbol{\varphi} - \frac{\lambda+\mu}{\mu}\,\mathrm{div}\,(\Gamma\boldsymbol{\varphi}) + \frac{\gamma}{\mu}\,\mathrm{grad}\,\psi, \tag{1.13}$$

$$\theta = \eta\partial_t\,\mathrm{div}\,\Box_2^2\,\boldsymbol{\varphi} + \frac{\lambda+2\mu}{\mu}\,\Box_1^2\,\psi, \tag{1.14}$$

where

$$\Gamma = D - \frac{\gamma}{\lambda+\mu}\,\eta\partial_t.$$

The functions $\boldsymbol{\varphi}$ and ψ are supposed to satisfy the following equations

$$\Box_2^2\,\Omega\boldsymbol{\varphi} + \frac{\mathbf{X}}{\lambda+2\mu} = 0, \tag{1.15}$$

$$\Omega\psi + \frac{Q\mu}{\varkappa(\lambda+2\mu)} = 0. \tag{1.16}$$

We obtain Eqs. (1.15) and (1.16) by substituting, respectively, (1.13) and (1.14) into the differential equations of thermoelasticity (1.1) and (1.2).

The representation (1.13) and (1.14) was first given by Kaliski [6] and somewhat later, in a different way, by Podstrigacz [7] and Rüdiger [8.]

Let us remark that the representation (1.13), (1.14) is a generalization on the problems of thermoelasticity of Galerkin's functions of stresses used in electrostatics [9] or of those given by Iacovache for classical elastokinetics [10].

However, the method of deriving the relations (1.13), (1.14) used in [6]—[8] does not ensure the completeness of the solutions. The demonstration of completeness of the solutions for the representation (1.13), (1.14) will be given in par. 2.

2. Theorem on the completeness of solutions

Let $\mathbf{u}(x, t)$ and $\theta(x, t)$ be the solutions of the system of Eqs. (1.1) and (1.2) in the region B for $-\infty < t < \infty$. Then there exists a vector function $\boldsymbol{\varphi}$ and a scalar function ψ such that the displacement $\mathbf{u}(x, t)$ and the temperature $\theta(x, t)$ are represented by (1.13) and (1.14), respectively, and the functions $\boldsymbol{\varphi}$ and ψ satisfy the wave Eqs. (1.15) and (1.16).

In virtue of the Stokes-Helmholtz solution there exist functions $\Phi(x, t)$ and $\boldsymbol{\chi}(x, t)$ such that

$$\mathbf{u}(x, t) = \operatorname{grad}\Phi + \operatorname{rot}\boldsymbol{\chi}, \quad \operatorname{div}\boldsymbol{\chi} = 0. \tag{2.1}$$

Let us substitute (2.1) into the system of differential equations of thermoelasticity (1.1) and (1.2). We obtain then the following equations

$$\operatorname{grad}\Box_1^2\Phi + \frac{\mu}{\lambda+2\mu}\operatorname{rot}\Box_2^2\boldsymbol{\chi} + \frac{\mathbf{X}}{\lambda+2\mu} = m\operatorname{grad}\theta, \tag{2.2}$$

$$D\theta - \eta\partial_t\nabla^2\Phi = -\frac{Q}{\varkappa}. \tag{2.3}$$

Now, if we eliminate from the above equations the temperature θ and make use of Eq. (1.16) we obtain

$$\operatorname{grad}(\Omega\Phi) + \frac{\mu}{\lambda+2\mu}\operatorname{rot}(D\Box_2^2\boldsymbol{\chi}) + \frac{D\mathbf{X}}{\lambda+2\mu} - \frac{\gamma}{\mu}\operatorname{grad}(\Omega\psi) = 0. \tag{2.4}$$

We shall now define the function $\boldsymbol{\varphi}$ in terms of the scalar function ζ and the vector function $\boldsymbol{\eta}$

$$\boldsymbol{\varphi} = \operatorname{grad}\zeta + \frac{\mu}{\lambda+2\mu}\operatorname{rot}\boldsymbol{\eta}. \tag{2.5}$$

Substituting (2.5) into Eq. (1.15) and performing the differentiation operation D on the latter, we arrive at the following result

$$\operatorname{grad}(D\Omega\Box_2^2\zeta) + \frac{\mu}{\lambda+2\mu}\operatorname{rot}(D\Omega\Box_2^2\boldsymbol{\eta}) + \frac{D\mathbf{X}}{\lambda+2\mu} = 0. \tag{2.6}$$

Comparison of Eq. (2.4) and Eq. (2.6) shows that the functions ζ and η have to satisfy the following differential equations

$$D\Box_2^2\zeta = \Phi - \frac{\gamma}{\mu}\psi, \tag{2.7}$$

$$\Omega\boldsymbol{\eta} = \boldsymbol{\chi}. \tag{2.8}$$

Let us perform the operation Ω on the relation (2.5). Then, taking into account Eq. (2.8), we get

$$\Omega\boldsymbol{\varphi} = \operatorname{grad}(\Phi\zeta) + \frac{\mu}{\lambda+2\mu}\operatorname{rot}\boldsymbol{\chi}. \tag{2.9}$$

We introduce the term $\operatorname{rot}\boldsymbol{\chi}$ from Eq. (2.9) into the formula (2.1). In this way, taking into account Eq. (2.7) we obtain the representation of the displacement $\mathbf{u}$ by means of the functions ψ, $\boldsymbol{\varphi}$ and ζ.

$$\mathbf{u} = \frac{\lambda+2\mu}{\mu}\Phi\boldsymbol{\varphi} + \frac{\gamma}{\mu}\operatorname{grad}\psi + \operatorname{grad}\left(D\Box_2^2 - \frac{\lambda+2\mu}{\mu}\Omega\right)\zeta. \tag{2.10}$$

Taking into consideration that

$$\Box_1^2 \zeta = \frac{\mu}{\lambda+2\mu} \Box_2^2 \zeta + \frac{\lambda+\mu}{\lambda+2\mu} \nabla^2 \zeta, \tag{2.11}$$

we reduce the operator at ζ in Eq. (2.10) to the form

$$\left(D\Box_2^2 - \frac{\lambda+2\mu}{\mu}\Omega\right)\zeta = -\frac{\lambda+\mu}{\mu}\left(D - \frac{\gamma\eta\partial_t}{\lambda+\mu}\right)\nabla^2\zeta = -\frac{\lambda+\mu}{\mu}\Gamma\nabla^2\zeta. \tag{2.12}$$

Substituting Eq. (2.12) into (2.10) and taking into account the following relation resulting from (2.5)

$$\operatorname{div} \boldsymbol{\varphi} = \nabla^2 \zeta \tag{2.13}$$

we reduce (2.10) to the form

$$\mathbf{u} = \frac{\lambda+2\mu}{\mu}\Omega\boldsymbol{\varphi} - \frac{\lambda+\mu}{\mu}\operatorname{div}(\Gamma\boldsymbol{\varphi}) + \frac{\gamma}{\mu}\operatorname{grad}\psi \tag{2.14}$$

which is in agreement with the representation (1.13). We have still to demonstrate the consistency of the representation (1.14). To this end, we perform on Eq. (1.1) the operation of divergence and make use of the relation $\operatorname{div}\mathbf{u} = \nabla^2 \Phi$, which results from (2.1). Thus, we obtain

$$\Box_1^2 \nabla^2 \Phi + \frac{1}{\lambda+2\mu}\operatorname{div}\mathbf{X} = \frac{\gamma}{\lambda+2\mu}\nabla^2\theta. \tag{2.15}$$

Then, taking advantage of Eq. (1.16) we eliminate the function Φ from Eqs. (2.3) and (2.15). As a result we obtain

$$\Omega\theta + \frac{\eta}{\lambda+2\mu}\partial_t \operatorname{div}\mathbf{X} = \frac{\lambda+2\mu}{\mu}\Box_1^2\Omega\psi. \tag{2.16}$$

Finally, let us perform the operation of divergence on Eq. (1.15). This leads to the formula

$$\frac{1}{\lambda+2\mu}\operatorname{div}\mathbf{X} = -\operatorname{div}(\Box_2^2\Omega\boldsymbol{\varphi}). \tag{2.17}$$

Substituting (2.17) into (2.16) and subjecting the equation transformed in this way to the operation Ω^{-1} we get the relation

$$\theta = \eta\partial_t \operatorname{div}\Box_2^2\boldsymbol{\varphi} + \frac{\lambda+2\mu}{\mu}\Box_1^2\psi. \tag{2.18}$$

which is in agreement with the representation (1.14).

This concludes the proof of the completeness.

A particular case of coupled thermoelasticity is the theory called technical theory of thermal stresses, wherein the coupling of Eqs. (1.1) and (1.2) is neglected by disregarding the term $\eta \operatorname{div}\dot{\mathbf{u}} = 0$ in the equation of heat conductivity. It may be easily shown — putting $\eta = 0$ in Eqs. (1.13)—(1.16) and introducing the functions $\mathbf{G} = D\boldsymbol{\varphi}$ — that the following representation holds:

$$\mathbf{u} = \frac{\lambda+2\mu}{\mu} \Box_1^2 \mathbf{G} - \frac{\lambda+\mu}{\mu} \operatorname{div} \mathbf{G} + \frac{\gamma}{\mu} \operatorname{grad} \psi, \tag{2.19}$$

$$\theta = \frac{\lambda+2\mu}{\mu} \Box_1^2 \psi. \tag{2.20}$$

The functions $\mathbf{G}$ and ψ have to satisfy the following equations

$$\Box_1^2 \Box_2^2 \mathbf{G} + \frac{\mathbf{X}}{\lambda+2\mu} = 0, \tag{2.21}$$

$$D\Box_1^2 \psi + \frac{Q\mu}{\varkappa(\lambda+2\mu)} = 0. \tag{2.22}$$

Finally, in the particular case of classical elastokinetics we have

$$\mathbf{u} = \frac{\lambda_s+2\mu_s}{\mu_s} \Box_1^2 \mathbf{G} - \frac{\lambda_s+\mu_s}{\mu_s} \operatorname{div} \mathbf{G}, \quad \theta = -\eta_T m_T \varkappa \operatorname{div} \mathbf{u}. \tag{2.23}$$

where the function $\mathbf{G}$ satisfies the repeated wave equation

$$\Box_1^2 \Box_2^2 \mathbf{G} + \frac{\mathbf{X}}{\lambda_s+2\mu_s} = 0. \tag{2.24}$$

The representation (2.23), (2.24) was given first by Iacovache [10] and the theorem on the completeness of this representation by Sternberg and Eubanks [11].

3. Connections between the potentials Φ, χ and the functions φ and ψ

Let us consider the homogeneous system of Eqs. (1.1), (1.2). Introducing the representation (1.13) and (1.14) we reduce this system to the wave equations

$$\Box_2^2 \Omega \boldsymbol{\varphi} = 0, \tag{3.1}$$

$$\Omega \psi = 0. \tag{3.2}$$

Remark that — in virtue of Boggio's theorem [12] — the solution of Eq. (3.1) may take on the following form

$$\boldsymbol{\varphi} = \boldsymbol{\varphi}' + \boldsymbol{\varphi}'' \tag{3.3}$$

the functions $\boldsymbol{\varphi}'$ and $\boldsymbol{\varphi}''$ satisfying the equations

$$\Omega \boldsymbol{\varphi}' = 0, \tag{3.4}$$

$$\Box_2^2 \boldsymbol{\varphi}'' = 0. \tag{3.5}$$

Introducing (3.3) into the representation (1.13) and taking into account Eqs. (3.4) and (3.5) we obtain

$$\mathbf{u} = \frac{\lambda+2\mu}{\mu} \Omega \boldsymbol{\varphi}'' - \frac{\lambda+\mu}{\mu} \operatorname{grad} \operatorname{div} \Gamma(\boldsymbol{\varphi}' + \boldsymbol{\varphi}'') + \frac{\gamma}{\mu} \operatorname{grad} \psi. \tag{3.6}$$

Let us transform now the term $\Omega\boldsymbol{\varphi}''$ taking into consideration $\Omega = \Box_1^2 D - m\eta\partial_t \nabla^2$ and making use of the relation (3.5). After some simple transformations, the relation (2.11) being taken into account, we get

$$\Omega\boldsymbol{\varphi}'' = \frac{\lambda+\mu}{\lambda+2\mu}(\nabla^2 \Gamma\boldsymbol{\varphi}''). \tag{3.7}$$

In this way, taking into consideration (3.7), we may write (3.6) in the form

$$\mathbf{u} = \frac{\lambda+\mu}{\mu}[\nabla^2 \Gamma\boldsymbol{\varphi}'' - \operatorname{grad}\operatorname{div}(\Gamma\boldsymbol{\varphi}' + \Gamma\boldsymbol{\varphi}'')] + \frac{\gamma}{\mu}\operatorname{grad}\psi. \tag{3.8}$$

Since there is

$$\operatorname{grad}\operatorname{div}(\Gamma\boldsymbol{\varphi}'') = \nabla^2(\Gamma\boldsymbol{\varphi}'') + \operatorname{rot}\operatorname{rot}(\Gamma\boldsymbol{\varphi}''),$$

we obtain from (3.8):

$$\mathbf{u} = \frac{\lambda+\mu}{\mu}[-\operatorname{rot}\operatorname{rot}(\Gamma\boldsymbol{\varphi}'') - \operatorname{grad}\operatorname{div}(\Gamma\boldsymbol{\varphi}')] + \frac{\gamma}{\mu}\operatorname{grad}\psi. \tag{3.9}$$

Now we may easily reduce the relation (3.9) to the Stokes-Helmholtz representation

$$\mathbf{u} = \operatorname{grad}\Phi + \operatorname{rot}\boldsymbol{\chi}, \quad \operatorname{div}\boldsymbol{\chi} = 0. \tag{3.10}$$

Comparison of Eqs. (3.9) and (3.10) leads to the following representation

$$\Phi = -\frac{\lambda+\mu}{\mu}\operatorname{div}(\Gamma\boldsymbol{\varphi}') + \frac{\gamma}{\mu}\psi, \tag{3.11}$$

$$\boldsymbol{\chi} = -\frac{\lambda+\mu}{\mu}\operatorname{rot}(\Gamma\boldsymbol{\varphi}''), \tag{3.12}$$

where the functions Φ, $\boldsymbol{\chi}$ are connected with the functions $\boldsymbol{\varphi}$, ψ.

Let us perform on the representation (1.14) the operation D. Making use of Eq. (3.5) we obtain the following relation

$$D\theta = \eta\partial_t \operatorname{div}(D\Box_2^2\boldsymbol{\varphi}') + \frac{\lambda+2\mu}{\mu}D\Box_1^2\psi. \tag{3.13}$$

We shall now transform Eq. (3.4) which may be also written in the form

$$\Box_1^2 D\boldsymbol{\varphi}' - m\eta\partial_t \nabla^2\boldsymbol{\varphi}' = 0. \tag{3.14}$$

Taking advantage of the relation (2.11) we obtain

$$D\Box_2^2\boldsymbol{\varphi}' = -\frac{\lambda+\mu}{\mu}\nabla\Gamma\boldsymbol{\varphi}'. \tag{3.15}$$

Substituting (3.15) into (3.13) and taking into consideration Eq. (3.2) we arrive at the equation

$$D\theta = \eta\partial_t \nabla^2\left[-\frac{\lambda+\mu}{\mu}\operatorname{div}(\Gamma\boldsymbol{\varphi}') + \frac{\gamma}{\mu}\psi\right]. \tag{3.16}$$

Since, in virtue of Eq. (3.11), the expression in square brackets in Eq. (3.16) equals Φ, we have

$$D\theta - \eta\partial_t \nabla^2 \Phi = 0. \tag{3.17}$$

This is the homogeneous equation of heat conductivity (1.7).

We have now to check whether the functions Φ, $\boldsymbol{\chi}$ expressed by the functions $\boldsymbol{\varphi}$ and ψ satisfy the homogeneous wave equations (1.9) and (1.10).

$$\Omega\Phi = 0, \quad \Box_2^2 \boldsymbol{\chi} = 0. \tag{3.18}$$

It may be easily verified, in virtue of (3.2) and (3.4), that

$$\Omega\Phi = \frac{\gamma}{\mu}\Omega\psi - \frac{\lambda+\mu}{\mu}\operatorname{div}(\Gamma\Omega\boldsymbol{\varphi}') = 0, \tag{3.19}$$

and also, in virtue of (3.5),

$$\Box_2^2 \boldsymbol{\chi} = -\frac{\lambda+\mu}{\mu}\operatorname{rot}(\Gamma\Box_2^2\boldsymbol{\varphi}'') = 0. \tag{3.20}$$

It was shown in par. **2** that the representation (1.13)—(1.16) leads to the completeness of the solutions. The reduction of (3.11) and (3.12) performed in this paragraph supplies an alternative proof for the completeness of solutions (1.3), (1.9), (1.10).

In the case of the theory of thermal stresses, where the coupling of the temperature field and displacement field is disregarded we obtain in lieu of the transformation (3.11), (3.12) the following one

$$\Phi = -\frac{\lambda+\mu}{\mu}\operatorname{div}\mathbf{G} + \frac{\gamma}{\mu}\psi, \tag{3.21}$$

$$\boldsymbol{\chi} = -\frac{\lambda+\mu}{\mu}\operatorname{rot}\mathbf{G}, \quad \mathbf{G} = D\boldsymbol{\varphi}. \tag{3.22}$$

The displacements $\mathbf{u}$ and the temperature θ are given here by the representations (2.19), (2.20). They have to satisfy the homogenenous equations

$$\Box_1^2\Box_2^2\mathbf{G} = 0, \quad D\Box_1^2\psi = 0. \tag{3.23}$$

It may also easily be shown that the functions Φ, $\boldsymbol{\chi}$ given by the relations (3.21) and (3.22) verify the following equations

$$\Box_1^2\Phi - m\theta = 0, \quad \Box_2^2\boldsymbol{\chi} = 0, \quad D\theta = 0. \tag{3.24}$$

In the case of classical elastokinetics the formulae (3.21) and (3.22) transform to

$$\Phi = -\frac{\lambda_s+\mu_s}{\mu_s}\operatorname{div}\mathbf{G}, \tag{3.25}$$

$$\boldsymbol{\chi} = -\frac{\lambda_s+\mu_s}{\mu_s}\operatorname{rot}\mathbf{G}. \tag{3.26}$$

in agreement with the result obtained by Sternberg [5].

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В. НОВАЦКИЙ, О ПОЛНОТЕ ФУНКЦИЙ НАПРЯЖЕНИЙ В ТЕРМОУПРУГОСТИ

В работе доказана полнота решения проблемы сопряженной термоупругости полученного в результате введения функций напряжения φ и ψ. Определены также связи между функциями φ, ψ и представлением перемещения u путем использования скалярного потенциала Φ и векторного потенциала χ.

В заключение дискутируется переход от проблемы сопряженной термоупругости к так называемой технической теории термических напряжений.