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APPLIED MECHANICS
(THERMOELASTICITY)

Couple-stresses in the Theory of Thermoelasticity. III

by

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1. Introduction

In our previous papers [1], [2] the theory of Cosserat's medium was extended on the problems of coupled thermoelasticity. The displacement vector $\vec{u}$ and temperature θ where assumed as in the papers quoted as independent functions. The rotation $\vec{\omega}$ and displacement $\vec{u}$ vectors were connected by the relation: $\vec{\omega} = \frac{1}{2} \operatorname{rot} \vec{u}$. In the present Note both vectors, $\vec{u}$ and $\vec{\omega}$ are considered as independent functions and so is the temperature θ, [3]—[7].

In what follows, constitutive equations are derived basing on the thermodynamics of irreversible processes and on fundamental equations of coupled thermoelasticity. Finally, variational theorem and the theorem on reciprocity are formulated.

2. Equations of energy and of entropy balance

The principle of conservation of energy referred to an arbitrary volume V of a body, bounded by the surface A, has the form:

$$\frac{d}{dt}\int_V \left[\frac{1}{2}(\varrho v_i v_i + J w_i w_i) + U\right] dV = \int_V (X_i u_i + Y_i w_i)\, dV + \int_A (p_i v_i + m_i w_i)\, dA - \int_A q_i n_i\, dA. \tag{2.1}$$

Here $v_i = \dot{u}_i$, $w_i = \dot{\omega}_i$, U denotes the internal energy, and q_i — the component of heat-flux vector. X_i denote the components of the body force vector referred to a volume unit, and Y_i — the components of the body couple vector. The quantities p_i and m_i are connected with the asymmetric tensors σ_{ji} and μ_{ji} by the following relations

$$p_i = \sigma_{ji} n_j, \quad m_i = \mu_{ji} n_j, \tag{2.2}$$

where m_i stands for the component of the unit normal vector, σ_{ji} is the force-stress tensor, μ_{ji} — the couple-stress tensor. Taking into consideration equations of motion [5]

$$\sigma_{ji,j}+X_i = \varrho\ddot{u}_i, \tag{2.3}$$

$$\epsilon_{ijk}\,\sigma_{jk}+\mu_{ji,j}+Y_i = J\ddot{\omega}_i, \tag{2.4}$$

and making use of the divergence theorem, we obtain the following equation

$$\int_V \{\dot{U}-[\sigma_{ji}(v_{i,j}-\varepsilon_{kji}\,w_k)+\mu_{ji}\,w_{i,j}]+q_{i,i}\}\,dV = 0. \tag{2.5}$$

If the integrand is continuous, then the relation

$$\dot{U} = \sigma_{ji}\,\dot{\gamma}_{ji}+\mu_{ji}\,\dot{\varkappa}_{ji}-q_{i,i} \tag{2.6}$$

holds locally. In Eq. (2.6) there is

$$\gamma_{ji} = u_{i,j}-\varepsilon_{kji}\,\omega_k, \quad \varkappa_{ji} = \omega_{i,j}. \tag{2.7}$$

The equation of entropy balance can be written in the form ([8], p. 29.)

$$\int_V \dot{S}\,dV = -\int_A \frac{q_i\,n_i}{T}\,dA+\int_V \Theta\,dV. \tag{2.8}$$

S being the entropy referred to the volume unit, T — absolute temperature. The second term on the right-hand side of (2.8) denotes the rate of production of energy, due to heat conduction.

Eq. (2.8) transformed, in accordance to the divergence theorem leads to the following local relation

$$\dot{S} = \Theta-\frac{q_{i,i}}{T}+\frac{q_i\,T_{,i}}{T^2}, \quad \Theta \geqq 0. \tag{2.9}$$

Eliminating $q_{i,i}$ from (2.6) and (2.9) and introducing therein the expression for the Helmholtz free energy $F = U-ST$, we obtain

$$\dot{F} = \sigma_{ji}\dot{\gamma}_{ji}+\mu_{ji}\,\dot{\varkappa}_{ji}+T\dot{S}-T\left(\Theta+\frac{q_i\,T_{,i}}{T^2}\right). \tag{2.10}$$

Since the free energy is a function of independent variables γ_{ji}, $\varkappa_{ji}$, T there is

$$\dot{F} = \frac{\partial F}{\partial\gamma_{ji}}\,\dot{\gamma}_{ji}+\frac{\partial F}{\partial\varkappa_{ji}}\,\dot{\varkappa}_{ji}+\frac{\partial F}{\partial T}\,dT. \tag{2.11}$$

Assuming that the functions Θ, q_i, σ_{ji}, μ_{ji} do not explicitly depend on time derivatives of the functions γ_{ji}, $\varkappa_{ji}$, T and defining the entropy as $S = -\dfrac{\partial F}{\partial T}$, we obtain, comparing (2.10) with (2.11), the following relations

$$\sigma_{ji} = \frac{\partial F}{\partial\gamma_{ji}}, \quad \mu_{ji} = \frac{\partial F}{\partial\varkappa_{ji}}, \quad S = -\frac{\partial F}{\partial T}, \quad \Theta+\frac{q_i\,T_{,i}}{T^2} = 0. \tag{2.12}$$

The second law of thermodynamics will be satisfied, if $\Theta \geqq 0$ or if

$$-\frac{T_{,i}\, q_i}{T^2} \geqq 0. \tag{2.13}$$

This inequality satisfies the Fourier law of thermal conductivity

$$k_{ij} \cdot T_{,j} = -q_i, \quad k_{ij}\, \theta_{,j} = -q_i, \quad T = T_0 + \theta, \tag{2.14}$$

T_0 denotes here the temperature of the body in natural state where stresses and deformations are equal zero. From (2.9) — taking into account the last relation of the group (2.12) — we have

$$T\dot{S} = -q_{i,i} = k_{ij}\, \theta_{,ij}. \tag{2.15}$$

For an isotropic and homogeneous body we get

$$T\dot{S} = k\theta_{,jj}, \tag{2.16}$$

where k is a constant.

3. Constitutive equations

Let us expand the expression for free energy $F(\gamma_{ji}, \varkappa_{ji}, T)$ in natural state of a body in the environment ($\gamma_{ji} = \varkappa_{ji} = 0, T = T_0$) into a Taylor series, we obtain for an isotropic centrosymmetric body, the following form

$$F = \frac{\mu+\alpha}{2} \gamma_{ji}\, \gamma_{ji} + \frac{\mu-\alpha}{2} \gamma_{ji}\, \gamma_{ij} + \frac{\lambda}{2} \gamma_{kk}\, \gamma_{nn} - \nu_1\, \gamma_{kk}\, \theta + \frac{\gamma+\varepsilon}{2} \varkappa_{ji}\, \varkappa_{ji} + \frac{\gamma-\varepsilon}{2} \varkappa_{ji}\, \varkappa_{ij} + \frac{\beta}{2} \varkappa_{kk}\, \varkappa_{nn} - \nu_2\, \varkappa_{kk}\, \theta - \frac{m}{2}\, \theta^2 + \ldots \tag{3.1}$$

Making use of relations (2.12), we have

$$\sigma_{ji} = (\mu+\alpha)\, \gamma_{ji} + (\mu-\alpha)\, \gamma_{ij} + (\lambda\gamma_{kk} - \nu_1\, \theta)\, \delta_{ij}, \tag{3.2}$$

$$\mu_{ji} = (\gamma+\varepsilon)\, \varkappa_{ji} + (\gamma-\varepsilon)\, \varkappa_{ij} + (\beta\varkappa_{kk} - \nu_2\, \theta)\, \delta_{ij}, \tag{3.3}$$

$$S = \nu_1\, \gamma_{kk} + \nu_2\, \varkappa_{kk} + m\theta + \ldots. \tag{3.4}$$

Eqs. (3.2) and (3.3) may be given the following forms

$$\sigma_{ji} = 2\mu\gamma_{(ij)} + 2\alpha\gamma_{[ij]} + (\lambda\gamma_{kk} - \nu_1\, \theta)\, \delta_{ij}, \tag{3.2'}$$

$$\mu_{ji} = 2\gamma\varkappa_{(ij)} + 2\varepsilon\varkappa_{[ij]} + (\beta\varkappa_{kk} - \nu_2\, \theta)\, \delta_{ij}. \tag{3.3'}$$

Here μ, λ are Lamé constants and $\alpha, \gamma, \varepsilon, \beta$ — new material constants. All these quantities refer to the isothermic state. ν_1 and ν_2 constants depend on mechanical as well as thermal properties of the body. The brackets ( ) and [ ] denote the symmetric and skew-symmetric parts of the tensor, respectively.

Solving Eqs. (2.3) and (3.3) with respect to γ_{ij} and $\varkappa_{ij}$, we get

$$\gamma_{ij} = a_1\, \delta_{ij}\, \theta + 2\mu'\, \sigma_{(ij)} + 2\alpha'\, \sigma_{[ij]} + \lambda'\, s\delta_{ij}, \tag{3.5}$$

$$\varkappa_{ij} = a_2\, \delta_{ij}\, \theta + 2\gamma'\, \mu_{(ij)} + 2\varepsilon'\, \mu_{[ij]} + \beta'\, q\delta_{ij}, \tag{3.6}$$

where

$$2\mu' = \frac{1}{2\mu}, \quad 2\alpha' = \frac{1}{2\alpha}, \quad 2\gamma' = \frac{1}{2\gamma}, \quad 2\varepsilon' = \frac{1}{2\varepsilon}, \quad s = \sigma_{kk}, \quad q = \mu_{kk},$$

$$\lambda' = -\frac{\lambda}{6\mu K}, \quad \beta' = -\frac{\beta}{6\gamma\Omega}, \quad \alpha_1 = \frac{\nu_1}{3K}, \quad \alpha_2 = \frac{\nu_2}{3\Omega},$$

$$K = 2\mu+3\lambda, \quad \Omega = 2\gamma+3\beta.$$

The symbol α_1 in Eqs. (3.5) stands for the coefficient of linear thermal dilatation; α_2 in Eqs. (3.6) denotes the coefficient of thermal rotation. Since the volume element free of strains and couple stresses on the surface undergoes—under the influence of temperature—but voluminal changes, we have

$$\gamma_{ij}^0 = \alpha_1\, \delta_{ij}\, \theta, \varkappa_{ij}^0 = \alpha_2\, \delta_{ij}\, \theta = 0,$$

Thus, we have to put formula (3.6) $\alpha_2 = 0$, while in formulae (3.3) and (3.4) $\nu_2 = 0$. In order to determine the coefficient m — undefined as yet in the formula (3.4) for the entropy — let us consider the differential equation

$$\text{(3.7)} \qquad dS = \left(\frac{\partial S}{\partial \gamma_{ji}}\right)_{\varkappa, T} d\gamma_{ji} + \left(\frac{\partial S}{d\varkappa_{ji}}\right)_{\mu, T} d\varkappa_{ji} + \left(\frac{\partial S}{\partial T}\right)_{\gamma, \varkappa} dT.$$

Since $\left(\frac{\partial S}{\partial T}\right)_{\gamma,\varkappa} = \frac{c_\varepsilon}{T}$, where c_ε denotes the specific heat of the body at constant deformation, we get

$$\text{(3.8)} \qquad dS = \nu_1\, d\gamma_{kk} + \frac{c_\varepsilon}{T}\, dT.$$

Integrating this expression under the assumption that $S = 0$ for the natural state of the body, we obtain

$$\text{(3.9)} \qquad S = \nu_1\, \gamma_{kk} + \log\frac{T}{T_0}.$$

Taking into consideration the formula

$$\text{(3.10)} \qquad T\dot{S} = \nu_1\, T\dot{\gamma}_{kk} + c_\varepsilon\, T$$

and comparing it with Eq. (2.16), we obtain the following equation

$$\text{(3.11)} \quad \theta_{,jj} - \frac{1}{\varkappa}\,\dot{\theta} - \eta_1\left(1+\frac{\theta}{T_0}\right)\dot{\gamma}_{kk} = 0, \quad \varkappa = k/c_\varepsilon, \quad \eta_1 = \nu_1\, T_0/k,$$

In linearizing this equation, we assume that θ/T_0 is small as compared with unity. Taking, moreover, into account heat sources within the body and denoting by W the quantity of heat generated per volume and time unit, we obtain the following expanded equation of thermal conductivity

$$\text{(3.12)} \qquad \theta_{,jj} - \frac{1}{\varkappa}\,\dot{\theta} - \eta_1\,\dot{\gamma}_{kk} = -\frac{Q}{\varkappa}, \quad Q = W/k.$$

4. Differential equations of thermoelasticity

The constitutive relations (3.2) and (3.3) enable to express the equation of motion (2.3), (2.4) in terms of displacement vector $\vec{u}$, rotation vector $\vec{\omega}$ and temperature θ. The following set of equations is obtained

$$(\mu+\alpha)\nabla^2 u_i+(\lambda+\mu-\alpha)u_{j,ji}+2\alpha\epsilon_{kij}\omega_{k,j}+X_i=\varrho\ddot{u}_i+\nu_1\theta_{,i}, \tag{4.1}$$

$$(\gamma+\varepsilon)\nabla^2\omega_i+(\beta+\gamma-\varepsilon)\omega_{j,ji}+2\alpha\epsilon_{kij}u_{k,j}-4\alpha\omega_i+Y_i=J\ddot{\omega}_i. \tag{4.2}$$

We have to supplement the above equation with the equation of heat conductivity

$$\nabla^2\theta-\frac{1}{\varkappa}\dot{\theta}-\eta_1\dot{\gamma}_{kk}=-\frac{Q}{\varkappa}. \tag{4.3}$$

Now, all the three equations here derived will read — when put in vector notation — as follows

$$(\mu+\alpha)\nabla^2\vec{u}+(\lambda+\mu-\alpha)\operatorname{grad}\operatorname{div}\vec{u}+2\alpha\operatorname{rot}\vec{\omega}+\vec{X}=\varrho\ddot{\vec{u}}+\nu_1\operatorname{grad}\theta, \tag{4.1'}$$

$$(\gamma+\varepsilon)\nabla^2\vec{\omega}+(\beta+\gamma-\varepsilon)\operatorname{grad}\operatorname{div}\vec{\omega}+2\alpha\operatorname{rot}\vec{u}-4\alpha\vec{\omega}+\vec{Y}=J\ddot{\vec{\omega}}, \tag{4.2'}$$

$$\nabla^2\theta-\frac{1}{\varkappa}\dot{\theta}-\eta_1\operatorname{div}\dot{\vec{u}}=-\frac{Q}{\varkappa}. \tag{4.3'}$$

These equations are interconnected and coupled. Only in the case of stationary heat flow the equation of heat conductivity becomes independent of the remaining two. Eqs. (4.1)—(4.3) should be supplemented with initial and boundary conditions. Initial conditions take the form

$$\begin{aligned} &u_i(\vec{x},0)=f_i(\vec{x}), \quad \dot{u}_i(\vec{x},0)=g_i(\vec{x}),\\ &\omega_i(\vec{x},0)=h_i(\vec{x}), \quad \dot{\omega}_i(\vec{x},0)=l_i(\vec{x}). \end{aligned} \tag{4.4}$$

If, on the boundary, the loadings p_i and moments m_i are prescribed, then

$$\begin{gathered} p_i(\vec{x},t)=\sigma_{ji}(\vec{x},t)\,n_j(\vec{x}), \quad m_i(\vec{x},t)=\mu_{ji}(\vec{x},t)\,n_j(\vec{x}),\\ \vec{x}\in A, \quad t>0. \end{gathered} \tag{4.5}$$

5. Variational principle

It is easy to show that the following equation holds

$$\int_V[(X_i-\varrho\ddot{u}_i)\,\delta u_i+(Y_i-J\ddot{\omega}_i)\,\delta\omega_i]\,dV+\int_A(p_i\,\delta u_i+m_i\,\delta\omega_i)\,dA=\\=\int_V(\sigma_{ji}\,\delta\gamma_{ji}+\mu_{ji}\,\delta\varkappa_{ji})\,dV. \tag{5.1}$$

In this equation the terms δu_i and $\delta\omega_i$ denote the virtual increments of the components of displacement and rotation vectors. Thus, introducing the relations (3.2) and (3.3) into the right-hand part of Eq. (5.1), we obtain the equation

$$\int_V[(X_i-\varrho\ddot{u}_i)\,\delta u_i+(Y_i-J\ddot{\omega}_i)\,\delta\omega_i]\,dV+\int_A(p_i\,\delta u_i+m_i\,\delta\omega_i)\,dA=\\=\delta W-\int_V\nu_1\theta\delta\gamma_{kk}\,dV, \tag{5.2}$$

where

$$\delta W = \int_V [(\mu+\alpha)\gamma_{ji}\,\delta\gamma_{ji} + (\mu-\alpha)\gamma_{ij}\,\delta\gamma_{ij} + \lambda\gamma_{kk}\,\delta\gamma_{kk} + \\ + (\gamma+\varepsilon)\varkappa_{ji}\,\delta\varkappa_{ji} + (\gamma-\varepsilon)\varkappa_{ji}\,\delta\varkappa_{ij} + \beta\varkappa_{kk}\,\delta\varkappa_{kk}]\,dV.$$

Eq. (5.2) ought to be supplemented with an additional formula since only four causes, namely X_i, Y_i, p_i, m_i, appear in this equation in explicit form. We introduce the vector $\vec{H}$ connected with the vector of neat flow $\vec{q}$ and the entropy S by the following relations

$$\vec{q} = T_0\,\dot{\vec{H}}, \quad S = -\operatorname{div}(\dot{\vec{H}}). \tag{5.3}$$

Taking into account the Fourier law of heat conductivity

$$\vec{q} = -k \operatorname{grad}\theta \tag{5.4}$$

and the relation for the rate of entropy (3.10)

$$-\operatorname{div}\vec{q} = T_0\,\dot{S} = \nu_1 T_0\,\dot{\gamma}_{kk} + c_\varepsilon\,\dot{\theta}, \tag{5.5}$$

we obtain the following connection of the vector $\vec{H}$ with temperature θ and γ_{kk}, $\varkappa_{kk}$

$$\dot{H}_i = -\frac{k}{T_0}\theta_{,i}, \quad -T_0 H_{i,i} = c_\varepsilon\,\theta + \nu_1\gamma_{kk}. \tag{5.6}$$

Multiplying formula (5.6) by δH_i and integrating it over the region V, we get

$$\int_V \left(\theta_{,i} + \frac{T_0}{k}\,\dot{H}_i\right)\delta H_i\,dV = 0. \tag{5.7}$$

Now, integrating (5.7) per parts, applying the divergence theorem and making use of the relation

$$-T_0\,\delta H_{i,i} = c_\varepsilon\,\delta\theta + \nu_1\,\delta\gamma_{kk}, \tag{5.8}$$

we get the following formula

$$\int_A \theta\delta H_n\,dA + \frac{c_\varepsilon}{T_0}\int_V \theta\delta\theta dV + \frac{T_0}{k}\int_V \dot{H}_i\,\delta H_i\,dV = -\int_V \theta\nu_1\,\delta\gamma_{kk}\,dV. \tag{5.9}$$

Connecting Eqs. (5.2) and (5.9), we obtain the final formulation of the variational theorem

$$\delta W + \delta P + \delta D = \int_V [(X_i - \varrho\ddot{u}_i)\,\delta u_i + (Y_i - J\ddot{\omega}_i)\,\delta\omega_i]\,dV + \\ + \int_A (p_i\,\delta u_i + m_i\,\delta\omega_i)\,dA - \int_A \theta\delta H_n\,dA. \tag{5.10}$$

Here there is

$$P = \frac{c_\varepsilon}{2T_0} \int_V \theta^2 \, dV, \quad D = \frac{T_0}{2k} \int_V \dot{H}_i \dot{H}_i \, dV, \tag{5.11}$$

where P is the heat potential and D denotes the dissipation function.

The variational principle, Eq. (5.10), may serve to derive the energetic theorem, if we compare the functions u_i, ω_i, θ at the point $\vec{x}$ and the moment t with those actually appearing in some other point after a dt time lapse.

Thus, introducing into Eq. (5.10) the expressions:

$$\delta u_i = \frac{\partial u_i}{\partial t} dt = v_i \, dt, \quad \delta \omega_i = \frac{\partial \omega_i}{\partial t} dt = w_i \, dt,$$

$$\delta\theta = \dot{\theta} dt, \quad \delta H_i = \dot{H}_i \, dt = -\frac{k}{T_0} \theta_{,i} \, dt, \quad \text{and so on},$$

we obtain the following formula

$$\frac{d}{dt}(K+W+P)+\chi_0 = \int_V (X_i v_i + Y_i w_i) \, dV + \int_A (p_i v_i + m_i w_i) \, dA + \frac{k}{T_0} \int_A \theta\theta_{,n} \, dA, \tag{5.12}$$

where

$$K = \frac{1}{2} \int_V (\varrho v_i v_i + J w_i w_i) \, dV, \quad \chi_0 = \frac{k}{T_0} \int_V \theta_{,i} \theta_{,i} \, dV \geqslant 0.$$

Theorem (5.12) may be used to demonstrate the uniqueness theorem for a simply connected body. Such a demonstration may be carried out similarly as that given in [2].

6. Theorem on reciprocity

Let us consider now two systems of agents and effects acting on an elastic body contained within the region V and bounded by the surface A.

As agents we regard: body forces X_i, body couples Y_i, heat sources Q, loadings p_i, m_i on surface A and the heating of this surface (prescribed temperature or heat flow).

As effects are regarded displacements u_i, components of the rotation vector ω_i and temperature θ.

The notations for the second system of agents and effects and of loadings will differ from those for the first system by mark "prim" ('). We assume the initial conditions of the problem to be homogeneous.

Let us perform on the equations of motion and the constitutive equations the Laplace transformation, where

$$\bar{u}_i(\vec{x}, p) = \mathcal{L}[u_i(\vec{x}, t)] = \int_0^\infty u_i(\vec{x}, t) e^{-pt} \, dt. \tag{6.1}$$

Now, we shall consider the integral

$$(6.2)\qquad I = \int_V (\bar{X}_i \bar{u}_i' - \bar{X}_i' \bar{u}_i)\, dV + \int_V (\bar{Y}_i \bar{\omega}_i' - \bar{Y}_i' \bar{\omega}_i)\, dV + \int_A (\bar{p}_i \bar{u}_i' - \bar{p}_i' \bar{u}_i)\, dA + \int_A (\bar{m}_i \bar{\omega}_i' - \bar{m}_i' \bar{\omega}_i)\, dA.$$

It may be shown the above integral may be reduced to the formula

$$(6.3)\qquad I = \int_V [\nu_1 \bar{\gamma}_{kk} \bar{\theta}' - \nu_1 \bar{\gamma}_{kk}' \bar{\theta}]\, dV.$$

Making use of equations of heat conductivity for both (i.e. with and without primes) states, we obtain the expression

$$(6.4)\qquad -\int_V (\bar{\theta} \nabla^2 \bar{\theta}' - \bar{\theta}' \nabla^2 \bar{\theta})\, dV - p \int_V \eta_1 \bar{\gamma}_{kk} \bar{\theta}'\, dV + p \int_V \eta_1 \bar{\gamma}_{kk}' \bar{\theta} dV + \frac{1}{\varkappa} \int_V (\bar{Q} \bar{\theta}' - \bar{Q}' \bar{\theta})\, dV = 0.$$

Since $\eta_1 = \dfrac{\nu_1 T_0}{k}$, we obtain from (6.4):

$$(6.5)\qquad \int_V [\nu_1 \bar{\gamma}_{kk} \bar{\theta}' - \nu_1 \bar{\gamma}_{kk}' \bar{\theta}]\, dV = \frac{k}{T_0 p} \left[\int_A (\bar{\theta}' \bar{\theta}_{,n} - \bar{\theta} \bar{\theta}'_{,n})\, dA + \frac{1}{\varkappa} \int_V (\bar{Q} \bar{\theta}' - \bar{Q}' \bar{\theta})\, dV \right].$$

Comparing (6.5) with (6.3) and (6.2), we arrive at the following equation

$$(6.6)\qquad p\zeta \left\{ \int_V (\bar{X}_i \bar{u}_i' - \bar{X}_i' \bar{u}_i)\, dV + \int_V (\bar{Y}_i \bar{\omega}_i' - \bar{Y}_i' \bar{\omega}_i)\, dV + \int_A (\bar{p}_i \bar{u}_i' - \bar{p}_i' \bar{u}_i)\, dA + \int_A (\bar{m}_i \bar{\omega}_i' - \bar{m}_i' \bar{\omega}_i)\, dA \right\} = \varkappa \int_A (\bar{\theta}' \bar{\theta}_{,n} - \bar{\theta} \bar{\theta}'_{,n})\, dA + \int_V (\bar{Q} \bar{\theta}' - \bar{Q}' \bar{\theta})\, dV, \qquad \zeta = \frac{T_0 \varkappa}{k}.$$

Performing the inverse Laplace transformation, we get the following final formation of the theorem on reciprocity

$$(6.7)\qquad \int_V dV(\vec{x}) \int_0^t \left[X_i(\vec{x}, t-\tau) \frac{\partial u_i'(\vec{x}, \tau)}{\partial \tau} - X_i'(\vec{x}, \tau) \frac{\partial u_i(\vec{x}, t-\tau)}{\partial \tau} \right] d\tau + \int_V dV(\vec{x}) \int_0^t \left[Y_i(\vec{x}, t-\tau) \frac{\partial \omega_i'(\vec{x}, \tau)}{\partial \tau} - Y_i'(\vec{x}, \tau) \frac{\partial \omega_i(\vec{x}, t-\tau)}{\partial \tau} \right] d\tau + \int_A dA(\vec{x}) \int_0^t \left[p_i(\vec{x}, t-\tau) \frac{\partial u_i'(\vec{x}, \tau)}{\partial \tau} - p_i'(\vec{x}, \tau) \frac{\partial u_i(\vec{x}, t-\tau)}{\partial \tau} \right] d\tau +$$

$$(6.7) \qquad + \int_A dA(\vec{x}) \int_0^t \left[m_i(\vec{x}, t-\tau) \frac{\partial \omega_i'(\vec{x}, \tau)}{\partial \tau} - m_i'(\vec{x}, \tau) \frac{\partial \omega_i(\vec{x}, t-\tau)}{\partial \tau} \right] d\tau =$$

$$= \frac{\varkappa}{\zeta} \int_A dA(\vec{x}) \int_0^t [\theta'(\vec{x}, \tau)\, \theta_{,n}(\vec{x}, t-\tau) - \theta(\vec{x}, t-\tau)\, \theta'_{,n}(\vec{x}, \tau)]\, d\tau +$$

$$+ \frac{1}{\zeta} \int_V dV(\vec{x}) \int_0^t [Q(\vec{x}, t-\tau)\, \theta'(\vec{x}, \tau) - Q'(\vec{x}, \tau)\, \theta(\vec{x}, t-\tau)]\, d\tau .$$

In the case of a stationary temperature field and static loads, we obtain the following set of two equations of reciprocity

$$(6.8) \qquad \int_V (X_i u_i' - X_i' u_i)\, dV + \int_V (Y_i \omega_i' - Y_i' \omega_i)\, dV + \int_A (p_i u_i' - p_i' u_i)\, dA +$$

$$+ \int_A (m_i \omega_i' - m_i' \omega_i)\, dA = \int_V [\nu_1 \gamma_{kk} \theta' - \nu_1 \gamma_{kk}' \theta]\, dV;$$

$$(6.9) \qquad \varkappa \int_V (\theta \theta'_{,n} - \theta' \theta_{,n})\, dA + \int_V (Q' \theta - Q \theta')\, dV = 0 .$$

Temperatures θ and θ' in Eq. (6.8) are considered as known functions, obtained from the corresponding equation of heat conductivity. Eq. (6.9) is a theorem on reciprocity for the problem of steady heat conduction.

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В. НОВАЦКИЙ, **МОМЕНТОВЫЕ НАПРЯЖЕНИЯ В ТЕОРИИ ТЕРМОУПРУГОСТИ**. III.

Рассматриваются взаимодействия деформационного и температурного полей в упругой среде Коссэрата, в которой полагается независимость вектора перемещений $\vec{u}$ от вектора вращения $\vec{\omega}$.

В настоящей заметке, являющейся обобщением работ [1] и [2] автора, выведены конститутивные уравнения и основное дифференциальное уравнение сопряженной термоупругости, а также приводятся вариационная теорема и теорема о взаимности.