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APPLIED MECHANICS

The Theorem on Reciprocity for Real Anisotropic Conductors in Thermo-magneto-elasticity

by

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In [1] was demonstrated the theorem on reciprocity of thermo-magneto-elasticity of isotropic bodies in the case of perfect electric conductivity of the body. In [2] we find an analogous theorem for isotropic bodies of finite electric conductivity. In the present paper the theorem of [2] will be generalized to the case of anisotropic bodies.

Fundamental equations

To begin with, we will consider the equations of linear theory of thermo-magneto-elasticity for slowly moving bodies, taking into account the thermoelectric effects [3]. The symmetry relations and thermodynamic discussion of the fundamental equations [3] are given in [4]. The results of the latter reference will be made use of in the present paper.

Similarly to [2] we shall start from the full, linearized set of equations of thermo-magneto-elasticity disregarding the second-order terms, which is admissible, if we assume that μ_{ik} and ε_{ik} do not differ very much from unity.

The equations of thermo-magneto-elasticity for a homogeneous anisotropic body are:

Equations of electrodynamics

$$\operatorname{rot} \boldsymbol{h} = \frac{4\pi}{c}\boldsymbol{j} + \frac{1}{c}\dot{\boldsymbol{D}}; \qquad \operatorname{rot} \boldsymbol{E} = -\frac{1}{c}\dot{\boldsymbol{b}};$$

$$b_i = \mu_{ik} h_k; \quad D_i = \varepsilon_{ik}\left[E_k + \frac{1}{c}(\dot{\boldsymbol{u}} \times \boldsymbol{B}_0)_k\right] - \frac{1}{c}(\dot{\boldsymbol{u}} \times \boldsymbol{H}_0) \approx \varepsilon_{ik} E_k;$$

$$\text{(1.1)} \qquad \operatorname{div} \boldsymbol{b} = 0; \quad \operatorname{div} \boldsymbol{D} = -4\pi\varrho_e \approx (\varepsilon_{ik} E_k)_{,i}; \quad \operatorname{div} \boldsymbol{j}_c = -\frac{\partial \varrho_e}{\partial t};$$

$$j_i = \eta_{ik} E_k - \varkappa_{ik} T_{,k} + \frac{\eta_{ik}}{c}(\dot{\boldsymbol{u}} \times \boldsymbol{B}_0)_k + \varrho_c u_i; \quad j_{ci} = j_i + j_{iz},$$

where, in agreement with the assumption that μ_{ik}, ε_{ik} are near unity, we shall use an approximate expression for D_i.

Equations of elasticity

$$-\varrho \ddot{u}_i+\sigma_{ik,k}+\frac{1}{c}(j\times B_0)_i+\varrho_e E_i - X_i = 0; \tag{1.2}$$
$$\sigma_{ik} = E_{ikmn}\, e_{mn} - \alpha_{ik} T.$$

Heat equations

$$\beta\dot{T}+\lambda_{ij}\,\dot{e}_{ij} - (k_{ij}\,T_{,j})_{,i}+(\pi_{ik}\, j_{ok})_{,i} = f. \tag{1.3}$$

The symbols are conventional and similar to those of [2]: $\boldsymbol{E}$, $\boldsymbol{h}$ are the perturbed electric and magnetic field, respectively, $\boldsymbol{j}$ — vector of current density, $\boldsymbol{j}_z$ — vector of external current density, ε_{ik}, μ_{ik} — tensors of electric and magnetic permeability, respectively, η_{ik} — tensor of electric conductivity, T — perturbed temperature, ϱ — density, E_{ikmn} — tensor of moduli of elasticity, α_{ik} — tensor of coefficients of linear thermal dilatation, λ_{ik} — tensor describing the influence of the strain on the temperature field, π_{ik} — tensor describing the influence of the density of the current vector on the density of heat flow, k_{ik} — tensor of coefficients of heat conduction, $\varkappa_{ik}$ — tensor, relating the gradient of the temperature field of body forces, f — function expressing external heat souices.

The coefficients λ_{ik} and α_{ik} in Eqs. (1.1)—(1.3) are interrelated by known relations obtained from the classical thermodynamic relations ($T_0\,\alpha_{ik} = \lambda_{ik}$). The symmetry relations for the coefficients η_{ik}, $\varkappa_{ik}$, π_{ik}, k_{ik}, etc., and their interrelations follow from the second law of thermodynamics and Onsager's relations [4].

In addition to the above equations for the body under consideration we have, in the case of a finite body in vacuum, the equations of field in vacuum

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)(\boldsymbol{h}^0, \boldsymbol{E}^0) = 0. \tag{1.4}$$

In the case of a finite body the set of equations should be supplemented with the boundary conditions on the bounding surface A, where there are prescribed: stresses or strains (or mixed conditions), the temperature or its normal derivatives and the continuity conditions of the field. In what follows we shall give a set of Eqs. (1.1)—(1.5) for the Laplace integral transformation:

$$\boldsymbol{u}(x_i, p) = \mathcal{L}\{u(x_i, t)\} = \int_{\infty}^{0} u(x_i, t)\, e^{-pt}\, dt, \quad \text{etc.} \tag{1.5}$$

It is assumed that the initial conditions are homogeneous and every action inducing the motion starts at $t = 0_+$. The theorem on reciprocity will be discussed for the integral transformations.

Theorem on reciprocity

We shall consider two sets of causes and effects. One of those sets will be denoted by "primes", the other will be written without "piimes". We start from the identity (1.2) after subjecting it to the Laplace transformation. We can easily find the equation:

$$(\bar{\sigma}_{ij}+\alpha_{ij}\,\bar{T})\,\bar{e}'_{ij} = (\bar{\sigma}'_{ij}+\alpha_{ij}\,\bar{T}')\,\bar{e}_{ij}. \tag{2.1}$$

On integrating (2.1) over the region B, we find

$$\int_B (\bar{\sigma}_{ij}\,\bar{e}'_{ij} - \bar{\sigma}'_{ij}\,\bar{e}_{ij})\,dV + \int_B \alpha_{ij}\,(\bar{T}\bar{e}'_{ij} - \bar{T}'\bar{e}_{ij})\,dV. \tag{2.2}$$

Bearing in mind that $\bar{\sigma}_{ij}\,\bar{\varepsilon}_{ij} = \bar{\sigma}_{ij}\,\bar{u}_{i,j}$, and making use of the divergence theorem and Eq. (1.2) written in the form

$$\sigma_{ij,\,j} + T_{ij,\,j} - \varrho\ddot{u}_i + X_i = 0\,, \tag{2.3}$$

where

$$T_{ij} = \frac{1}{4\pi}\,[b_i\,H_j + H_i\,b_j - \delta_{ij}\,H_k\,b_k] \tag{2.4}$$

is the Maxwell's tensor, we obtain

$$\int_B (\bar{X}_i\,\bar{u}'_i - \bar{X}'\,\bar{u}_i)\,dV + \int_A (\bar{p}^*_i\,\bar{u}'_i - \bar{p}^{*\prime}_i\,\bar{u}_i)\,dA + \int_B \alpha_{ij}\,(\bar{T}\,\bar{e}'_{ij} - \bar{T}'\,\bar{e}_{ij})\,dV = \int_B (\bar{T}_{ij}\,\bar{e}'_{ij} - \bar{T}'_{ij}\,\bar{e}_{ij})\,dV, \tag{2.5}$$

where

$$\bar{p}^*_i = (\bar{\sigma}_{ij} + \bar{T}_{ij})\,n_j\,. \tag{2.6}$$

Eq. (2.5) can be transformed thus

$$\int_B (\bar{X}_i\,\bar{u}'_i - \bar{X}'_i\,\bar{u}_i)\,dV + \int_A (\bar{p}_i\,\bar{u}'_i - \bar{p}'_i\,\bar{u}_i)\,dA + \int_B \alpha_{ij}\,(\bar{T}\bar{e}'_{ij} - \bar{T}'\bar{e}_{ij})\,dV = - \int_B (\bar{T}_{ij,\,j}\,\bar{u}'_i - \bar{T}'_{ij,\,j}\,\bar{u}_i)\,dV = -\frac{1}{c}\int_B [(\bar{j}'_c \times \boldsymbol{B}_0)_i\,\bar{u}'_i - (\bar{j}'_c \times \boldsymbol{B}_0)_i\,\bar{u}_i]\,dV, \tag{2.7}$$

where

$$\bar{p}_i = \bar{\sigma}_{ij}\,n_j\,.$$

From the heat equation we find in turn:

$$\int_B [k_{ij}\,\bar{T}_{,ji}\,\bar{T}' - k_{ij}\,\bar{T}'_{,ji}\,\bar{T}]\,dV = p\int_B \lambda_{ij}\,(\bar{e}_{ij}\,\bar{T}' - \bar{e}'_{ij}\,\bar{T})\,dV + \int_B [\pi_{ik}\,\bar{j}_{ck,\,i}\,\bar{T}' - \pi_{ik}\,\bar{j}'_{ck,\,i}\,\bar{T}]\,dV - \int_B (\bar{f}\,\bar{T}' - \bar{f}'\,\bar{T})\,dV. \tag{2.8}$$

Making use of Green's identity and the relation (1.1), we transform

$$\int_B \lambda_{ij}\,(\bar{e}'_{ij}\,\bar{T} - \bar{e}_{ij}\,\bar{T}')\,dV = \frac{1}{p}\int_B (\bar{f}'\,\bar{T} - \bar{f}\,\bar{T}')\,dV + \frac{1}{p}\int_A (\bar{T}\,k_{ij}\,\bar{T}'_{,j} + - \bar{T}'\,k_{ij}\,\bar{T}_{,\,j})\,n_i\,dA + \frac{\pi_{ik}}{p}\int_B (\bar{j}_{ck,\,i}\,\bar{T}' - \bar{j}'_{ck,\,i}\,\bar{T})\,dV. \tag{2.9}$$

On substituting (2.9) into (2.7), we find:

$$T_0\,p\,[\int_B (\bar{X}_i\,\bar{u}'_i - \bar{X}'_i\,\bar{u}_i)\,dV + \int_A (\bar{p}_i\,\bar{u}'_i - \bar{p}'_i\,\bar{u}_i)\,dA] + - \int_B (\bar{f}\,\bar{T}' - \bar{f}'\bar{T})\,dV - \int_A (\bar{T}'\,\bar{T}_{,j} - \bar{T}\,\bar{T}'_{,j})\,k_{ij}\,n_i\,dA + - \pi_{ik}\int_B (\bar{T}\,\bar{j}'_{ck,\,i} - \bar{T}'\,\bar{j}_{ck,\,i})\,dV = -\frac{T_0\,p}{c}\int_B [(\bar{j}_c \times \boldsymbol{B}_0)_i\,\bar{u}'_i - (\bar{j}'_c \times \boldsymbol{B}_0)_i\,\bar{u}_i]\,dV, \tag{2.10}$$

where use has been made of the familiar thermodynamic relation $T_0 a_{ij} = \lambda_{ij}$ [4]. Let us consider now the set of equations (1.1). The sets of equations with and without "primes" can be written thus

$$\varepsilon_{ikl}\bar{h}_{l,k} = \frac{4\pi}{c}\bar{j}_{ci} + \frac{p}{c}\varepsilon_{ik}\bar{E}_k = \frac{4\pi}{c}\bar{j}_{zi} + \frac{1}{c}(4\pi\eta_{ik}+p\varepsilon_{ik})\bar{E}_k + \frac{4\pi p\eta_{ik}}{c^2}(\bar{u}'\times B_0)_k - \frac{4\pi}{c}\varkappa_{ik}\bar{T}_{,k}, \tag{2.11.1}$$

$$\varepsilon_{ikl}\bar{E}_{l,k} = -\frac{p}{c}\mu_{ik}\bar{h}_k;$$

$$\begin{aligned}\varepsilon_{ikl}\bar{h}'_{l,k} &= \frac{4\pi}{c}\bar{j}'_{zi} + \frac{1}{c}(4\pi\eta_{ik}+p\varepsilon_{ik})\bar{E}'_k + \frac{4\pi p}{c^2}\eta_{ik}(\bar{u}'\times B_0)_k - \frac{4\pi}{c}\varkappa_{ik}\bar{T}'_{,k}\\ \varepsilon_{ikl}\bar{E}'_{lk} &= -\frac{p}{c}\mu_{ik}\bar{h}'_k,\end{aligned} \tag{2.11.2}$$

where ε_{ikl} is the unite pseudo-tensor.

The term $\dot{u}_l \varrho_e$ has been omitted in the expressions (2.11), in agreement with the assumption of linearity. On multiplying scalarly the first of Eqs. (2.11.1) by $\bar{E}'$, and the second of Eqs. (2.11.2) by $\bar{h}$ and substracting, we obtain bearing in mind the familiar equation

$$div\,(A\times B) = B\,\mathrm{rot}\,A - A\,\mathrm{rot}\,B \tag{2.12}$$

an expression for $\mathrm{div}(\bar{E}'\times\bar{h})$. Proceeding in the same manner with the remaining pair of Eqs. (2.11), we obtain an expression for div $(\bar{E}\times\bar{h}')$.

On subtracting these expressions, we find after some transformations

$$\frac{c}{4\pi}[\mathrm{div}(\bar{E}'\times\bar{h}) - \mathrm{div}(\bar{E}\times\bar{h}')] = (\bar{j}'_c\bar{E} - \bar{j}_c\bar{E}'), \tag{2.13}$$

where use has been made of the symmetry relations for the electric and magnetic permeability $\varepsilon_{ik}(B) = \varepsilon_{ki}(-B)$; $\mu_{ik}(B) = \mu_{ki}(-B)$, which follow directly from thermodynamic considerations (cf. [4]).

Bearing in mind the relations

$$\bar{j}_{ci} = \eta_{ik}\left[\bar{E}_k + \frac{p}{c}(\bar{u}\times B_0)_k\right] - \varkappa_{ik}\bar{T}_{,k} + \bar{j}_{zi},$$

$$\bar{j}'_{ci}\bar{E}_i - \bar{j}_{ci}\bar{E}'_i = \frac{p}{c}\eta_{ik}[(\bar{u}'\times\bar{B}_0)_k\bar{E}_i - (\bar{u}\times B_0)_k\bar{E}'_i] + \varkappa_{ik}(\bar{T}_{,k}\bar{E}'_i - \bar{T}'_{,k}\bar{E}_i) + \bar{j}'_{zi}\bar{E}_i - \bar{j}_{zi}\bar{E}_i \tag{2.15}$$

and also

$$\begin{aligned}(\bar{j}_c\times B_0)_i\bar{u}'_i - (\bar{j}'_c\times B_0)_i\bar{u}_i &= (\bar{u}\times B_0)_i\bar{j}'_{ci} - (\bar{u}'\times B_0)_i\bar{j}_{ci} =\\ &= (\bar{j}_z\times B_0)_i\bar{u}'_i - (\bar{j}'_z\times B_0)\bar{u}_i + \eta_{ik}[(\bar{u}\times B_0)_i\bar{E}'_k - (\bar{u}'\times B_0)_i\bar{E}_k] +\\ &\quad + \varkappa_{ik}[(\bar{u}'\times B_0)_i\bar{T}_{,k} - (\bar{u}\times B_0)_i\bar{T}'_{,k}]\end{aligned} \tag{2.16}$$

we can proceed to formulate the reciprocity relation.

Let us observe that the symmetry conditions have been used in the relations (2.15) (cf. [4]) $\eta_{ik}(B) = \eta_{ki}(-B)$.

Combining (2.10) and (2.13) and taking into consideration (2.15) and (2.16), we obtain

$$(2.17)\quad T_0 p \int_B (\bar{X}_i \bar{u}_i' - \bar{X}_i' u_i)\, dV - \int_B (\bar{f}\, \bar{T}' - \bar{f}' \bar{T})\, dV - \pi_{ik} \int_B (\bar{T} \bar{j}'_{ck,\,i} + - \bar{T}' \bar{j}_{ck,\,i})\, dV + \frac{T_0 p}{c} \int_B [(\bar{j}_z \times \boldsymbol{B}_0)_i\, \boldsymbol{u}_i - (\bar{j}_z' \times \boldsymbol{B}_0)\, \bar{u}_i]\, dV + + \frac{T_0 p}{c} \varkappa_{ik} \int_V [(\bar{\boldsymbol{u}}' \times \boldsymbol{B}_0)_i\, \bar{T}_{,k} - (\bar{\boldsymbol{u}} \times \boldsymbol{B}_0)_i\, \bar{T}_{,k}']\, dV + T_0 \varkappa_{ik} \int_B (\bar{T}_{,k} \bar{E}_i' + - \bar{T}_{,k}' \bar{E}_i)\, dV + T_0 \int_V (\bar{j}_{zi}' \bar{E}_i - \bar{j}_{zi} \bar{E}_i')\, dV = - T_0\, p \int_A (\bar{p}_i \bar{u}_i' - \bar{p}_i' \bar{u}_i)\, dA + + \int_A (\bar{T}' \bar{T}_{,j} - \bar{T} \bar{T}_{,j}')\, k_{ij}\, n_i\, dA + \frac{c}{4\pi} T_0 \int_A [(\bar{\boldsymbol{E}}' \times \bar{\boldsymbol{h}}) - (\bar{\boldsymbol{E}} \times \bar{\boldsymbol{h}}')]_i\, n_i\, dA\,.$$

Eq. (2.17) can now be transformed thus:

$$(2.18)\quad T_0 p \int_B (\bar{X}_i \bar{u}_i' - \bar{X}_i' \bar{u}_i) dV - \int_B (\bar{f}\, \bar{T}' - \bar{f}' \bar{T})\, dV + T_0 \varkappa_{ik} \int_B (\bar{T}_{,k} \bar{E}_{0i}' + - \bar{T}_{,k}' \bar{E}_{0i})\, dV + T_0 \int_B (\bar{j}_{zi}' \bar{E}_{0i} - \bar{j}_{zi} \bar{E}_{0i}')\, dV - \pi_{ik} \int_B (\bar{T} \bar{j}'_{ck,\,i} - \bar{T}' \bar{j}_{ck,\,i})\, dV = - T_0 p \int_A (\bar{p}_i \bar{u}_i' - \bar{p}_i' \bar{u}_i)\, dA + k_{ij} \int_A (\bar{T}' \bar{T}_{,j} - \bar{T} \bar{T}_{,j}')\, n_i\, dA + + \frac{c T_0}{4\pi} \int_A [(\bar{\boldsymbol{E}}' + \bar{\boldsymbol{h}}) - (\bar{\boldsymbol{E}} \times \bar{\boldsymbol{h}}')]_i\, n_i\, dA\,,$$

where

$$(2.19)\quad \bar{E}_{0i} = \bar{E}_i + \frac{p}{c} (\bar{\boldsymbol{u}} \times \boldsymbol{B}_0)_i$$

is the field in the system at rest.

Writing out the last term of the left-hand member of (2.19), we obtain finally, making use of (2.14):

$$(2.20)\quad T_0 p \int_B (\bar{X}_i \bar{u}_i' - \bar{X}_i' \bar{u}_i)\, dV - \int_B (\bar{f}\, \bar{T}' - \bar{f}' \bar{T})\, dV + T_0 \varkappa_{ik} \int_B (\bar{T}_{,k}' \bar{E}_{0i} + - \bar{T}_{,k}' \bar{E}_{0i})\, dV + T_0 \int_B (\bar{j}_{zi}' \bar{E}_{0i} - \bar{j}_{zi} \bar{E}_{0i}')\, dV - \pi_{ik} \int_B (\bar{T} \bar{j}'_{zk,\,i} - \bar{T}' \bar{j}_{zk,\,i})\, dV + - \pi_{ik} \eta_{kl} \int_V (\bar{T} \bar{E}'_{0l,\,i} - \bar{T}' E_{0l,\,i})\, dV - T_{ik} \varkappa_{kl} \int_V (\bar{T} \bar{T}_{,li}' - \bar{T}' \bar{T}_{,li})\, dV = = - T_0 p \int_A (\bar{p}_i \bar{u}_i' - \bar{p}_i' \bar{u}_i)\, dA + k_{ij} \int_A (\bar{T}' \bar{T}_{,j} - \bar{T} \bar{T}_{,j}')\, n_i\, dA + + \frac{c\, T_0}{4\pi} \int_A [(\bar{\boldsymbol{E}}' \times \bar{\boldsymbol{h}}) - (\bar{\boldsymbol{E}} \times \bar{\boldsymbol{h}}')]_i\, n_i\, dA\,.$$

Eq. (2.20) represents the final form of the theorem on reciprocity of magneto-thermo-elasticity for anisotropic bodies having finite electric conductivity. Its physical discussion is analogous to [1] and [2]. Eq. (2.20) becomes, for isotropic bodies, the corresponding Eq. (2.16) of [2], if we use, in addition, the relation (1.1) div $\boldsymbol{j}_c = -\dot{\varrho}_e$.

In the case of a finite body B Eq. (2.20) should be supplemented with that for the vacuum surrounding the body B and the boundary conditions — relations of continuity of field. The considerations are the same as in [2], therefore they will be omitted, referring the reader to that paper. If B becomes infinite, the surface integrals fall off. If the field in vacuum is disregarded and 3 field components are prescribed in B in an explicit form and if we assume that the boundary conditions are homogeneous in an appropirate manner, we can also make the surface integrals vanish.

If we set $\varkappa_{ik} = \pi_{ik} = 0$ in (2.20) we obtain the theorem on reciprocity of magneto-thermo-elasticity without thermoelectric effects. If, moreover, we pass with η_{ik} to infinity and disregard the external currents we obtain the theorem on reciprocity for perfect conductors. If now the body is assumed to be isotropic, we obtain the theorem in the same form as in [1]. For obtaining the explicit form of the theorem on reciprocity we should perform on (2.20) the inverse Laplace transformation.

Let us write as an example the first term of the Eq. (2.20). We have

$$T_0 \int_A dV \int_0^t \left[[X_i(\boldsymbol{x}, t-\tau) \frac{\partial u_i'(\boldsymbol{x}, \tau)}{\partial \tau} - X_i'(\boldsymbol{x}, t-\tau) \frac{\partial u_i(\boldsymbol{x}, \tau)}{\partial \tau} \right] d\tau + \ldots \tag{2.21}$$

Similarly to the analogous equation of [2].

For applications of the theorem on reciprocity, in particular for the construction of integral equations of boundary value problems, etc., the final remarks of [2] remain valid.

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С. КАЛИСКИЙ и В. НОВАЦКИЙ, **ТЕОРЕМА ВЗАИМНОСТИ ДЛЯ РЕАЛЬНЫХ АНИЗОТРОПНЫХ ПРОВОДНИКОВ В МАГНИТО-ТЕРМОУПРУГОСТИ.**

В работе приводится доказательство теоремы взаимности в магнито-термоупругости для реальных анизотропных проводников с учетом термоэлектрических эффектов. Работа представляет собой расширение работ [1] и [2], где аналогичные теоремы получены для изотропных проводников идеальных и реальных.

Теорема может быть использована для решения ряда практических задач, как напр. конструкция интегральных уравнений краевых проблем и т.п.