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APPLIED MECHANICS

Problem of Linear Coupled Magneto-thermo-elasticity. II. Variational Formulation for Magneto-thermo-elasticity

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1. Initial equations

Three groups of equations of magneto-thermo-elasticity will constitute the starting point for our subsequent considerations. They are:

a) equations of electrodynamics of slow-moving media, [1]—[2]

$$\operatorname{rot} \boldsymbol{h} = \frac{4\pi}{c} \boldsymbol{j}, \tag{1.1}$$

$$\operatorname{rot} \boldsymbol{E} = -\frac{\mu_0}{c} \frac{\partial \boldsymbol{h}}{\partial t}, \tag{1.2}$$

$$\boldsymbol{j} = \lambda_0 \left[\boldsymbol{E} + \frac{\mu_0}{c} \left(\frac{\partial \boldsymbol{u}}{\partial t} \times \boldsymbol{H} \right) \right], \tag{1.3}$$

$$\operatorname{div} \boldsymbol{h} = 0, \tag{1.4}$$

b) equations of motion

$$\sigma_{ij,j} + T_{ij,j} + X_i = \varrho \ddot{u}_i, \quad i, j = 1, 2, 3, \tag{1.5}$$

and

c) coupled equation of heat conductivity, [3]

$$\nabla^2 \theta - \frac{1}{\varkappa} \dot{\theta} - \eta \operatorname{div} \dot{\boldsymbol{u}} = 0. \tag{1.6}$$

In Eqs. (1.1)—(1.6) the following notations are used: $\boldsymbol{h}$ and $\boldsymbol{E}$ stand for the vectors of magnetic and electric fields, respectively, $\boldsymbol{j}$ is to denote the vector of the current density, $\boldsymbol{H}$ means the vector of primary constant field, $\boldsymbol{u}$ — the displacement vector, μ_0 — the magnetic permeability, c — the velocity of light and λ_0 — the electric conductivity.

The stress tensor is denoted by σ_{ij} and the Maxwell tensor of the electromagnetic field — by T_{ij}. $\mathbf{X}$ is the vector of body forces and ϱ — the density.

The term $\theta = T - T_0$ is the difference between the absolute temperature, T, and that characterizing the natural thermic state of the body, T_0; c_ε means the specific heat of the body, its deformation being assumed constant. $\varkappa = \lambda_0/\varrho c_\varepsilon$ is a coefficient, λ_0 denoting the heat conductivity of the body.

Making use of Eqs. (1.1)–(1.3) – after eliminating the functions $\boldsymbol{j}$ and $\boldsymbol{E}$ – and of Eq. (1.4), we obtain the following equation:

$$\nabla^2 \boldsymbol{h} - \beta \dot{\boldsymbol{h}} = -\beta \operatorname{rot}(\dot{\boldsymbol{u}} \times \boldsymbol{H}), \qquad \beta = \frac{4\pi\mu_0 \lambda_0}{c^2}. \tag{1.7}$$

In the sequel we shall take advantage of the Duhamel–Neumann's relations

$$\sigma_{ij} = 2\mu\varepsilon_{ij} + (\lambda e - \gamma\theta)\,\delta_{ij}, \qquad e = \varepsilon_{kk}, \tag{1.8}$$

as well as of the relations between the deformations and displacements

$$\varepsilon_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}), \qquad i, j = 1, 2, 3. \tag{1.9}$$

Maxwell's tensor appearing in (1.5) may be expressed by the components of the $\boldsymbol{h}$ and $\boldsymbol{H}$ vectors:

$$T_{ij} = \frac{\mu_0}{4\pi}\left[h_i H_j + h_j H_i - \delta_{ij}(h_k H_k)\right], \qquad i, j, k = 1, 2, 3. \tag{1.10}$$

Equations of the electrodynamics, (1.1)–(1.5) refer to a body with finite electric conductivity. An elastic body is here considered – as may be seen from the formulae (1.8) – as isotropic and homogeneous.

2. Variational principle of magneto-thermo-elasticity

Let us consider the expression

$$W = \int_B \left(\mu\varepsilon_{ij}\varepsilon_{ij} + \frac{\lambda}{2} e^2\right) dV, \tag{2.1}$$

where – beneath the symbol of the integral – the homogeneous quadratic function of deformation is given. Let us now compare this expression with the neighbouring state where the displacements u_i undergone changes by the virtual quantity δu_i, and the deformations ε_{ij} – by $\delta\varepsilon_{ij}$. In this way we obtain

$$\delta W = \int_B (2\mu\varepsilon_{ij}\,\delta\varepsilon_{ij} + \lambda e \delta e)\, dV. \tag{2.2}$$

Taking into consideration the Duhamel–Neumann's relations, (1.8), we transform the variation δW to the form

$$\delta W = \int_B \sigma_{ij}\,\delta\varepsilon_{ij}\, dV + \gamma \int_B \theta \delta e\, dV. \tag{2.3}$$

Performing a transformation on the first of the integrals and taking advantage of the theorem on divergence and making use of Eq. (1.5), we get

$$(2.4)\qquad \int_B X_i\,\delta u_i\,dV + \int_A p_i\,\delta u_i\,dA - \varrho\int_B \ddot{u}_i\,\delta u_i\,dV = \\ = \delta W - \gamma\int_B \theta\delta e\,dV + \int_B T_{ij}\,\delta\varepsilon_{ij}\,dV,$$

where $(\sigma_{ij}+T_{ij})\,n_j = p_i$.

Eq. (2.4) is a generalization of the d'Alembert principle of virtual works on the problems of magneto-thermo-elasticity.

This equation ought to be supplemented with two further equations since only two causes, namely X_i and p_i, appear in this equation in explicit form.

We have to adjoin to Eq. (2.4) the relation

$$(2.5)\qquad -\gamma\int_B \theta\delta e\,dV = \int_A \theta n_i\,dS_i\,dA + \frac{c_\varepsilon}{T_0}\int_B \theta\delta\theta\,dV + \frac{T_0}{\lambda_0}\int_B \dot{S}_i\,\delta S_i\,dV,$$

derived from the equation of heat conductivity (1.6) by M. A. Biot.

The symbol $\mathbf{S}$ in Eq. (2.5) denotes the vector connected with the vector of the heat flow q and the entropy s by the following relations

$$(2.6)\qquad \mathbf{q} = T_0\,\dot{\mathbf{S}},\qquad s = -\operatorname{div}(\mathbf{S}).$$

Taking into account the Fourier law of the heat conductivity

$$(2.7)\qquad \mathbf{q} = -\lambda_0\operatorname{grad}\theta,$$

and the relation for the increment of entropy with time [3]

$$(2.8)\qquad -\operatorname{div}\mathbf{q} = \dot{s}T_0 = c_\varepsilon\dot{\theta} + \gamma T_0\,\dot{e},$$

we obtain the following connection of the vector $\mathbf{S}$ with temperature and dilatation

$$(2.9)\qquad \dot{S}_i = -\frac{\lambda_0}{T_0}\theta_{,i},\qquad S_{i,i} = -\frac{c_\varepsilon}{T_0}\theta - \frac{\gamma}{T_0}e.$$

Introducing (2.5) into (2.4), we have

$$(2.10)\qquad \delta(W+P+D) = \int_B X_i\,\delta u_i\,dV + \int_A p_i\,\delta u_i\,dA - \varrho\int_B \ddot{u}_i\,\delta u_i\,dV - \\ - \int_A \theta n_i\,\delta S_i\,dA - \int_B T_{ij}\,\delta\varepsilon_{ij}\,dV.$$

We have introduced here the function of heat energy P and the dissipation function D introduced already by M. A. Biot. There is

$$(2.11)\qquad P = \frac{c_\varepsilon}{2T_0}\int_B \theta^2\,dV,\qquad D = \frac{T_0}{2\lambda_0}\int_B (\dot{S}_i)^2\,dV,\qquad \delta D = \frac{T_0}{\lambda_0}\int_B \dot{S}_i\,\delta S_i\,dV.$$

For $T_{ij} \to 0$ Eq. (2.10) reduces to the variational equation of coupled thermoelasticity.

It remains to find an expression, for the last term of Eq. (2.10) by the function h_j, making use of Eq. (1.7). For the sake of simplicity (with no prejudice to the generality) we assume the vector $\boldsymbol{H}$ to be directed along the x_3-axis, what means that $\boldsymbol{H} = (0, 0, H)$.

The direction of the $\boldsymbol{H}$-vector being assumed as already stated we will express the last integral of Eq. (2.11) — making use of the relation (1.10) — in the form as below

$$(2.12) \quad \int_B T_{ij}\, \delta\varepsilon_{ij}\, dV = \frac{\mu_0 H}{4\pi} \int_B [h_j(\delta u_{j,3} + \delta u_{3,j}) - h_3\, \delta e]\, dV = \\ = \frac{\mu_0 H}{4\pi} \int_B (h_j\, \delta u_{j,3} - h_3\, \delta e)\, dV + \frac{\mu_0 H}{4\pi} \int_B h_j\, n_j\, \delta u_3\, dV.$$

Assuming $\boldsymbol{H} = (0, 0, H)$, we may write Eq. (1.7) as follows

$$(2.13) \quad \nabla^2 h_j - \beta \dot{h}_j = -\beta H (\dot{u}_{j,3} - \delta_{j3}\, \dot{e}), \quad j = 1, 2, 3.$$

Let us now introduce the symmetrical tensor Φ_{ij} chosen in such a manner as to have

$$(2.14) \quad \dot{\Phi}_{ij} = -h_{j,i}, \quad i, j = 1, 2, 3.$$

Introducing (2.14) into (2.13), we obtain

$$(2.15) \quad \dot{\Phi}_{ij,i} = \beta [(\dot{u}_{j,3} - \delta_{j3}\, \dot{e}) H - \dot{h}_j],$$

whence

$$(2.16) \quad \Phi_{ij,i} = \beta [(u_{j,3} - \delta_{j3}\, e) H - h_j], \qquad \delta\Phi_{ij,i} = \beta [(\delta u_{j,3} - \delta_{j3}\, \delta e) H - \delta h_j].$$

Multiplying Eq. (2.14) by $\delta\Phi_{ij}$ and integrating it over the region of the body, we get

$$(2.17) \quad \int_B (\dot{\Phi}_{ij} + h_{j,i})\, \delta\Phi_{ij}\, dV = 0.$$

We make use of the theorem on divergence to present Eg. (2.17) in the form

$$(2.18) \quad \int_B \dot{\Phi}_{ij}\, \delta\Phi_{ij}\, dV + \int_A h_j\, \delta\Phi_{ij}\, n_i\, dA - \int_B h_j\, \delta\Phi_{ij,i}\, dV = 0.$$

Introducing the second relation of (2.16) into Eq. (2.18) and summing up with respect to j, we obtain the following equation

$$(2.19) \quad \int_B \dot{\Phi}_{ij}\, \delta\Phi_{ij}\, dV + \int_A h_j\, \delta\Phi_{ij}\, n_i\, dA - \beta H \int_B h_j (\delta u_{j,3} - h_3\, \delta e)\, dV + \\ + \beta \int_B h_j\, \delta h_j\, dV = 0.$$

Introducing the functions

$$(2.20) \quad F = \frac{\mu_0}{8\pi\beta} \int_B \dot{\Phi}_{ij}\, \dot{\Phi}_{ij}\, dV, \qquad L = \frac{\mu_0}{8\pi} \int_B h_j\, h_j\, dV,$$

and eliminating from Eqs. (2.12) and (2.19) the integral $\int_B (h_j, \delta u_{j,3} - h_3 \delta e)\, dV$ and subsequently introducing Eq. (2.12) into Eq. (2.10), we obtain a general form for the variational principle for the problems of magneto-thermo-elasticity

$$(2.21) \quad \delta(W+P+D+F+L) = \int_B X_i \delta u_i\, dV + \int_A p_i \delta u_i\, dA - \varrho \int_B \ddot{u}_i \delta u_i\, dV - \\ - \int_A \theta n_i \delta S_i\, dA - \frac{\mu_0}{4\pi\beta} \int_A h_j \delta \Phi_{ij} n_i\, dA - \frac{\mu_0 H}{4\pi} \int_A h_j n_j \delta u_3\, dV.$$

In a particular case of an ideal conductor there is $\lambda_0 = \infty$, consequently, $\beta = \infty$ also and Eq. (2.21) simplifies to the form

$$(2.22) \quad \delta(W+P+D+L) = \int_B X_i \delta u_i\, dV + \int_A p_i \delta u_i\, dA - \varrho \int_B \ddot{u}_i \delta u_i\, dA - \\ - \int_A \theta n_i \delta S_i\, dA - \frac{\mu_0 H}{4\pi} \int_A h_j n_j \delta u_3\, dV.$$

Neglecting the terms representing the electromagnetic processes we reduce Eq. (2.21) to the variational equation of thermoelasticity. Assuming, in addition, that the mechanical vibrations occur in adiabatic conditions ($\theta = -\eta\varkappa e$) we arrive at

$$(2.23) \quad \delta W + \varrho \int_B \ddot{u}_i \delta u_i\, dV = \int_B X_i \delta u_i\, dV + \int_A p_i \delta u_i\, dA, \qquad p_i = \sigma_{ij} n_j,$$

what means that we obtain the expression for the variational formula of classical elastokinetics.

The variational principle, Eq. (2.21), may serve to derive the energetic theorem, if we compare the functions u_i, θ. h_i in the point (x) at the moment t with those actually appearing in the same point after a dt time lapse.
Thus, introducing into Eq. (2.21)

$$(2.24) \quad \begin{aligned} &\delta u_i = \frac{\partial u_i}{\partial t} dt = v_i\, dt, \qquad \delta\theta = \dot{\theta} dt, \qquad \delta S_i = \dot{S}_i dt = -\frac{\lambda_0}{T_0} \theta_{,i}, \\ &\delta \Phi_{ij} = \dot{\Phi}_{ij}\, dt = -h_{j,i} dt, \qquad \delta W = \dot{W} dt, \text{ and so on,} \end{aligned}$$

we obtain the following formula [4]

$$(2.25) \quad \frac{d}{dt}(K+W+P+L) + \chi_\theta + \chi_h = \int_B X_i v_i\, dV + \int_A p_i v_i\, dA + \\ + \frac{\lambda_0}{T_0} \int_A \theta \theta_{,n}\, dA + \frac{\mu_0}{4\pi\beta} \int_A h_j h_{j,n}\, dA - \frac{\mu_0 H}{4\mu} \int_A \dot{u}_3 h_j n_j\, dA,$$

where

$$\chi_\theta = \lambda_0 T_0 \int_B \left(\frac{\theta_{,i}}{T_0}\right)^2 dV, \qquad \chi_h = \frac{\mu_0}{4\pi\beta} \int_B h_{j,i}\, h_{j,i}\, dV.$$

In the particular case of an ideal electric conductor there is $\lambda_0 = \infty$ and $\beta = \infty$. Then Eq. (2.25) simplifies to the form

$$\frac{d}{dt}(K+W+P+L)+\chi_\theta = \int_B X_i v_i\, dV + \int_A p_i v_i\, dA + \\ + \frac{\lambda_0}{T_0} \int_A \theta\theta_{,n}\, dA - \frac{\mu_0 H}{4\pi} \int_A \dot{u}_3 h_j n_j\, dA.$$

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В. НОВАЦКИЙ, **ПРОБЛЕМА ЛИНЕЙНОЙ СВЯЗАННОЙ МАГНИТО-ТЕРМО-УПРУГОСТИ. II. ВАРИАЦИОННАЯ ФОРМУЛИРОВКА МАГНИТО-ТЕРМО-УПРУГОСТИ.**

Обосновываясь на уравнениях движения (1.5) и используя уравнения теплопроводности (1.6), а также уравнение (1.7) для вектора интенсивности магнитного поля, выведена общая вариационная формулировка для проблем магнито-термо-упругости (2.21).

На основании этой формулировки выведена основная энергетическая зависимость (2.25).