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APPLIED MECHANICS

Two-dimensional Problem of Magnetothermoelasticity III.

by

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In our previous papers, [1] and [2], the dynamic problem was considered concerning the propagation of magnetothermoelastic waves in a perfectly conductive medium, the latter being in a constant primary magnetic field. In the present paper, we drop the assumption of a perfect conductivity of the medium considering a medium with finite conductivity. To begin with, we take as our starting point three groups of equations. The first of them is composed of equations of electrodynamics of slow-moving media, [3], namely:

$$\text{rot}\,\vec{h} = \frac{4\pi}{c}\vec{j}, \tag{1}$$

$$\text{rot}\,\vec{H} = -\frac{\mu_0}{c}\frac{\partial \vec{h}}{\partial t}, \tag{2}$$

$$\vec{j} = \lambda_0\left[\vec{E} + \frac{\mu_0}{c}\left(\frac{\partial \vec{u}}{\partial t} \times \vec{H}\right)\right], \tag{3}$$

$$\text{div}\,\vec{h} = 0. \tag{4}$$

In Eqs. (1)—(4) the symbols $\vec{h}$, $\vec{E}$ stand for the vectors of the magnetic and electric field intensities, respectively, $\vec{j}$ denotes the vector of the current density, $\vec{H}$ — the vector of primary, constant magnetic field, $\vec{u}$ — the displacement vector, μ_0 — the magnetic permeability factor, c — the velocity of light, and, finally, λ_0 — the electric conductivity.

The second group consists of equations of motion of an elastic medium supplemented with terms derived from Lorentz forces

$$\mu\,\Delta^2\vec{u} + (\lambda+\mu)\,\text{grad div}\,\vec{u} - \gamma\,\text{grad}\,\theta + \vec{X} + \frac{\mu_0}{c}[\vec{j}\times\vec{H}] = \varrho\frac{\partial^2\vec{u}}{\partial t^2}, \tag{5}$$

and of the expanded equation of heat conductivity

$$\nabla^2\theta - \frac{1}{\varkappa}\frac{\partial\theta}{\partial t} - \eta\frac{\partial}{\partial t}\text{div}\,\vec{u} = -\frac{Q}{\varkappa}. \tag{6}$$

The notations used in the two last equations have the following meaning: in Eq. (5) $\vec{X}$ denotes the vector of the body force and θ — the temperature referred to the natural, undeformed and unstressed state of the body; μ, λ are Lamé's isothermic constants and $\gamma = (3\lambda+2\mu)\,a_t$, where a_t is the coefficient of linear expansion. Now, in Eq. (6) $\varkappa$ denotes the thermal diffusivity, $\theta = W/\varrho c$, where W means the quantity of heat produced per time and volume unit and c_ε is the specific heat of constant deformation. Finally, $\eta = \gamma T_0/k$ stands for the coefficient describing the coupling of the field of temperature with that of deformation, T_0 denoting the absolute temperature of the body in its natural state (i.e. for $\theta = 0$); k is the coefficient of thermal conductivity.

Eliminating the quantities $\vec{E}$ and $\vec{j}$ from Eqs. (1)—(3), we obtain the following relation

$$\text{rot rot}\,\vec{h} = -\beta\frac{\partial\vec{h}}{\partial t} + \beta\,\text{rot}\left(\frac{\partial\vec{u}}{\partial t}\times\vec{H}\right), \qquad \beta = \frac{4\mu\lambda_0\,\mu_0}{c^2}. \tag{7}$$

Taking into account that

$$\text{rot rot}\,\vec{h} = \text{grad div}\,\vec{h} - \Delta^2\vec{h}, \tag{8}$$

as well as relation (4) we reduce Eq. (7) to the form

$$\nabla^2\vec{h} - \beta\frac{\partial\vec{h}}{\partial t} = -\beta\,\text{rot}\left(\frac{\partial\vec{u}}{\partial t}\times\vec{H}\right). \tag{8'}$$

Eqs. (5), (6) and (8) describe the propagation of magnetothermoelastic waves in a medium with finite conductivity.

In the sequel we assume (without loss of generality) the primary magnetic field to be reduced to the component $\vec{H} = (0, 0, H_3)$ acting along the x_3-axis.

In this case we have

$$\begin{aligned}
\vec{j} &= \frac{c}{4\pi}\{\partial_2 h_3 - \partial_3 h_2,\ \partial_3 h_1 - \partial_1 h_3,\ \partial_1 h_2 - \partial_2 h_1\}.\\
\dot{\vec{h}} &= -\frac{c}{\mu_0}\{\partial_2 E_3 - \partial_3 E_2,\ \partial_3 E_1 - \partial_1 E_3,\ \partial_1 E_2 - \partial_2 E_1\},\\
\vec{j} &= \lambda_0\left(E_1 + \frac{\mu_0 H_3}{c}\dot{u}_2,\ E_2 - \frac{\mu_0 H_3}{c}\dot{u}_1,\ E_3\right), \qquad \dot{u}_i = \frac{\partial u_i}{\partial t}.
\end{aligned} \tag{9}$$

Equations of displacement (5) take the following form:

$$\begin{aligned}
&\mu\nabla^2 u_1 + (\lambda+\mu)\,\partial_1 e - \gamma\partial_1\theta + X_1 + \frac{\mu_0 H_3}{4\pi}j_2 = \varrho\ddot{u}_1,\\
&\mu\nabla^2 u_2 + (\lambda+\mu)\,\partial_2 e - \gamma\partial_2\theta + X_2 - \frac{\mu_0 H_3}{4\pi}j_1 = \varrho\ddot{u}_2,\\
&\mu\nabla^2 u_3 + (\lambda+\mu)\,\partial_3 e - \gamma\partial_3\theta + X_3 = \varrho\ddot{u}_3,\\
&\qquad c = \varepsilon_{kk} = \partial_j u_j.
\end{aligned} \tag{10}$$

Equation of heat conductivity (6) undergoes no changes and the system of Eqs. (8) reduces to a sole equation

$$\nabla^2 h_3 - \beta \frac{\partial \vec{h}}{\partial t} = \beta H_3 \frac{\partial e}{\partial t}. \tag{11}$$

Now, passing from the spatial problem to the two-dimensional one we assume that all causes inducing the wave propagation in an unbounded space are independent of x_3. Thus, assuming $Q = Q(x_1, x_2, t)$, $x_j = x_j(x_1, x_2, t)$, $j = 1, 2$, $x_3 = 0$ the third equation from the equation group (10) will be dropped. In the remaining equations all derivatives of functions with respect to x_3 should be equalized to 0.

With these assumptions the magnetothermoelastic waves are described by the following set of equations

$$\nabla^2 h_3 - \beta \dot{h}_3 = \beta H_3 \dot{e}, \tag{12}$$

$$\mu \nabla^2 u_1 + (\lambda+\mu)\,\partial_1 e - \gamma \partial_1 \theta + X_1 - \frac{\mu_0 H_3}{4\pi} \partial_1 h_3 = \varrho \ddot{u}_1, \tag{13}$$

$$\mu \nabla^2 u_2 + (\lambda+\mu)\,\partial_2 e - \gamma \partial_2 \theta + X_2 - \frac{\mu_0 H_3}{4\pi} \partial_2 h_3 = \varrho \ddot{u}_2, \tag{14}$$

$$\nabla_1^2 \theta - \frac{1}{\varkappa} \dot{\theta} - \eta \dot{e} = 0 \tag{15}$$

where $e = \partial_1 u_1 + \partial_2 u_2$, $\nabla_1^2 = \partial_1^2 + \partial_2^2$.

The system of Eqs. (12)—(15) can be partially disjoined, namely by introducing a decomposition of the displacement vector $\vec{u} = (u_1, u_2, 0)$ and of the body forces vector $\vec{X} = (X_1, X_2, 0)$ into two parts: potential and rotational.

$$u_1 = \partial_1 \Phi - \partial_2 \psi, \quad u_2 = \partial_2 \Phi + \partial_1 \psi, \tag{16}$$

$$X_1 = \varrho(\partial_1 \vartheta - \partial_2 \chi), \quad X_2 = \varrho(\partial_2 \vartheta + \partial_1 \chi). \tag{17}$$

Introducing Eqs. (16) and (17) into Eqs. (12)—(15), we obtain the following system of equations

$$\left(\nabla_1^2 - \frac{1}{c_2^2} \frac{\partial^2}{\partial t^2}\right) \psi = -\frac{1}{c_2^2} \chi, \tag{18}$$

$$\left(\nabla_1^2 - \frac{1}{c_1^2} \frac{\partial^2}{\partial t^2}\right) \Phi - m\theta - \frac{\mu_0 H_3}{4\pi \varrho c_1^2} h_3 = -\frac{1}{c_1^2} \vartheta, \tag{19}$$

$$\left(\nabla_1^2 - \frac{1}{\varkappa} \frac{\partial}{\partial t}\right) \theta - \eta \nabla_1^2 \frac{\partial \Phi}{\partial t} = -\frac{Q}{\varkappa}, \tag{20}$$

$$\left(\nabla_1^2 - \beta \frac{\partial}{\partial t}\right) h_3 - \beta H_3 \nabla_1^2 \frac{\partial \Phi}{\partial t} = 0, \quad c_1 = \left(\frac{\lambda+2\mu}{\varrho}\right)^{\frac{1}{2}}, c_2 = \left(\frac{\mu}{\varrho}\right)^{\frac{1}{2}}, m = \frac{\gamma}{c_1^2 \varrho}. \tag{21}$$

Let us observe that Eq. (18) may be solved independently of other equations, which are conjugate. If in an unbounded space we have $Q = 0$ and $\vartheta = 0$, then the only factor inducing a motion are the body forces, $\vec{X} = \varrho(-\partial_2 \chi, \partial_1 \chi, \partial)$. Thus, $\Phi = 0$,

$\theta = 0$ and $h_3 = 0$. In the unbounded space only a transverse wave, purely elastic, propagates with constant velocity $c_2 = (u/\varrho)^{1/2}$. In this case we have

$$e = \partial_1 u_1 + \partial_2 u_2 = 0, \quad \omega_3 = \tfrac{1}{2}(\partial_1 u_2 - \partial_2 u_1) = \tfrac{1}{2}\nabla^2\psi,$$

and the state of stress described in a general form by the formula

$$\sigma_{ij} = 2\mu\,\varepsilon_{ij} + (\lambda e - \gamma\theta)\,\delta_{ij} \tag{22}$$

reduces to three quantities

$$\sigma_{11} = \sigma_{22} = -2\mu\partial_1\partial_2\psi, \quad \sigma_{12} = \mu(\partial_1^2 - \partial_2^2)\,\psi. \tag{23}$$

From the formulae (9) we have

$$\vec{j} = 0, \quad E_1 = -\frac{\mu_0 H_3}{c}\partial_1\dot{\psi}, \quad E_2 = -\frac{\mu_0 H_3}{c}\partial_2\dot{\psi}, \quad \partial_2 E_1 - \partial_1 E_2 = 0.$$

Let us now consider the case, where $\chi = 0$ and in the unbounded space sources of heat Q are acting and body forces $\vec{X} = \varrho(\partial_1, \vartheta, \partial_2, \vartheta, 0)$ derived from the potential ϑ. In this case we have $\psi = 0$ in each point of the unbounded space. We have at our disposal the conjugate equations (19) and (21). Then in the unbounded region longitudinal magnetothermoelastic waves Φ, h_3 and Q will arise.

We shall now eliminate the function h_3 from Eqs. (19)—(21). As a result we obtain a system of two equations; their right-hand sides represent the causes inducing the wave movement.

$$D_1\,\Box_1^2\,\Phi - mD_1\,\theta - \alpha\beta\,\nabla_1^2\,\dot{\Phi} = -\frac{1}{c_1^2}D_1\,\vartheta, \tag{24}$$

$$D_2\,\theta - \eta\nabla_1^2\,\dot{\Phi} = -\frac{Q}{\varkappa}. \tag{25}$$

We have introduced here the following notations

$$D_1 = \nabla_1^2 - \beta\frac{\partial}{\partial t}, \quad D_2 = \nabla_1^2 - \frac{1}{\varkappa}\frac{\partial}{\partial t}, \quad \Box_1^2 = \nabla^2 - \frac{1}{c_1^2}\frac{\partial^2}{\partial t^2},$$
$$a_0^2 = \frac{\mu_0 H_3^2}{4\pi\varrho}, \quad \alpha = \frac{a_0^2}{c_1^2}. \tag{26}$$

Here, the symbol a_0 stands for what is called the Alfven velocity. The disturbance provoked by the existence of the primary magnetic field is characterized by the term $\alpha\beta\nabla_1^2\Phi$ in Eq. (24). Let us observe that in the particular case of perfect conductivity, i.e. for $\lambda_0 = \infty$ and, consequently, $\beta = \infty$, Eq. (24) reduces to the form

$$(1+\alpha)\,\nabla_1^2\,\Phi - \frac{1}{c_1^2}\ddot{\Phi} - m\theta = -\frac{1}{c_1^2}\vartheta, \tag{24'}$$

or

$$\nabla^2\Phi - \frac{1}{c_0^2}\ddot{\Phi} - m_0\,\theta = -\frac{1}{c_1^2}\vartheta. \quad c_0^2 = c_1^2 + a_0^2 = c_1^2(1+\alpha), \quad m_0 = \frac{\gamma}{c_0^2\varrho} \tag{24''}$$

in conformity with the equation derived in [1]. If, in addition, $H_3 = 0$, Eqs. (24) and (25) transform into the known equation of thermoelasticity. To determine the potential Φ we may make use of the equation

$$(27)\qquad \left[D_1 D_2 \Box_1^2 - \frac{1}{\varkappa}\partial_t \nabla_1^2 (\varepsilon_T D_1 + \varepsilon_H D_2)\right]\Phi = -\frac{m}{\varkappa} D_1 Q - \frac{1}{c_1^2} D_1 D_2 \vartheta .$$

The following notations have been introduced here

$$\varepsilon_T = \eta m \varkappa, \qquad \varepsilon_H = \alpha\beta\varkappa, \qquad \partial_t = \frac{\partial}{\partial t}.$$

In the equation of longitudinal wave, i.e. in Eq. (27), the coefficient ε_T characterizes the conjugation of the temperature field with that of deformation, while the coefficient ε_H describes the conjugation of the electromagnetic field with that of deformation. True, Eq. (27) is very complicated, however, it appears from its very structure that the magnetothermoelastic wave Φ is a damped wave and undergoes dispersion.

After determining the function Φ as a particular integral of Eq. (27) in an unbounded space, we substitute $\nabla_1^2 \Phi$ into Eqs. (20) and (21). The solution of these equations leads to the determination of functions θ and h_3. The function Φ being known, we are able to determine certain mechanical quantities.

$$(28)\qquad u_i = \partial_i \Phi, \qquad \varepsilon_{ij} = \partial_i \partial_j \Phi, \qquad e = \nabla^2 \Phi, \qquad \omega_3 = \tfrac{1}{2}(\partial_1 u_2 - \partial_2 u_1) = 0 .$$

The stresses σ_{ij} may be obtained from the formula (22), Eq. (19) being taken into account. The corresponding formulae read as follows

$$(29)\qquad \begin{aligned} \sigma_{ij} &= 2\mu(\Phi_{ij} - \delta_{ij}\nabla_1^2 \Phi) + \varrho(\ddot{\Phi} - \vartheta) + \frac{\mu_0 H_3 h_3}{4\pi}, \qquad i,j = 1,2, \\ \sigma_{33} &= -2\mu \nabla_1^2 \Phi + \varrho(\ddot{\Phi} - \vartheta) + \frac{\mu_0 H_3 h_3}{4\pi}. \end{aligned}$$

Electromagnetic quantities are given by formulae (9).

The function θ may be obtained equally as a particular integral of an equation derived by means of elimination of the function Φ from Eqs. (24) and (25)

$$(30)\qquad \left[D_1 D_2 \Box_1^2 - \frac{1}{\varkappa}\partial_t \nabla_1^2 (\varepsilon_T D_1 + \varepsilon_H D_2)\right]\theta = -\frac{1}{\varkappa}(D_1 \Box_1^2 - \alpha\beta\partial_t \nabla_1^2) Q - \frac{1}{c_1^2}\eta \partial_t \nabla_1^2 D_1 \vartheta .$$

Similarly, eliminating functions Φ and θ from Eqs. (19)—(21), we obtain the following equation

$$(31)\qquad \left[D_1 D_2 \Box_1^2 - \frac{1}{\varkappa}\partial_t \nabla_1^2 (\varepsilon_T D_1 + \varepsilon_H D_2)\right] h_3 = \frac{m\beta H_3}{\varkappa}\partial_t \nabla_1^2 Q + \frac{\beta H_3}{c_1^2}\partial_t \nabla_1^2 \vartheta .$$

To solve Eqs. (27), (30) and (31) it is a difficult and cumbersome operation. We will attempt to simplify them. First, we obtain a notable simplification if we consider the wave propagation at $Q = 0$ as an adiabatic process. In such a case, we have $\theta = -\eta\varkappa e$ and, consequently, the Eq. (19) will be reduced to the following one

$$\left(\nabla_1^2 - \frac{1}{c_1^2}\frac{\partial^2}{\partial t^2}\right)\Phi - \frac{\mu_0 H_3 h_3}{4\pi\varrho \bar{c}_1^2} = -\frac{1}{\bar{c}_1^2}\vartheta, \tag{32}$$

where $\bar{c}_1 = \frac{(\lambda_1+2\mu_1)^{\frac{1}{2}}}{\varrho}$, where μ_1, λ_1 are Lamé's constants measured in adiabatic conditions.

Eliminating the function h_3 from Eqs. (32) and (21), we obtain

$$(D_1 \Box_1^2 - \alpha\beta\partial_t \nabla_1^2)\,\Phi = -\frac{1}{c_1^2} D_1 \vartheta. \tag{33}$$

The quantity c_1 appearing in the operators as well as in the right-hand side of Eq. (33) is regarded as an adiabatic quantity. Eq. (33) describes the longitudinal magneto-elastic wave. After determining the function Φ from Eq. (33) we determine the function h_3 from Eq. (21).

A further simplification of Eq. (33) may be obtained if we assume $\alpha = a_0^2/c_1^2 \ll 1$, i.e. if we assume the primary magnetic field to be of low value $H = (0, 0, H_3)$. Then, considering α as a small parameter and expanding the function Φ into a power series, with respect to the quantities

$$\Phi = \Phi_0 + \alpha\Phi_1 + \alpha^2\Phi_2 + \ldots, \tag{34}$$

we determine the functions appearing in (34) from the following set of equations

$$\begin{aligned} &\Box_1^2 \Phi_0 = -\frac{1}{c_1^2}\vartheta, \\ &\cdots\cdots\cdots\cdots \\ &D_1 \Box_1^2 \Phi_r = \beta\partial_t \nabla_1^2 \Phi_{r-1} \\ &\cdots\cdots\cdots\cdots \end{aligned} \tag{35}$$

If the wave motion is induced by a heat source, a considerable simplification of Eq. (2.7) will be obtained by disregarding the conjugation of the field of deformation with that of temperature, i.e. assuming $\varepsilon_T = 0$. In this way we get

$$D_2 [D_1 \Box_1^2 - \frac{\varepsilon_H}{\varkappa}\partial_t \nabla_1^2]\,\Phi = -\frac{m}{\varkappa} D_1 Q, \qquad \varepsilon_H = \alpha\beta\varkappa. \tag{36}$$

If, here again, $\alpha = a_0^2/c_1^2 \ll 1$ then, applying the perturbation method and expressing Φ by the series (34), we obtain the following system of equations

$$\begin{aligned} &D_1 \Box_1^2 \Phi_0 = -\frac{m}{\varkappa} Q, \\ &\cdots\cdots\cdots\cdots \\ &D_1 \Box_1^2 \Phi_r = \beta\partial_t \nabla_1^2 \Phi_{r-1}, \\ &\cdots\cdots\cdots\cdots \end{aligned}$$

To illustrate our considerations let us quote a simple example. Assume that in an unbounded space act body forces distributed uniformly along the x_3-axis. The body forces are due to the potential ϑ and there is $\vartheta = \vartheta_0 e^{i\omega t}\,\delta(r)/2\pi r$. Then the particular integral of Eq. (27) may be presented with the help of the Hankel's integral

$$(37)\quad \Phi(r,t) = \frac{\vartheta_0\, e^{i\omega t}}{2\pi c_1^2} \int_0^\infty \frac{(\zeta^2+i\beta\omega)\,(\zeta^2+q)\,\zeta J_0(r\zeta)\,d\zeta}{\{(\zeta^2+i\beta\omega)(\zeta^2+q)(\zeta^2-\gamma^2)+q\zeta^2[\varepsilon_T(\zeta^2+1\beta\omega)+\varepsilon_H(\zeta^2+q)]\}},$$

$$q = \frac{i\omega}{\varkappa}, \qquad \sigma = \frac{\omega}{c_1}.$$

The complexity of this integral is obvious. Proceeding by approximation characterized by Eq. (31) we arrive at the following result

$$(38)\qquad \Phi(r,t) = \frac{\vartheta_0\, e^{i\omega t}}{2\pi c_1^2} \int_0^\infty \frac{(\zeta^2+i\beta\omega)\,\zeta J_0(\zeta r)\,d\zeta}{(\zeta^2+k_1^2)\,(\zeta^2+k_2^2)},$$

where

$$k_1^2+k_2^2 = \beta i\omega+q\varepsilon_H-\sigma^2, \quad k_1^2k_2^2 = -\,i\beta\omega\sigma^2.$$

The quantities k_1 and k_2 are roots of the equation

$$k^4+k^2\,[\beta i\omega+q\varepsilon_H-\sigma^2]-i\beta\omega\sigma^2 = 0,$$

they are conjugated quantities and are chosen so as to have

$$k_\beta = a_\beta+ib_\beta, \quad a_\beta>1, \quad b_\beta>1, \quad \beta = 1, 2,$$

Under this assumption the conditions of radiation in infinity will be satisfied. The function Φ may we presented in a closed form:

$$(39)\quad \Phi = \frac{\vartheta_0}{2\pi c_1^2}\,\mathrm{Re}\left\{\frac{e^{i\omega t}}{k_1^2-k_2^2}\,[(k_1^2-i\beta\omega)\,K_0(k_1 r)-(k_2^2-i\beta\omega)\,K_0(k_2 r)]\right\},$$

where $K_0(z)$ is the modified Bessel's function of the third kind.

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В. НОВАЦКИЙ, О ДВУХМЕРНОЙ ПРОБЛЕМЕ МАГНИТОТЕРМОУПРУГОСТИ.

В заметке обсуждается проблема распространения двухмерных волн, вызванных в неограниченной среде действием массовых сил, а также источников тепла. Упругая среда находится в постоянном первичном магнитном поле, так что факторы, вызывающие волновое движение приводят к образованию температурного и электромагнитного полей, сопряженных с полем деформации. Полагая, что массовые силы и источники тепла не зависимы от переменной x_3, предполагая далее, что первичное магнитное поле действует вдоль оси x_3, получается разделение волнового движения на продольные и поперечные волны.

Поперечная волна не подвергается затуханию ни дисперсии; в противоположность тому, продольные магнитотермоупругие волны подвергаются затуханию и дисперсии.

Наконец, предложен метод приближенного решения уравнения продольной волны при использовании метода пертурбаций.