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APPLIED MECHANICS

Mixed Boundary Problems of Elastodynamics

by

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In the previous paper [1], the author presented a method of solving problems of elastokinetics with mixed boundary conditions. In the present report our considerations will be extended to cover dynamic problems of the theory of elasticity.

Let us consider a simply connected elastic body B, bounded by the surface S. Let this surface consist of three smooth surfaces intersecting along the curves α and β (Fig. 1). Body forces $\vec{X}$ are acting inside the body, while the surface S_2 is subject to surface loading $\vec{q}$. Let us assume that the elastic body is perfectly fixed on the surfaces S_1 and S_3. Thus, the components u_i of the displacement vector $\vec{u}$ vanish on these surfaces *).

We assume that the surface loadings and the body forces, which are variable in time, start acting at the instant $t = 0^+$.

The components of the displacement vector, the functions $u_i(x, t)$, should satisfy the Lamé equations of motion, in terms of displacements,

$$\mu u_{i,jj} + (\lambda + \mu) u_{j,ji} + X_i = \varrho \ddot{u}_i, \quad i, j = 1, 2, 3 \tag{1}$$

the boundary conditions

$$u_i(\xi, t) = 0 \quad \text{on} \quad S_1 \text{ and } S_3, \quad \sigma_{ij}(\xi, t)\, n_j = q_i(\xi, t) \quad \text{on} \quad S_2, \tag{2}$$

and the initial conditions

$$u_i(x, 0) = 0, \quad \dot{u}_i(x, 0) = 0. \tag{3}$$

We shall employ for the motion equations (1) the abbreviated operator notation:

$$D_{ij}(u_i(x, t)) + X_i(x, t) = 0, \quad x \in B, \tag{1'}$$

where

$$D_{ij} = \mu \delta_{ij} \Box_2^2 + (\lambda + \mu)\, \partial_i \partial_j, \quad i, j = 1, 2, 3,$$

$$\Box_2^2 = \nabla^2 - \frac{1}{c_2^2} \partial_t^2, \quad c_2^2 = \frac{\mu}{\varrho}.$$

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*) We have made such an assumption for the sake of clarity. Nothing prevents, however, for assuming on S_1 and S_3 non-homogeneous boundary conditions in terms of displacements.

In what follows we shall denote by $x \equiv (x_1, x_2, x_3)$ the point of the region B, and by $\xi \equiv (\xi_1, \xi_2, \xi_3)$ the point on the surface S. The symbols μ and λ, occurring in Eqs. (1) and (1′), are the Lamé constants, while the components of the stress tensor and the components of the unit normal vector of the surface S_2, occurring in the boundary conditions (2), will be denoted by σ_{ij} and n_j, respectively.

The stresses σ_{ij} are connected with the displacements u_i by the Hooke relation

$$\sigma_{ij} = \mu (u_{i,j} + u_{j,i}) + \lambda \delta_{ij} u_{k,k}, \quad i, j, k = 1, 2, 3. \tag{4}$$

Thus, the boundary conditions (2) can be presented in the form

$$u_i(\xi, t) = 0 \quad \text{on} \quad S_1 \text{ and } S_3 \tag{2'}$$

$$q_i(\xi, t) = \mu (u_{i,j} + u_{j,i}) n_j + \lambda n_i \operatorname{div} \vec{u} = L_i(\vec{u}) \quad \text{on} \quad S_2.$$

Having formulated, in this way, the boundary-value problem, we shall now endeavour to reduce it to solving simpler problems where on two adjacent surfaces, e.g., S_1 and S_2, there occur boundary conditions of the same type. To this end we introduce what is called the "fundamental system", i.e., an elastic body of the same shape presented in Fig. 1 but perfectly fixed only on the surface S_3, and free from loadings on the surfaces S_1 and S_2.

Let us determine in the chosen "fundamental system" the Green tensor of the displacement field $G(x, x', t) = \{G_{ik}(x, x', t)\}$. We construct this tensor field of displacements as follows. Let us apply at the point $x' \in B$ of our "fundamental system" an instantaneous concentrated force parallel to the axis x_k. This force produces in the "fundamental system" the displacement vector $\vec{G}^{(k)}$ with the components G_{ik} $(i = 1, 2, 3)$. Directing, successively, this instantaneous concentrated force parallel to the axes x_1, x_2 and x_3; thus assuming for k the successive indices $k = 1, 2, 3$, we obtain nine quantities G_{ik} $i, k = 1, 2, 3)$ which yield the symmetric Green tensor of displacements, since $G_{ik} = G_{ki}$.

The Green functions G_{ik} should satisfy the motion equations,in terms of displacements,

$$D_{ij}\big(G_{ik}(x, x', t)\big) + \delta(x - x')\,\delta(t)\,\delta_{ik} = 0, \quad i, j, k = 1, 2, 3, \quad x, x' \in B, \tag{5}$$

with the boundary conditions

$$G_{ik}(\xi, x', t) = 0 \quad \text{on} \quad S_3, \quad \sigma_{ij}^{(k)}(\xi, x', t)\, n_j = 0 \quad \text{on} \quad S_1 \quad \text{and} \quad S_2 \tag{6}$$

and the initial conditions

$$G_{ik}(x, x', 0) = 0, \quad \dot{G}_{ik}(x, x', 0) = 0. \tag{7}$$

The relations (5) represent three sets of equations (for $k = 1, 2, 3$). In these equations the quantity

$$\delta(x - x')\,\delta(t) \equiv \delta(x_1 - \xi_1)\,\delta(x_1 - \xi_2)\,\delta(x_3 - \xi_3)\,\delta(t)$$

denotes the Dirac function expressing the instantaneous concentrated force which acts at the point $x' \in B$, while δ_{ik} is the Kronecker symbol. The expression

$\delta(x-x')\,\delta(t)\,\delta_{ik}$ indicates the concentrated force parallel to the axis x_k. The quantity $\sigma_{ij}^{(k)}$, occurring in the boundary conditions (6), denotes the components of the stresses produced by the instantaneous concentrated force, applied at the point x' and parallel to the x_k-axis. The boundary conditions (6) can also be represented in the form

$$G_{ik}=0 \quad \text{on} \quad S_3, \tag{8}$$

$$\mu\,(G_{ik,j}+G_{jk,i})\,n_j+\lambda n_i\,G_{jk,j}=0 \quad \text{on} \quad S_1 \quad \text{and} \quad S_2.$$

In what follows, we shall assume that the Green functions in the fundamental system have been determined, thus we shall consider these quantities as known.

In our further considerations we shall use Betti's reciprocal theorem, in the form generalized for dynamic problems,

$$\int_B (X_i u_i' - X_i' u_i)\,dV + \int_S (p_i u_i' - p_i' u_i)\,dS - \varrho \int_B (\ddot{u}_i u_i' - u_i \ddot{u}_i')\,dV = 0, \tag{9}$$

where X_i, p_i and u_i are the components of the body forces, surface loadings and displacements in the first system of forces, while X_i', p_i' and u_i' denote the respective quantities in the second system of forces.

Let us apply to Eq. (9) the Laplace transformation according to the relations

$$\bar{u}_i(x,p)=\int_0^\infty u_i(x,t)\,e^{-pt}\,dt, \quad \bar{p}_i(x,p)=\int_0^\infty p_i(x,t)\,e^{-pt}\,dt, \quad \text{etc.} \tag{10}$$

Then we obtain

$$\int_B (\bar{X}_i \bar{u}_i' - \bar{X}_i' \bar{u}_i)\,dV + \int_S (\bar{p}_i \bar{u}_i' - \bar{p}_i' \bar{u}_i)\,dS - \tag{9'}$$

$$-\varrho \int_B \{[p^2 \bar{u}_i - p u_i(x,0) - \dot{u}_i(x,0)]\,\bar{u}_i' - [p^2 \bar{u}_i' - p u_i'(x,0) - \dot{u}_i'(x,0)]\,\bar{u}_i\}\,dV = 0.$$

However, by our assumption concerning the loading of the body at the instant $t=0^+$, and the homogeneous initial conditions which follow from this assumption, $u_i(x,0)=\dot{u}_i(x,0)=u_i'(x,0)=\dot{u}_i'(x,0)=0$, Eq. (9′) takes the simplified form

$$\int_B (\bar{X}_i \bar{u}_i' - \bar{X}_i' \bar{u}_i)\,dV + \int_S (\bar{p}_i \bar{u}_i' - \bar{p}_i' \bar{u}_i)\,dS = 0. \tag{10'}$$

Let us apply the formula (10′) to the displacements u_i described by Eq. (1) and to the displacements G_{ik} satisfying Eq. (5). Substituting into (10′) the quantities:

$$\bar{u}_i(x,p)=\bar{G}_{ik}(x,x',p), \quad \bar{X}_i'=\delta(x-x')\,\delta_{ik}\cdot 1,$$

we obtain — since $\int_0^\infty \delta(t)\,e^{-pt}\,dt=1$ —

$$\int_B (\bar{X}_i(x,p)\,\bar{G}_{ik}(x,x',p) - \delta(x-x')\,\delta_{ik}\,\bar{u}(x,p))\,dV(x) + \tag{11}$$

$$+\int_{S_2} \bar{q}_i(\xi,p)\,\bar{G}_{ik}(\xi,x',p)\,dS(\xi) + \int_{S_1} \bar{R}_i(\xi)\,\bar{G}_{ik}(\xi,x',p)\,dS(\xi) = 0.$$

Here we have denoted by $\bar{R}_i(\xi, p)$, $\xi \in S_1$, the Laplace transform of the function $R_i(\xi, t)$ representing the distributed reaction on the surface S_1. Bearing in mind the relation

$$\int_B \delta(x-x')\,\delta_{ik}\,\bar{u}_i(x, p)\,dV(x) = \bar{u}_k(x', p), \tag{12}$$

we can write Eq. (11) in the form

$$\bar{u}_k(x', p) = \bar{u}_k^0(x', p) + \int_{S_1} \bar{R}_i(\xi, p)\,\bar{G}_{ik}(\xi, x', p)\,dS(\xi), \tag{13}$$

where we have denoted by $\bar{u}_k^0$ the expression

$$\bar{u}_k^0(x', p) = \int_B \bar{X}_i(x, p)\,\bar{G}_{ik}(x, x', p)\,dV(x) + \int_{S_2} \bar{q}_i(\xi, p)\,\bar{G}_{ik}(\xi, x', p)\,dS(\xi). \tag{14}$$

This is a known quantity since all functions occurring under the sign of integration are known. The quantity $\bar{u}_k^0(x', p)$ can be regarded as the Laplace transform of the function $u_k^0(x', t)$ satisfying, in the "fundamental system", the equation of motion

$$D_{ij}\left(u_j^0(x, t)\right) + X_i(x, t) = 0, \quad x \in B \tag{15}$$

with the boundary conditions

$$\sigma_{ij}^0(\xi, t)\,n_j = 0 \quad \text{on} \quad S_1, \quad \sigma_{ij}^0(\xi, t)\,n_j = q_i/\xi, t \quad \text{on} \quad S_2, \tag{16}$$
$$u_i^0(\xi, t) = 0 \quad \text{on} \quad S_3,$$

and the initial conditions

$$u_i^0(x, 0) = 0, \quad \dot{u}_i(x, 0) = 0. \tag{17}$$

Here we have denoted by σ_{ij}^0 the stresses related to the displacements u_i^0. Let us apply Eq. (10) to the displacements u_i^0 and G_{ik}. Then, we obtain

$$\int_B \left(\bar{X}_i(x, p)\,\bar{G}_{ik}(x, x', p) - \delta(x-x')\,\delta_{ik}\,\bar{u}_i^0(x, p)\right) dV(x) + \tag{18}$$
$$+ \int_{S_2} \bar{q}_i(\xi, p)\,\bar{G}_{ik}(x, x', p)\,dS(\xi) = 0,$$

whence we arrive at relation (14).

Let us now revert to formula (13), in which the transform of the displacement $\bar{u}_k(x', p)$ depends on the unknown functions $\bar{R}_i(\xi, p)$ occurring under the surface integral. In order to determine the function $\bar{R}_i(\xi, p)$, we use the first boundary condition of the set (2), thus $u_i(\xi, t) = 0$ on S_1. Hence, we pass in (13) with the point $x' \in B$ to the point ξ' on the surface S_1. In this way we arrive at the system of integral equations

$$\bar{u}_k(\xi', p) = 0 = \bar{u}_k^0(\xi', p) + \int_{S_1} \bar{R}_i(\xi, p)\,\bar{G}_{ik}(\xi, \xi', p)\,dS(\xi), \qquad i, k = 1, 2, 3. \tag{19}$$

Having determined from this system of integral equations the function $\bar{R}_i(\xi, p)$, we can obtain the displacements $\bar{u}_k(x', p)$ by the formula (13). Applying to (13) the inverse Laplace transformation, we have

$$(20) \qquad u_k(x', t) = u_k^0(x, t) + \int_0^t d\tau \int_{S_1} R(\xi, \tau)\, G_{ik}(\xi, \xi', t-\tau)\, dS(\xi).$$

Formulae (13) and Eqs. (19) can also be represented in another, very convenient, form. To do this we introduce a new displacement tensor $U_{ik}(x, \xi, t)$ which satisfies in the "fundamental system" the equations of motion

$$(21) \qquad D_{ij}\left(U_{ik}(x, \xi, t)\right) = 0, \quad x \in B, \quad \xi \in S_1, \quad i, j, k = 1, 2, 3,$$

with the boundary conditions

$$(22) \qquad \hat{\sigma}_{ij}^{(k)}(\xi', \xi, t) = \delta(\xi' - \xi)\, \delta_{ik}\, \delta(t) \quad \text{on} \quad S_1,$$

$$\hat{\sigma}_{ij}^{(k)}(\xi', \xi, t) = 0 \quad \text{on} \quad S_2, \quad u_{ik}(\xi', \xi, t) = 0 \quad \text{on} \quad S_3,$$

and the initial conditions

$$(23) \qquad U_{ik}(x, \xi, 0) = 0, \qquad \dot{U}_{ik}(x, \xi, 0) = 0.$$

Here $\hat{\sigma}_{ij}^{(k)}(x, \xi, t)$ denote the stresses related to the functions $U_{ik}(x, \xi, t)$.

The first of the initial conditions (22) denotes that at the point $\xi \in S_1$ there acts an instantaneous concentrated force parallel to the axis x_k $(k = 1, 2, 3)$.

Solving Eq. (21) with the boundary conditions (22) and initial conditions (23), we obtain nine functions U_{ik} which reduce to six functions, owing to the symmetry of the displacement tensor $(U_{ik} = U_{ki})$. Let us apply Betti's theorem (10) to the functions G_{ik} and U_{ik}. Then we arrive at the relation:

$$-\int_B \delta(x - x')\, \delta_{ik}\, \bar{U}_{ik}(x, \xi, p)\, dV(x) + \int_{S_1} \delta(\xi' - \xi)\, \delta_{ik}\, \bar{G}_{ik}(\xi', x', p)\, dS(\xi') = 0,$$

whence it follows

$$(24) \qquad \bar{U}_{ik}(x', \xi, p) = \bar{G}_{ik}(\xi, x', p).$$

In particular, passing with the point $x' \in B$ to the point $\xi' \in S_1$, we obtain

$$(25) \qquad \bar{U}_{ik}(\xi', \xi, p) = \bar{G}_{ik}(\xi, \xi', p).$$

Thus, Eq. (20) can be represented in the form

$$(26) \qquad u_k(x', t) = u_k^0(x', t) + \int_0^t d\tau \int_{S_1} R(\xi, \tau)\, U_{ik}(x', \xi, t-\tau)\, dS(\xi).$$

In a similar way, applying to the integral Eqs. (19) the inverse Laplace transformation, we can represent them in the form

$$(27) \qquad u_k(\xi, t) = 0 = u_k^0(\xi', t) + \int_0^t d\tau \int_{S_1} R(\xi, \tau)\, U_{ik}(\xi' \xi, t-\tau)\, dS(\xi).$$

The procedure presented above can be extended to solve the problem of an elastic body with an arbitrary number of surface supports $S_1, S_3, S_5 \ldots$. Here we use a "fundamental system" in which the body has only one supported surface.

Repeating these proceedings for a larger number of surface supports, we can obtain, from the conditions of vanishing of the redundant displacements on these surface supports, a system of integral equations. The number of the unknown functions representing the surface supports will be equal to the number of the equations.

Let us now return to Eqs. (27), and observe that the surface S_1 may be constructed so as to carry only surface reactions in the normal direction. In this case there occur additional linear relations between the components R_i $(i = 1, 2, 3)$ allowing to reduce the number of the unknown functions to one redundancy, and the system of relations (27) reduces to one equation. Let us also observe that our considerations include such cases where the surface S_1 degenerates into a curve.

Let us, finally, examine the case where the external loadings are harmonic functions of time

(28) $$X(x, t) = X^*(x)\, e^{i\omega t}, \quad q(x, t) = q^*(x)\, e^{i\omega t}.$$

The displacements $u_i(x, t)$ and $G_{ik}(x, x', t)$ are also harmonic functions of the time variable

(29) $$u_i(x, t) = u_i^*(x, \omega)\, e^{i\omega t}, \quad G_{ik}(x, x', t) = G_{ik}^*(x, x', \omega)\, e^{i\omega t}.$$

The equations of motion (1′) and (5) assume the form

(30) $$D_{ij}^*\left(u_i^*(x, \omega)\right) + X_i^*(x) = 0,$$

(31) $$D_{ij}^*\left(G_{ik}^*(x, x', \omega)\right) + \delta(x - x')\, \delta_{ik} = 0,$$

respectively, where

$$D_{ij}^* = \mu\left(\nabla^2 + \frac{\omega^2}{c_2^2}\right)\delta_{ij} + (\lambda + \mu)\, \partial_i\, \partial_j,$$

and the appropriate boundary conditions also take the corresponding form.

In the case of harmonic vibrations the reciprocal theorem (9) yields

(32) $$\int_B (X_i^* u_i^{*\prime} - X_i^{*\prime} u_i^*)\, dV + \int_S (p_i^* u_i^{*\prime} - p_i^{*\prime} u_i^*)\, dS.$$

Applying Eq. (32) to the displacement amplitudes u_i^* and G_{ik}^* and to the appropriate stresses and boundary conditions, we obtain

(33) $$u_k^*(x', \omega) = u_k^{*0}(x'\, \omega) + \int_{S_1} R^*(\xi)\, G_{ik}^*(\xi, x', \omega)\, dS(\xi).$$

Passing with the point $x' \in B$ to the point $\xi' \in S_1$, we arrive at the system of three integral equations

(34) $$u_k^{*0}(\xi', \omega) + \int_{S_1} R_i^*(\xi)\, G_{ik}^*(\xi, \xi', \omega)\, dS(\xi) = 0, \quad i, k = 1, 2, 3,$$

whence we can determine the unknown functions $R_i^*(\xi)$, and then we obtain the required amplitudes of the displacements by the formula (33).

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В. НОВАЦКИЙ, **О ЗАДАЧАХ УПРУГО-ДИНАМИКИ СО СМЕШАННЫМИ КРАЕВЫМИ УСЛОВИЯМИ**

В одной из предыдущих работ [2] автор представил метод решения задач упруго-динамики со смешанными краевыми условиями. В настоящей работе представлен аналогичный метод для динамических задач теории упругости. Используя функции перемещений Грина а также опираясь на теорему взаимности Бэтти, проблема сводится к решению системы интегральных уравнений первого рода.