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THERMODYNAMICS

Mixed Boundary Problems in Heat Conduction

by

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In paper [1] the author considered the problem of mixed boundary conditions in a solid body in the case of steady heat flow. The method employed in that paper will here be extended to non-homogeneous mixed boundary conditions and to the problem of unsteady heat flow. However, the formulation of the method in the present paper differs from that given in [1].

Let us consider the simply connected region B bounded by the surface S, and let this surface consist of three smooth surfaces S_1, S_2 and S_3 intersecting along α and β (Fig. 1). We assume that inside the region B there act heat sources $W(P, t)$, $P \in B$, varying in time, and that the surfaces S_i $(i = 1, 2, 3)$ are subject to heating. The appropriate temperature field $T(P, t)$ is described by the heat conduction equation

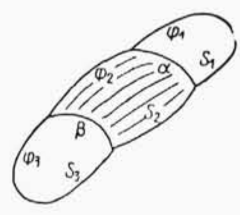

Fig. 1

$$\varkappa \nabla^2 T(P, t) - \dot{T}(P, t) = -M(P, t), \tag{1}$$
$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2},$$

where $\varkappa = \dfrac{\lambda}{\varrho c}$ is a material constant, λ—the coefficient of heat conductivity, ϱ—the density and c—the specific heat. The function $M(P, t)$ represents the intensity of the heat sources, thus $M(P, t) = \dfrac{1}{\varrho c} W(P, t)$, where $W(P, t)$ is the amount of heat generated in the unit volume per unit time. By $\dot{T}$ we have denoted the time derivative of the temperature, $\dot{T} = \partial T/\partial t$.

We assume that $T(P, t)$ satisfies the initial condition

$$T(P, 0) = f(P), \quad P \in B \tag{2}$$

and the following boundary conditions

$$
(3)\qquad
\begin{aligned}
&T(R_1, t) = \varphi_1(R_1, t) && \text{on the surface} \quad S_1, \quad R_1 \in S_1,\\
&\frac{\partial T(R_2, t)}{\partial n} = \psi_2(R_2, t) && \text{,,} \qquad \text{,,} \qquad S_2, \quad R_2 \in S_2,\\
&T(R_3, t) = \varphi_3(R_3, t) && \text{,,} \qquad \text{,,} \qquad S_3, \quad R_3 \in S_3.
\end{aligned}
$$

We introduce the general notation

$$
T(R_i, t) = \varphi_i(R_i, t), \qquad \frac{\partial T(R_i, t)}{\partial n} = \psi_i(R_i, t), \qquad i = 1, 2, 3.
$$

Observe that the functions $\varphi_1, \psi_2, \varphi_3$ given on the surfaces S_1, S_2, S_3 are known, while ψ_1, φ_2 and ψ_3 are unknown functions on the same surfaces. One of these functions, $\psi_1(R_1, t) = \dfrac{\partial T(R_1, t)}{\partial n}$ on the surface S_1, will be assumed as the unknown function of the problem considered.

Let us determine the Green function $G(P, Q, t)$ which satisfies the heat conduction equation

$$
(4)\qquad \varkappa \nabla^2 G(P, Q, t) - \dot{G}(P, Q, t) = -\delta(P - Q)\,\delta(t), \qquad P, Q \in B,
$$

with the homogeneous initial condition

$$
(5)\qquad G(P, Q, 0) = 0
$$

and the homogeneous boundary conditions

$$
(6)\qquad
\begin{aligned}
&\frac{\partial G(R_1, Q, t)}{\partial n} = 0 && \text{on the surface} \quad S_1, \quad R_1 \in S_1,\\
&\frac{\partial G(R_2, Q, t)}{\partial n} = 0 && \text{,,} \qquad \text{,,} \qquad S_2, \quad R_2 \in S_2,\\
&G(R_3, Q, t) = 0 && \text{,,} \qquad \text{,,} \qquad S_3, \quad R_3 \in S_3.
\end{aligned}
$$

We determine the function G in what is called the "fundamental system", where the surfaces S_1 and S_2 are thermally insulated, and the temperature on the surface S_3 is zero. The Dirac function occurring on the right-hand side of Eq. (4) expresses the instantaneous concentrated heat source acting at the point Q. The integral from the right-hand side of Eq. (4), over the volume B and the time t is equal to -1.

Let us perform on Eqs. (1) and the boundary conditions (3) the one-side Laplace transformation given by the formula

$$
(7)\qquad \bar{F}(P, p) = \int_0^\infty F(P, t)\, e^{-pt}\, dt, \qquad p > 0.
$$

Here we assume that the action of the heat sources and the heating on the surface S begin at the instant $t = 0+$.

Then we obtain Eq. (1) in the transformed form

$$\varkappa\nabla^2 \overline{T}(P,p) - [p\overline{T}(P,p) - T(P,0)] = -\overline{M}(P,p), \quad T(P,0) = f(P). \tag{8}$$

The Laplace transformation performed on the boundary conditions (3) yields

$$\begin{cases} \overline{T}(R_1,p) = \overline{\varphi}_1(R_1,p), & R_1 \in S_1, \\ \dfrac{\partial \overline{T}(R_2,p)}{\partial n} = \overline{\psi}_2(R_2,p), & R_2 \in S_2, \\ \overline{T}(R_3,p) = \overline{\varphi}_3(R_3,p), & R_3 \in S_3. \end{cases} \tag{9}$$

Proceeding similarly with Eq. (4) and the boundary conditions (6), we obtain:

$$\varkappa\nabla^2 \overline{G}(P,Q,p) - p\overline{G}(P,Q,p) = -\delta(P-Q), \tag{10}$$

$$\begin{cases} \dfrac{\partial \overline{G}(R_1,Q,p)}{\partial n} = 0, & R_1 \in S_1, \\ \dfrac{\partial \overline{G}(R_2,Q,p)}{\partial n} = 0, & R_2 \in S_2, \\ \overline{G}(R_3,Q,p) = 0, & R_3 \in S_3, \end{cases} \tag{11}$$

Let us use the Green formula

$$\iiint\limits_B (\overline{G}\nabla^2\overline{T} - \overline{T}\nabla^2\overline{G})\,dB = \iint\limits_S \left(\overline{G}\frac{\partial\overline{T}}{\partial n} - \overline{T}\frac{\partial\overline{G}}{\partial n}\right)dS, \tag{12}$$

and introduce into (12) the appropriate quantities from the formulae (8)—(11).

Bearing in mind that

$$\iiint\limits_B \overline{T}(P,p)\,\delta(P-Q)\,dB_P = \overline{T}(Q,p),$$

we obtain from (12)

$$\begin{aligned} \overline{T}(Q,p) = {} & \iiint\limits_B f(P)\,\overline{G}(P,Q,p)\,dB_P + \iiint\limits_B \overline{G}(P,Q,p)\,\overline{M}(P,p)\,dB_P + \\ & + \varkappa\iint\limits_{S_1} \overline{\psi}_1(R_1,p)\,\overline{G}(R_1,Q,p)\,dS_{R_1} + \varkappa\iint\limits_{S_2} \overline{\psi}_2(R_2,p)\,\overline{G}(R_2,Q,p)\,dS_{R_2} + \\ & - \varkappa\iint\limits_{S_3} \overline{\varphi}_3(R_3,p)\frac{\partial\overline{G}(R_3,Q,p)}{\partial n}\,dS_{R_3}. \end{aligned} \tag{13}$$

Observe that the integrals occurring on the right-hand side of (13), except for $\iint\limits_{S_1} \overline{\psi}_1\,\overline{G}dS_{R_1}$, can be determined, since the functions ψ_2, φ_3, f and M are given, and G has been obtained from (4). Eq. (13) can be represented in the form

$$\overline{T}(Q,p) = \overline{T}_0(Q,p) + \varkappa\iint\limits_{S_1} \overline{\psi}_1(R_1,p)\,\overline{G}(R_1,Q,p)\,dS_{R_1}, \tag{14}$$

since the temperature $T_0(Q, t)$ can be regarded as the solution of the equation

$$(15) \qquad \varkappa \nabla^2 T_0(P, t) - \dot{T}(P, t) = -M(P, t), \quad P \in B.$$

with the initial condition $T(P, 0) = f(P)$ and the boundary conditions

$$(16) \qquad \frac{\partial T_0(R_1, t)}{\partial n} = 0, \quad R_1 \in S_1; \qquad \frac{\partial T_0(R_2, t)}{\partial n} = \psi_2(R_2, t), \quad R_2 \in S_2,$$

$$T_0(R_3, t) = \varphi_3(R_3, t), \quad R_3 \in S_3,$$

for the fundamental system described above.

Performing on Eq. (15) and the boundary conditions (16) the one-side Laplace transformation, and using the Green formula

$$(17) \qquad \iiint_B (\bar{G}\nabla^2 \bar{T}_0 - \bar{T}_0 \nabla^2 \bar{G})\, dB = \iint_S \left(\bar{G} \frac{\partial \bar{T}_0}{\partial n} - \bar{T}_0 \frac{\partial \bar{G}}{\partial n} \right) dS,$$

we arrive, after simple rearrangements, at the relation

$$(18) \qquad \bar{T}_0(Q, p) = \iiint_B f(P)\, \bar{G}(P, Q, p)\, dB_P + \iiint_B \bar{M}(P, p)\, \bar{G}(P, Q, p)\, dB_P +$$
$$+ \varkappa \iint_{S_2} \bar{\psi}_2(R_2, p)\, \bar{G}(R_2, Q, p)\, dS_{R_2} - \varkappa \iint_{S_3} \bar{\varphi}_3(R_3, p) \frac{\partial \bar{G}(R_3, Q, p)}{\partial n} dS_{R_3}.$$

Performing on the expression (18) the inverse Laplace transformation, we obtain

$$(19) \qquad T_0(Q, t) = \iiint_B f(P)\, G(P, Q, t)\, dB_P +$$
$$+ \int_0^t d\tau \iiint_B M(P, \tau)\, G(P, Q, t-\tau)\, dB_P +$$
$$+ \varkappa \int_0^t d\tau \iint_{S_2} \psi_2(R_2, \tau)\, G(R_2, Q, t-\tau)\, dS_{R_2} -$$
$$- \varkappa \int_0^t d\tau \iint_{S_3} \varphi(R_3, \tau) \frac{\partial G(R_3, Q, t-\tau)}{\partial n} dS_{R_3}.$$

Consider now the expression (14) and perform on it the inverse Laplace transformation to obtain

$$(20) \qquad T(Q, t) = T_0(Q, t) + \varkappa \int_0^t d\tau \iint_{S_1} \psi_1(R_1, \tau)\, G(R_1, Q, t-\tau)\, dS_{R_1}.$$

In this formula an unknown function $\psi, (R_1, \tau)$ occurs on the S_1 surface.

Let us now pass with the point $Q \in B$ to the point $R_1' \in S_1$, on the surface S_1. Bearing in mind that $T(R_1', t) = \varphi_1(R_1', t)$ is the boundary condition on S_1, we obtain from relation (20)

$$(21) \qquad \varphi_1(R_1', t) = T_0(R_1', t) + \varkappa \int_0^t d\tau \iint_{S_1} \psi_1(R_1, \tau)\, G(R_1, R_1', t-\tau)\, dS_{R_1}.$$

The only unknown function in this integral equation is $\psi_1(R_1, t)$. Having determined this function, we obtain the temperature $T(Q, t)$ from Eq. (20).

Let us observe that the Green function $G(P, Q, t)$ can be expressed in terms of another Green function which is defined as follows. Let us consider the function $K(P, R_1', t)$ satisfying the heat conduction equation

$$\varkappa \nabla^2 K(P, R_1', t) - \dot{K}(P, R_1', t) = 0 \tag{22}$$

with the initial condition $K(P, R_1', 0) = 0$ and the boundary conditions

$$\begin{cases} \dfrac{\partial K(R_1, R_1', t)}{\partial n} = \delta(R_1 - R_1')\,\delta(t) \quad R_1, R_1' \in S_1, \\ \dfrac{\partial K(R_2, R_1', t)}{\partial n} = 0, \quad R_2 \in S_2, \\ K(R_3, R_1', t) = 0, \quad R_3 \in S_3. \end{cases} \tag{23}$$

The first condition of (23) expresses the fact that there exists in the thermal insulation a concentrated and instantaneous gap at the point $R_1' \in S_1$, thus

$$\int_0^t d\tau \iint_{S_1} \delta(R_1 - R_1')\,\delta(\tau)\, dS_{R_1} = 1.$$

Applying now the Green formula (12) to the functions $\overline{K}$ and $\overline{G}$, we arrive at the following relation:

$$\iiint_B \overline{K}(P, R_1', p)\,\delta(P - Q)\, dB_P = \varkappa \iint_{S_1} \overline{G}(R_1, Q, p)\,\delta(R_1 - R_1')\, dS_{R_1}, \tag{24}$$

whence we obtain

$$K(Q, R_1', t) = \varkappa G(R_1', Q, t). \tag{25}$$

Introducing the latter relation into Eq. (21), we have

$$\varphi_1(R_1', t) = T_0(R_1', t) + \int_0^t d\tau \iint_{S_1} \psi_1(R_1, \tau)\, K(R_1, R_1', t - \tau)\, dS_{R_1}. \tag{26}$$

We shall now present another version of the method of solving Eq. (1) with the mixed boundary conditions. As the unknown function of the problem we choose the function $\varphi_2(R_2, t) = T(R_2, t)$, on the surface S_2.

The problem will be solved by means of the Green function $G^*(P, Q, t)$ satisfying the heat conduction equation

$$\varkappa \nabla^2 G^*(P, Q, t) - \dot{G}^*(P, Q, t) = -\delta(P - Q)\,\delta(t) \tag{27}$$

with the initial condition $G^*(P, Q, 0) = 0$ and the boundary conditions

$$G^*(R_1, Q, t) = 0 \quad R_1 \in S_1; \quad G^*(R_2, Q, t) = 0, \quad R_2 \in S_2, \tag{28}$$
$$G^*(R_3, Q, t) = 0, \quad R_3 \in S_3.$$

The fundamental system is here represented by the region B with zero-temperatures on the surfaces S_1 and S_2. Employing now the Green formula (12) for the functions $\bar{T}$ and $\bar{G}^*$, and taking into account the boundary conditions (3) and (28), we obtain

$$(29)\qquad \bar{T}(Q,p) = \iiint_B f(P)\,\bar{G}^*(P,Q,p)\,dB_P + \iiint_B \bar{M}(P,p)\,\bar{G}^*(P,Q,p)\,dB_P + \\ - \varkappa \iint_{S_1} \bar{\varphi}_1(R_1,p)\frac{\partial \bar{G}^*(R_1,Q,p)}{\partial n}\,dS_{R_1} - \varkappa \iint_{S_2} \bar{\varphi}_2(R_2,p)\frac{\partial G^*(R_2,Q,p)}{\partial n}\,dS_{R_2} + \\ - \varkappa \iint_{S_3} \bar{\varphi}_3(R_3,p)\frac{\partial \bar{G}^*(R_3,Q,p)}{\partial n}\,dS_{R_3},$$

which can also be written in the form

$$(30)\qquad \bar{T}(Q,p) = \bar{T}_0^*(Q,p) - \varkappa \iint_{S_2} \bar{\varphi}_2(R_2,p)\frac{\partial \bar{G}^*(R_2,Q,p)}{\partial n}\,dS_{R_2}.$$

Observe that the quantities occurring in the function $T^*(Q,p)$ are known. The function $T_0^*(Q,t)$ can be determined by solving the heat conduction equation

$$(31)\qquad \varkappa\nabla^2 T_0^*(P,t) - \dot{T}_0^*(P,t) = -M(P,t), \quad P \in B,$$

with the initial condition $T_0^*(P,0) = f(P)$ and the boundary conditions

$$(32)\qquad T_0^*(R_1,t) = \varphi_1(R_1,t) \quad R_1 \in S_1; \\ T_0^*(R_2,t) = 0, \quad R_2 \in S_2; \quad T_0^*(R_3,t) = \varphi_3(R_3,t), \quad R_3 \in S_3.$$

Performing on Eqs. (27), (28) and (32) the one-side Laplace transformation, we obtain from the Green formula a relation which, being subject to the inverse transformation, takes the form:

$$(33)\qquad T_0^*(Q,t) = \iiint_B f(P)\,G^*(P,Q,t)\,dB_P + \\ + \int_0^t d\tau \iiint_B M(P,\tau)\,G^*(P,Q,t-\tau)\,dB_P + \\ - \varkappa \int_0^t d\tau \iint_{S_1} \varphi_1(R_1,\tau)\frac{\partial G^*(R_1,Q,t-\tau)}{\partial n}\,dS_{R_1} + \\ - \varkappa \int_0^t d\tau \iint_{S_3} \varphi_3(R_3,\tau)\frac{\partial G^*(R_3,Q,t-\tau)}{\partial n}\,dS_{R_3}.$$

Performing the inverse transformation on Eq. (30), we have

$$(34)\qquad T(Q,t) = T_0^*(Q,t) - \varkappa \int_0^t d\tau \iint_{S_2} \varphi_2(R_2,\tau)\frac{\partial G^*(R_2,Q,t-\tau)}{\partial n}\,dS_{R_2}.$$

Now, on the surface S_2 the boundary condition

$$(35)\qquad \frac{\partial T(R_2,t)}{\partial n} = \psi_2(R_2,t); \quad R_2 \in S_2.$$

is given. Let us use this condition passing from the point $Q \in B$ to the current point $R_2' \in S_2$, and perform the operation $\partial/\partial n'$. In this way we obtain from Eq. (34)

$$(36) \qquad \psi_2(R_2', t) = \frac{\partial T_0^*(R_2', t)}{\partial n'} - \varkappa \int_0^t d\tau \iint_{S_2} \varphi_2(R_2, \tau) \frac{\partial^2 G^*(R_2, R_2', t-\tau)}{\partial n' \, \partial n} dS_{R_2}.$$

Here we have denoted by $\partial/\partial n'$ the normal derivative at the point $R_2' \in S_2$. The unknown function $\varphi_2(R_2, t)$ will now be determined from the integral equation (36). The temperature $T(Q, t)$ can be obtained from the formula (34).

Let us further consider the Green function $K^*(P, R_2', t)$ which satisfies, in the fundamental system considered, the heat conduction equation

$$(37) \qquad \varkappa \nabla^2 K^*(P, R_2', t) - \dot{K}^*(P, R_2', t) = 0, \quad P \in B,$$

with the initial condition $K^*(P, R_2', 0) = 0$ and the boundary conditions

$$(38) \qquad K^*(R_1, R_2', t) = 0 \quad R_1 \in S_1; \quad K^*(R_2, R_2', t) = \delta(R_2 - R_2')\,\delta(t) \quad R_2, R_2' \in S_2$$
$$K^*(R_3, R_2', t) = 0, \quad R_3 \in S_3,$$

where

$$\int_0^t d\tau \iint_{S_2} \delta(R_2 - R_2')\, dS_{R_2} = 1.$$

Performing the one-side Laplace transformation on the functions G^* and K^*, and using the Green formula for the transforms, we arrive at the relation

$$(39) \qquad \iiint_B \bar{K}^*(P, R_2', p)\, \delta(P - Q)\, dB_P = - \varkappa \iint_{S_2} \delta(R_2 - R_2') \frac{\partial G^*(R_2, Q, p)}{\partial n} dS_{R_2},$$

whence, by the inverse Laplace transformation, we obtain

$$(40) \qquad K^*(Q, R_2', t) = -\varkappa \frac{\partial G^*(R_2', Q, t)}{\partial n}.$$

By virtue of (40) we can represent Eq. (36) in the form

$$(41) \qquad \psi_2(R_2', t) = \frac{\partial T_0^*(R_2', t)}{\partial n'} + \int_0^t d\tau \iint_{S_2} \varphi_2(R_2, \tau) \frac{\partial K^*(R_2, R_2', t-\tau)}{\partial n'} dS_{R_2}.$$

However, in solving the problems considered here, preference should be given to the first version of the method, since the kernels G^* and K^* exhibit stronger singularities than the functions G and K.

In the case of steady heat flow the problem simplifies considerably. The temperature and the Green functions are now independent of time. Thus, in the first version, we obtain for the temperature the following formula:

$$(42) \qquad T(Q) = T_0(Q) + \varkappa \iint_{S_1} \psi_1(R_1)\, G(R_1, Q)\, dS_{R_1},$$

where

$$(43)\qquad T_0(Q) = \iiint_B M(P)\, G(P, Q)\, dB_P + \varkappa \iint_{S_2} \psi_2(R_2)\, G(R_2, Q)\, dS_{R_2} + \\ - \varkappa \iiint_B \varphi_3(R_3) \frac{\partial G(R_3, Q)}{\partial n}\, dS_{R_3}.$$

and the unknown function $\psi_1(R_1)$ is determined from the integral equation

$$(44)\qquad \varphi_1(R_1') = T_0(R_1') + \varkappa \iint_{S_1} \psi_1(R_1)\, G(R_1, R_1')\, dS_{R_1}.$$

The here presented method of solving the problems of heat conduction with mixed boundary conditions can be extended also to problems with discontinuous boundary conditions. Let us consider, as an example of such a problem, the semi-infinite cylinder insulated on the portion S_2 of its end cross-section. Let on S_1 be prescribed the temperature $\theta(t)$, on S_3 the temperature $T = 0$, and on S_2—thermal insulation $\left(\frac{\partial T}{\partial n} = 0\right)$. In this case we choose the fundamental system in the form of an infinite cylinder with the boundary conditions $T = 0$ on S_3, and $\partial T/\partial z = 0$ on S_1 and S_2. In this system we determine the temperature T_0 and the Green function K. The unknown function $\psi_1 = \partial T/\partial z$ will be determined from the integral equation (26) which assumes, in the case considered, the form

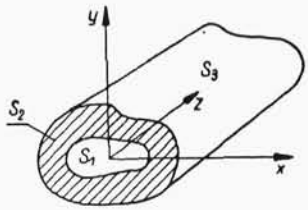

Fig. 2

$$\theta(t) = T_0(x, y, 0, t) + \int_0^t d\tau \iint_{S_1} \psi(\xi, \eta, \tau)\, K(x, y, 0; \xi, \eta, 0, t-\tau)\, d\xi\, d\eta\ .$$

A more detailed discussion of the method presented in this paper, supplemented with examples, will be given in Archiwum Mechaniki Stosowanej.

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В. НОВАЦКИЙ, СМЕШАННАЯ КРАЕВАЯ ЗАДАЧА ДЛЯ УРАВНЕНИЯ ТЕПЛОПРОВОДНОСТИ

В настоящей работе приводятся, при применении метода, представленного в [1], два варианта решения смешанной краевой задачи для уравнения теплопроводности, описывающего нестационарный тепловой поток.

Решение задачи в обоих вариантах приводит к интегральному уравнению первого порядка.