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Application of Difference Equations in Structural Mechanics. II

by

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The generalization of Galerkin's orthogonalization method to finite difference equations

Let us consider a linear non-homogeneous difference equation

$$(1.1)\qquad L_{xy}(w_{xy}) = U_{xy}; \qquad x = 0, 1, 2, ..., n; \qquad y = 0, 1, 2, ..., m,$$

where L_{xy} is the linear difference operator, w_{xy} — the sought for function and U_{xy} — a known function. The Eq. (1.1) is a partial difference equation of two variables x and y with constant or variable coefficients. Equations of such a type appear in the theory of gridworks, plane frame systems and in the theory of plates and discs. Let us assume that the function w_{xy} satisfies homogeneous boundary conditions.

We seek for an approximate solution of (1.1) in the form of the double series

$$(1.2)\quad w^*_{xy} = \sum_{\nu\mu}^{j,f} A_{\nu\mu}\,\varphi^{\nu\mu}_{xy}, \qquad \nu = 0, 1, 2, ..., j, \qquad \mu = 0, 1, 2, ..., f, \qquad j < n, f < m.$$

In the expression (1.2) the quantities $A_{\nu\mu}$ are constant quantities sought for, and $\varphi^{\nu\mu}_{xy}$ are linearly mutually independent functions chosen beforehand and satisfying at the edge the same boundary conditions as the function w_{xy}. The functions $\varphi^{\nu\mu}_{xy}$ show the first $(j+f)$ functions of the set $[\varphi^{\nu\mu}_{xy}]$ $\nu = 0, 1, 2, ..., n$, $\mu = 0, 1, 2, ..., m$ complete and orthogonal in the region under consideration. The functions $\varphi^{\nu\mu}_{xy}$ do not satisfy Eq. (1.1).

The requirement of the Eq. (1.1) to be satisfied by the series (1.2) is identical with the requirement of orthogonality of the function $L_{xy}(w^*_{xy}) - U_{xy}$ to every function $\varphi^{\nu\mu}_{xy}$ ($\nu = 0, 1, 2, ..., n$; $\mu = 0, 1, 2, ..., m$). However, having only $j+f$ function $\varphi^{\nu\mu}_{xy}$ and the same number of the constants $A_{\nu\mu}$, we can satisfy only $j+f$ orthogonality conditions. These conditions have the form

$$\sum_{x,y}^{n,m} [L_{xy}(w^*_{xy}) - U_{xy}]\,\varphi^{ik}_{xy} = 0,$$

or

$$(1.3)\qquad \sum_{x,y}^{n,m}\Big[L_{xy}\Big(\sum_{\nu,\mu}^{j,f} A_{\nu\mu}\,\varphi_{xy}^{\nu\mu}\Big)-U_{xy}\Big]\varphi_{xy}^{ik}=0.$$

Changing the summation order, Eq. (1.3) becomes

$$(1.4)\qquad \sum_{\nu,\mu}^{j,f} A_{\nu\mu}\,b_{\nu\mu ik}=q_{ik},\qquad i=0,1,2,\dots,j,\qquad k=0,1,2,\dots,f,$$

where

$$(1.5)\qquad b_{\nu\mu ik}=\sum_{x,y}^{n,m}\varphi_{xy}^{ik}\,L_{xy}(\varphi_{xy}^{\nu\mu}),\qquad q_{ik}=\sum_{x,y}^{n,m}U_{xy}\,\varphi_{xy}^{ik}.$$

Thus a system of $j+f$ linear equations is obtained for determination of the same number of unknown coefficients $A_{\nu\mu}$. Knowing the quantities $A_{\nu\mu}$, we find from relation (1.2) the function w_{xy}^*, constituting an approximate solution of the difference equation (1.1).

The procedure just described constitutes a full analogy to Galerkin's orthogonalization method for partial differential equations.

If the solution of (1.1) is sought for in the form of a series:

$$(1.6)\qquad w_{xy}=\sum_{\nu,\mu}^{n,m} A_{\nu\mu}\,\varphi_{xy}^{\nu\mu},$$

where $[\varphi_{xy}^{\nu\mu}]$ is a complete set of eigenfunctions of the following equation

$$(1.7)\qquad L_{xy}(\varphi_{xy}^{\nu\mu})=\sigma_{\nu\mu}\,\varphi_{xy}^{\nu\mu},\qquad \nu=0,1,2,\dots,n;\qquad \mu=0,1,2,\dots,m,$$

then, applying the above procedure, we obtain the accurate solution.

The equations (1.4) take the form:

$$(1.8)\qquad \sum_{\nu,\mu}^{n,m} A_{\nu\mu}\,b_{\nu\mu ik}=q_{ik},\qquad i=0,1,2,\dots,n,\qquad k=0,1,2,\dots,m,$$

where

$$q_{ik}=\sum_{x,y}^{n,m}U_{xy}\,\varphi_{xy}^{ik},$$

and

$$(1.9)\qquad b_{\nu\mu ik}=\sigma_{\nu\mu}\sum_{x,y}^{n,m}\varphi_{xy}^{ik}\,\varphi_{xy}^{\nu\mu}=\sigma_{\nu\mu}\,\delta_{i\nu}\,\delta_{k\mu},$$

in view of the orthogonality of the functions $\varphi_{xy}^{\nu\mu}$. In the expression (1.9) $\delta_{i\nu}$ and $\delta_{k\mu}$ are Kronecker's deltas. Inserting (1.9) into (1.8), we obtain

$$(1.10)\qquad A_{ik}\,\sigma_{ik}=q_{ik},\qquad i=0,1,2,\dots,n,\qquad k=0,1,2,\dots,m,$$

and therefore, in agreement with (1.6), we have

$$(1.11)\qquad w_{xy}=\sum_{\nu,\mu}^{n,m}\frac{q_{\nu\mu}}{\sigma_{\nu\mu}}\,\varphi_{xy}^{\nu\mu}.$$

This is an accurate solution of the difference equation (1.1).

The procedure just described may be extended to systems of partial difference equations. Such systems appear, for instance, for multi-storey frames and plane gridworks.

Let us consider the system of difference equations

$$(1.12) \qquad \begin{cases} L_{xy}(w_{xy}) + D_{xy}(f_{xy}) = U_{xy} \\ G_{xy}(w_{xy}) + H_{xy}(f_{xy}) = V_{xy}, \qquad x = 0, 1, 2, \ldots, n, \qquad y = 0, 1, 2, \ldots, m, \end{cases}$$

where w_{xy}, f_{xy} are unknown functions, L_{xy}, D_{xy}, G_{xy} H_{xy}, linear difference operators and U_{xy}, V_{xy} — known functions in the region considered. By introducing two new functions ξ_{xy}, η_{xy} connected with the functions w_{xy}, f_{xy} by means of the relations

$$(1.13) \qquad w_{xy} = H_{xy}(\xi_{xy}) - D_{xy}(\eta_{xy}),$$

$$(1.14) \qquad f_{xy} = L_{xy}(\xi_{xy}) - G_{xy}(\eta_{xy})$$

the system of equations (1.12) is reduced to two equations containing the functions ξ_{xy} and η_{xy} only.

$$(1.15) \qquad (H_{xy} L_{xy} - G_{xy} D_{xy})\, \xi_{xy} = U_{xy},$$

$$(1.16) \qquad (H_{xy} L_{xy} - G_{xy} D_{xy})\, \eta_{xy} = V_{xy},$$

or

$$(1.15') \qquad F_{xy}(\xi_{xy}) = U_{xy},$$

$$(1.16') \qquad F_{xy}(\eta_{xy}) = V_{xy},$$

where

$$F_{xy} = H_{xy} L_{xy} - G_{xy} D_{xy}.$$

The approximate solution of (1.15′) and (1.16′) will be sought for in the form of double series

$$(1.17) \qquad \xi^*_{xy} = \sum_{\nu,\mu}^{j,f} A_{\nu\mu}\, \varphi^{\nu\mu}_{xy}; \qquad \eta^*_{xy} = \sum_{\nu,\mu}^{\alpha,\beta} B_{\nu\mu}\, \chi^{\nu\mu}_{xy},$$

$$\nu = 0, 1, 2, \ldots, j, \qquad \mu = 0, 1, 2, \ldots, f; \qquad \nu = 0, 1, 2, \ldots, \alpha, \qquad \mu = 0, 1, 2, \ldots, \beta,$$

where

$$j, \alpha < n, \qquad f, \beta < m.$$

The functions $\varphi^{\nu\mu}_{xy}$ should sati fy the same homogeneous boundary conditions as the functions ξ_{xy}, and the functions $\chi^{\nu\mu}_{xy}$ should satisfy the same boundary conditions as the functions η_{xy}. .

The approximate solution of (1.15′) and (1.16′) can be obtained in the form

$$(1.18) \qquad \sum_{\nu,\mu}^{j,f} A_{\nu\mu}\, b_{\nu\mu ik} = q_{ik}, \qquad i = 0, 1, 2, \ldots, j; \qquad k = 0, 1, 2, \ldots, f,$$

where

$$b_{\nu\mu ik} = \sum_{x,y}^{n,m} \varphi^{ik}_{xy}\, F_{xy}(\varphi^{\nu\mu}_{xy}), \qquad q_{ik} = \sum_{x,y}^{n,m} U_{xy}\, \varphi^{ik}_{xy}.$$

and

(1.19) $$\sum_{\nu,\mu}^{\alpha,\beta} B_{\nu\mu}\, d_{\nu\mu ik} = p_{ik}, \qquad i = 0, 1, 2, \ldots, \alpha, \qquad k = 0, 1, 2, \ldots, \beta\ldots,$$

where

$$d_{\nu\mu ik} = \sum_{x,y}^{n,m} \chi_{xy}^{ik}\, F_{xy}(\chi_{xy}^{\nu\mu}), \qquad p_{ik} = \sum_{x,y}^{n,m} V_{xy}\, \chi_{xy}^{ik}.$$

If for the functions $\varphi_{xy}^{\nu\mu}$ and $\chi_{xy}^{\nu\mu}$ the eigenfunctions of Eqs. (1.15′) and (1.16′) are assumed, that is functions satisfying the equations

(1.20′) $$F_{xy}(\varphi_{xy}^{\nu\mu}) = \sigma_{\nu\mu}\, \varphi_{xy}^{\nu\mu}, \qquad \nu = 0, 1, 2, \ldots, n, \qquad \mu = 0, 1, 2, \ldots, m,$$

with the same boundary conditions as the functions ξ_{xy} and

(1.20″) $$F_{xy}(\chi_{xy}^{\nu\mu}) = \tau_{\nu\mu}\, \chi_{xy}^{\nu\mu}, \qquad \nu = 0, 1, 2, \ldots, n, \qquad \mu = 0, 1, 2, \ldots, m,$$

with the same boundary conditions as the functions η_{xy}, we obtain the accurate solution of Eqs. (1.15′) and (1.16′). It has the form

(1.21′) $$\xi_{xy} = \sum_{\nu,\mu}^{n,m} \frac{q_{\nu\mu}}{\sigma_{\nu\mu}}\, \varphi_{xy}^{\nu\mu}, \qquad \eta_{xy} = \sum_{\nu,\mu}^{n,m} \frac{p_{\nu\mu}}{\tau_{\nu\mu}}\, \chi_{xy}^{\nu\mu}.$$

The solution method just described may be generalized to the case of more than two equations and of a greater number of independent variables.

Let us return to Eq. (1.1). In many cases this equation can be reduced, by means of separation of variables, to an ordinary difference equation.

Let us assume that the solution of (1.1) can be represented in the form

(1.22) $$w_{xy} = \sum_{\nu=0}^{\nu=n} X_x^{\nu}\, F_y^{\nu},$$

where $[X_x^{\nu}]$ is a complete set of orthonormal functions satisfying the same boundary conditions as the functions w_{xy} for $x = 0$ and $x = n$.

From the requirement of orthogonality of the function $L_{xy}(w_{xy}) - U_{xy}$ to each of the functions X_x^{ν} $(\nu = 0, 1, 2, \ldots, n)$, we obtain n conditions

(1.23) $$\sum_{x=0}^{n} \Big[L_{xy}\Big(\sum_{\nu=0}^{n} X_x^{\nu}\, F_y^{\mu}\Big) - q_{xy}\Big]\, X_x^{i} = 0.$$

In view of the separability assumption of variables we have

(1.24) $$L_{xy}(X_x^{\nu} F_y^{\nu}) = X_x^{\nu}\, K_y(F_y),$$

where K_y is a linear difference operator of y.

Let us observe that

$$\sum_{x=0}^{n} X_x^{i} X_x^{\nu} = \delta_{i\nu}, \qquad \sum_{\nu=0}^{n} \delta_{i\nu}\, K_y(F_y^{\nu}) = L_y(F_y^{i}).$$

By introducing the symbol

$$q_y^{i} = \sum_{x=0}^{n} q_{xy}\, X_x^{i},$$

Eq. (1.23) is reduced to

$$K_y(F_y^i) = q_y^i, \qquad i = 0, 1, 2, \ldots, n; \qquad y = 0, 1, 2, \ldots, m. \tag{1.25}$$

This is an ordinary non-homogeneous linear difference equation, which can be solved in an approximate manner by means of the orthogonalization method.

Let us seek for the solution of (1.15) in the form of the finite series

$$F_y^i = \sum_{\mu}^{f} A_\mu^i \varphi_y^{\mu i}, \qquad f < m, \qquad i = 0, 1, 2, \ldots, n, \tag{1.26}$$

where $\varphi_y^{\mu i}$ are functions satisfying the same boundary conditions as the function F_y^i.

By a similar procedure as in the case of approximate solution of (1.1), we obtain the system of equations:

$$\sum_{\mu=1}^{f} A_\mu^i b_{\mu k}^i = q_k^i, \qquad k = 0, 1, 2, \ldots, f, \tag{1.27}$$

where

$$b_{\mu k}^i = \sum_{y}^{m} \varphi_y^{ki} K_y(\varphi_y^{\mu i}).$$

Knowing the coefficients A_μ^i, the approximate solution of (1.25) will be found from (1.26) and the approximate form of Eq. (1.1) — from (1.22).

The above approximate method of solving difference equations may be applied to the determination of the eigenvalues of a homogeneous difference equation.

Let us consider the homogeneous difference equation

$$L_{xy}(w_{xy}) - \omega D_{xy}(w_{xy}) = 0, \tag{1.28}$$

where ω is a parameter to be determined for the homogeneous boundary conditions, which are prescribed.

The approximate solution of (1.28) is sought for in the form

$$w_{xy}^* = \sum_{\nu, \mu}^{j, f} A_{\nu\mu} \varphi_{xy}^{\nu\mu}, \qquad j < n, \qquad f < m. \tag{1.29}$$

By requiring that the function $(L_{xy} - \omega D_{xy}) w_{xy}$ be orthogonal to each of the functions $\varphi_{xy}^{\nu\mu}$, we obtain the following system of homogeneous equations

$$\sum_{\nu, \mu}^{j, f} A_{\nu\mu}(b_{\nu\mu ik} - \omega c_{\nu\mu ik}) = 0, \qquad i = 1, 2, \ldots, j; \qquad k = 1, 2, \ldots, f, \tag{1.30}$$

where

$$b_{\nu\mu ik} = \sum_{x, y}^{n, m} \varphi_{xy}^{ik} L_{xy}(\varphi_{xy}^{\nu\mu}), \qquad c_{\nu\mu ik} = \sum_{x, y}^{n, m} \varphi_{xy}^{ik} D_{xy}(\varphi_{xy}^{\nu\mu}). \tag{1.31}$$

By setting the determinant of (1.30) equal to zero, we obtain $j+f$ values $\omega_{\nu\mu}$.

Example of application of the orthogonalization method

Let us consider the partial difference equation

$$L_{xy}(w_{xy}) = \varkappa q_{xy}, \qquad x = 0, 1, 2, \ldots, n, \qquad y = 0, 1, 2, \ldots, m, \tag{2.1}$$

where

$$L_{xy} = \Delta_x^4 + 2\varepsilon^2 \Delta_x^2 \Delta_y^2 + \varepsilon^4 \Delta_y^4, \qquad \varkappa = \frac{a^4}{Nn^4}, \qquad \varepsilon = \frac{\Delta x}{\Delta y} = \frac{am}{bn}. \tag{2.2}$$

Eq. (2.1) will be obtained by replacing with difference quotients the derivatives in the differential equation of plate deflection $N\Delta^4 w(x, y) = q(x, y)$.

In relations (2.2) Δ_x^2 denotes the second difference in the x-direction and Δ_y^4 — the fourth difference in the same direction, and

$$\begin{cases} \Delta_x^2(w_{xy}) = w_{x-1,y} - 2w_{xy} + w_{x+1,y}, \\ \Delta_x^4(w_{xy}) = w_{x-2,y} - 4w_{x-1,y} + 6w_{xy} - 4w_{x+1,y} + w_{x+2,y}. \end{cases} \tag{2.3}$$

Next, N is the flexural rigidity of the plate and q_{xy} — the load. The quantities $a = n\Delta x$, $b = m\Delta y$ are the side lengths of the rectangular plate.

Let us assume that the plate is simply supported on the edges and that the load is uniform.

Let us assume the approximate solution of Eq. (2.1) in the form of the sum

$$w_{xy}^* = \sum_{\nu=1}^{j} \sum_{\mu=1}^{f} A_{\nu\mu} \varphi_{xy}^{\nu\mu}, \qquad \nu = \underset{j<n,}{1, 2, \ldots, j}, \qquad \mu = \underset{f<m,}{1, 2, \ldots, f}, \tag{2.4}$$

where the functions

$$\varphi_{xy}^{\nu\mu} = \frac{2}{\sqrt{mn}} \sin \alpha_\nu x \sin \beta_\mu y, \qquad \alpha_\nu = \frac{\nu\pi}{n}, \qquad \beta_\mu = \frac{\mu\pi}{m}, \tag{2.5}$$

satisfy the boundary conditions of the problem. The coefficients $A_{\nu\mu}$ will be obtained from (1.4), and

$$b_{\nu\mu ik} = \sum_{x,y}^{n,m} \varphi_{xy}^{ik} L_{xy}(\varphi_{xy}^{\nu\mu}) = (r_\nu + \varepsilon^2 t_\mu)^2 \sum_{x,y}^{n,m} \varphi_{xy}^{ik} \varphi_{xy}^{\nu\mu}, \tag{2.6}$$

where

$$r_\nu = 2(\cos \alpha_\nu - 1), \qquad t_\mu = 2(\cos \beta_\mu - 1).$$

Eq. (1.4) takes the form

$$A_{ik} = \frac{q_{ik}\varkappa}{(r_i + \varepsilon^2 t_k)^2}, \tag{2.7}$$

where

$$q_{ik} = q \sum_{x,y}^{n-1,m-1} \varphi_{xy}^{ik} = \frac{2q}{\sqrt{nm}} \operatorname{ctg} \frac{i\pi}{2n} \operatorname{ctg} \frac{k\pi}{2m}.$$

Substituting A_{ik} into (2.4), we find

$$w^*_{xy} = \frac{4\varkappa q}{nm}\sum_{\nu,\mu}^{j,f} \frac{\operatorname{ctg}\frac{\nu\pi}{2n}\operatorname{ctg}\frac{\mu\pi}{2m}}{(r_\nu+\varepsilon^2 t_\mu)^2}\sin\alpha_\nu x\sin\beta_\mu y. \tag{2.8}$$

It can easily be observed that the function $w^*_{x,y}$ represents $j+f$ terms of the accurate solution, which has the form

$$w_{xy} = \frac{4 q\varkappa}{nm}\sum_{\nu,\mu}^{n-1,\,m-1} \frac{\operatorname{ctg}\frac{\nu\pi}{2n}\operatorname{ctg}\frac{\mu\pi}{2m}}{(r_\nu+\varepsilon^2 t_\mu)^2}\sin\alpha_\nu x\sin\beta_\mu y. \tag{2.9}$$

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