BULLETIN
DE
L'ACADÉMIE POLONAISE DES SCIENCES

SÉRIE DES SCIENCES TECHNIQUES

Volume IX, Numéro 7

VARSOVIE 1961

BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Série des sciences techniques
Volume IX, No. 7 — 1961

APPLIED MECHANICS (ELASTICITY)

The Three-dimensional Dynamic Problem of Thermoelasticity

by

W. NOWACKI

Presented on May 6, 1961

1. General equations

Let us consider an elastic solid with a non-steady-state temperature field $T(x_r, t)$, due to surface heating or the action of thermal sources. As a result displacements u_i, strain ε_{ij} and stress σ_{ij} are produced in the body.

The stresses are connected with the strains by the relations

$$\sigma_{ij} = 2\mu\,\varepsilon_{ij} + \delta_{ij}(\lambda\varepsilon_{kk} - \gamma T), \quad \gamma = (3\lambda + 2\mu)\,a_t, \quad i, j = 1, 2, 3, \tag{1.1}$$

where λ, μ are the Lamé constants and a_t — the coefficient of thermal dilatation. The relations between the strains and the displacements are

$$\varepsilon_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}). \tag{1.2}$$

If, into the equations of motion

$$\sigma_{ij,j} = \varrho\ddot{u}_i, \quad i, j = 1, 2, 3, \tag{1.3}$$

the stresses of (1.1) are inserted and expressed in terms of displacements, we obtain a system of three equations

$$\mu u_{i,kk} + (\lambda + \mu)\,u_{k,ki} - \gamma T_{,i} = \varrho\ddot{u}_i. \tag{1.4}$$

The solution for u_i will be composed of two components: a particular integral u_i^0 and the general integral u_i', where u_i^0 can be expressed by what is called the potential of thermoelastic displacement $\Phi : u_i^0 = \Phi_{,i}$. Thus, the following system of two equations is obtained

$$\mu\Phi_{,kki} + (\lambda + \mu)\,\Phi_{,kki} - \gamma T_{,i} = \varrho\ddot{\Phi}_i, \tag{1.5}$$

$$\mu u'_{i,kk} + (\lambda + \mu)\,u'_{k,ki} = \varrho\ddot{u}'_i. \tag{1.6}$$

Integrating (1.5) with respect to x_i, we obtain the following equation (see [1]),

$$\Box_1^2\Phi = mT, \tag{1.7}$$

where

$$m = \frac{\gamma}{\lambda + 2\mu}, \quad \Box_1^2 = \nabla^2 - \frac{1}{c_1^2}\partial_t^2, \quad c_1^2 = \frac{\lambda + 2\mu}{\varrho},$$

and ϱ is the density and c_1 — the velocity of propagation of longitudinal waves.

The temperature field should satisfy the heat conductivity equation:

$$\square_3^2 T = -Q/\varkappa, \quad \square_3^2 = \nabla^2 - \frac{1}{\varkappa}\partial_t, \tag{1.8}$$

where $\varkappa = \lambda'/c\varrho$, $Q = W/\varrho c$ and λ' is the coefficient of heat conduction, W — the quantity of heat produced per unit time and volume, and c — the specific heat. The problem is considered to be non-coupled. The member proportional to the dilatation rate $\dot{\varepsilon}_{kk}$ is disregarded in the heat equation.

The particular integral Φ is the solution of the problem for an infinite elastic space. Knowing the function Φ, we determine the stresses σ_{ij}^0 from Eqs. (1.1).

Bearing in mind that $\varepsilon_{ij}^0 = \Phi_{,ij}$, and making use of (1.7), we find

$$\sigma_{ij}^0 = 2\mu(\Phi_{,ij} - \delta_{ij}\Phi_{,kk}) + \varrho\ddot{\Phi}. \tag{1.9}$$

For a bounded body the general integral u_i^0 of (1.6) should be added to the particular integral u_i'. Introducing three functions φ_i $i = 1, 2, 3$ related with the displacements u_i' by the following relations (see [2]—[4]),

$$u_i' = [(\square_2^2 + a\nabla^2)\delta_{ij} - a\partial_i\partial_j]\varphi_j, \quad i, j = 1, 2, 3, \quad a = \frac{\lambda+\mu}{\mu}, \tag{1.10}$$

we obtain a system of three bi-wave equations

$$\square_1^2\square_2^2\varphi_i = 0, \quad i = 1, 2, 3, \quad \square_2^2 = \nabla^2 - \frac{1}{c_2^2}\partial_t^2, \quad c_2^2 = \mu/\varrho, \tag{1.11}$$

where c_2 is the velocity of propagation of a transversal wave.

The functions φ_i can be considered to be B. G. Galerkin's displacement functions generalized to dynamic problems.

In many cases (semi-space or elastic layer, for instance) a single function suffices to determine the displacements u_i'. Assuming that $\varphi_1 = 0$, $\varphi_2 = 0$, $\varphi_3 = \varphi$, we obtain

$$u_i' = -a\partial_i\partial_3\varphi + \delta_{i3}(\square_2^2 + a\nabla^2)\varphi, \tag{1.12}$$

$$\varepsilon_{ij}' = -a\partial_i\partial_j\partial_3\varphi + \tfrac{1}{2}(\square_2^2 + a\nabla^2)(\delta_{i3}\partial_j + \delta_{j3}\partial_i)\varphi, \tag{1.13}$$

and

$$\sigma_{ij}' = (\lambda\square_2^2\delta_{ij} - 2(\lambda+\mu)\partial_i\partial_j)\partial_3\varphi + \mu(\delta_{i3}\partial_j + \delta_{j3}\partial_i)(\square_2^2 + a\nabla^2)\varphi. \tag{1.14}$$

The function φ satisfies the bi-wave equation

$$\square_1^2\square_2^2\varphi = 0. \tag{1.15}$$

Eq. (1.15) may be replaced by the system of equations

$$\square_1^2\chi_1 = 0, \quad \square_2^2\chi_2 = 0, \quad \varphi = \chi_1 + \chi_2. \tag{1.16}$$

If the body is bounded, it is convenient to determine the particular integral Φ thus: Let us eliminate the function T from Eqs. (1.7) and (1.8). Then

$$\square_1^2\square_3^2\Phi = -\frac{mQ}{\varkappa}. \tag{1.17}$$

This equation can be transformed to the following form

$$(\square_1^2-\square_3^2)\,\Phi=-\frac{m}{\varkappa}\,[(\square_3^2)^{-1}-(\square_1^2)^{-1}]\,Q. \tag{1.18}$$

Since

$$\square_1^2-\square_3^2=\frac{1}{\varkappa}\,\partial_t-\frac{1}{c_1^2}\,\partial_t^2,\qquad T=-\frac{1}{\varkappa}\,(\square_3^2)^{-1}\,Q,$$

therefore

$$\left(\frac{1}{\varkappa}\,\partial_t-\frac{1}{c_1^2}\,\partial_t^2\right)\Phi=m\,(T-S), \tag{1.19}$$

where the function S is an auxiliary function satisfying the equation

$$\square_1^2\,S=-Q/\varkappa, \tag{1.20}$$

it being assumed that the function S satisfies the same boundary conditions as the function T. If Eq. (1.19) is subject to the Laplace transformation and assuming that for $t\leqslant 0$ the body was in the natural unstressed state, we obtain

$$\widetilde{\Phi}=\frac{m\,(\widetilde{T}-\widetilde{S})}{(p/\varkappa-p^2/c_1^2)},\qquad \widetilde{\Phi}=\int\limits_0^\infty \Phi e^{-pt}\,dt,\qquad \text{etc.} \tag{1.21}$$

Performing the inverse Laplace transformation, we obtain the function Φ. If the temperature varies harmonically in time, then, from Eq. (1.18), we obtain

$$\hat{\Phi}=\frac{m\,(U-V)}{i\eta+k_1^2},\qquad \eta=\omega/\varkappa,\qquad k_1^2=\omega^2/c_1^2. \tag{1.22}$$

where the following symbols have been introduced

$$\Phi=e^{i\omega t}\hat{\Phi},\qquad T=e^{i\omega t}\,U,\qquad S=e^{i\omega t}\,V. \tag{1.23}$$

The method of determining the function Φ, just described, has the advantage of having $\Phi=0$ on the edge, owing to the fact that the same boundary conditions are assumed for the functions U and V.

In many cases (a semi-space, a layer, a cube) this condition is univocal with vanishing of the normal stress.

The above procedure may also be applied to the problem of plane strain. Assuming that the temperature, the displacements and the stress do not depend on x_3, Eqs. (1.7), (1.8) and (1.11) remain valid, provided that the operator ∇^2 is replaced by $\nabla_1^2=\partial_1^2+\partial_2^2$ and $i,j=1,2$. For the displacements u_i' we obtain the relation

$$u_i'=[(\square_2^2+a\nabla_1^2)\,\delta_{ij}-a\partial_i\,\partial_j]\,\varphi_j,\qquad i,j=1,2. \tag{1.24}$$

Assuming that $\varphi_1=0$, $\varphi_2=\varphi$, we obtain

$$u_i'=-a\partial_i\,\partial_2\,\varphi+\delta_{i2}\,(\square_2^2+a\nabla_1^2)\,\varphi, \tag{1.25}$$

$$\varepsilon_{ij}'=-a\partial_i\,\partial_j\,\partial_2\,\varphi+\tfrac{1}{2}\,[\delta_{i2}\,\partial_j+\delta_{j2}\,\partial_i]\,[\square_2^2+a\nabla_1^2)\,\varphi, \tag{1.26}$$

$$\sigma_{ij}'=-2\mu\partial_i\partial_j\partial_2\varphi+\lambda\square_2^2\,\partial_2\,\varphi\delta_{ij}+\mu\,[\partial_j\delta_{i2}+\partial_i\delta_{j2}]\,[\square_2^2+a\nabla_1^2)\,\varphi. \tag{1.27}$$

The above method for solving dynamic non-coupled problems of thermoelasticity enables to pass directly to quasi-statical problems.

The particular integral Φ will be obtained from the Poisson's equation

$$\nabla^2 \Phi = mT, \tag{1.28}$$

where T should satisfy Eq. (1.8).

The functions φ_i $i = 1, 2, 3$ should satisfy the following B. G. Galerkin's biharmonic equation

$$\nabla^4 \varphi_i = 0. \tag{1.29}$$

The stresses σ^0_{ij} connected with the potential of thermoelastic displacement are given by the equations

$$\sigma^0_{i,j} = 2\mu (\Phi,_{ij} - \delta_{ij} \nabla^2 \Phi), \quad i,j = 1, 2, 3, \tag{1.30}$$

and the displacements u'_i are connected with the functions φ_i by the following relations

$$u'_i = [(1+a)\, \delta_{ij} \nabla^2 - a\partial_i \partial_j]\, \varphi_j, \quad i,j = 1, 2. \tag{1.31}$$

2. Examples. A heat source acting in the infinite space. A heated elastic layer

In the problem of the infinite space the knowledge of the function Φ suffices to obtain the stresses. It may be found in the simplest way from (1.21) for arbitrary time process or from (1.22) for a temperature field varying harmonically in time.

Let us consider the case of instantaneous concentrated heat source acting at the origin.

Let

$$Q(x_r, t) = Q_0\, \delta(x_r)\, \delta(t), \quad \delta(x_r) = \delta(x_1)\, \delta(x_2)\, \delta(x_3). \tag{2.1}$$

The resulting temperature field is

$$T = \frac{Q_0}{(4\pi\varkappa t)^{3/2}} \exp\left(-\frac{R^2}{4\varkappa t}\right), \quad R = (x_1^2+x_2^2+x_3^2)^{1/2}. \tag{2.2}$$

Performing the Laplace transformation on the Eq. (2.2), we obtain

$$\tilde{T} = \frac{Q_0}{4\pi\varkappa R} \exp(-R\sqrt{p/\varkappa}). \tag{2.3}$$

Bearing in mind the analogy between (1.8) and (1.20), we obtain

$$\tilde{S} = \frac{Q_0}{4\pi\varkappa R} e^{-Rp\sigma}, \quad \sigma = 1/c_1. \tag{2.4}$$

Inserting (2.3) and (2.4) into (1.21), we find

$$\tilde{\Phi} = \frac{Q_0 m}{4\pi\varkappa R\,(p/\varkappa - p^2\sigma^2)} (e^{-R\sqrt{p/\varkappa}} - e^{-Rp\sigma}). \tag{2.5}$$

Thus the function $\tilde{\Phi}$ has been obtained in another way than in [1]. The inverse Laplace transformation can easily be done [1], [5].

If a concentrated heat source varying harmonically in time acts at the origin $Q(x_r, t) = e^{i\omega t} Q_0 \delta(x_r)$, then, from (1.22), we obtain

$$\Phi = e^{i\omega t}\hat{\Phi} = \frac{Q_0 m e^{i\omega t}}{4\pi\varkappa R(i\eta+k_1^2)}\left(e^{-R\sqrt{i\eta}} - e^{-Rk_1}\right) \tag{2.6}$$

$$\eta = \omega/\varkappa, \quad k_1 = \omega/c_1.$$

From the comparison between the equations for the function U and V

$$\begin{cases} (\nabla^2 - i\eta)\, U = -\dfrac{Q_0}{\varkappa}\delta(x_r), \\ (\nabla^2 + k_1^2)\, V = -\dfrac{Q_0}{\varkappa}\delta(x_r), \end{cases} \tag{2.7}$$

and the equations for $\widetilde{T}$ and $\widetilde{S}$ in the case of an instantaneous heat source

$$(\nabla^2 - p/\varkappa)\widetilde{T} = -\frac{Q_0}{\varkappa}\delta(x_r), \quad \left(\nabla^2 - \frac{p^2}{c_1^2}\right)\widetilde{S} = -\frac{Q_0}{\varkappa}\delta(x_r), \tag{2.8}$$

it follows that for the determination of the function Φ use can be made of (2.5), by replacing p with $i\omega$.

Let the heat source vary harmonically in time and also harmonically along the lines x_1 and x_2.

$$Q(x_r, t) = Q_0 e^{i\omega t}\delta(x_3) e^{i(\alpha_n x_1 + \beta_m x_2)} \tag{2.9}$$

$$\alpha_n = \frac{n\pi}{a}, \quad \beta_m = \frac{m\pi}{b}, \quad n, m = 0, \pm 1, \pm 2, \ldots,.$$

A solution of the equation

$$(\nabla^2 - i\eta)\, U = -\frac{Q_0}{\varkappa}\delta(x_3) e^{i(\alpha_n x_1 + \beta_m x_2)}, \tag{2.10}$$

is the function

$$U = \frac{Q_0}{2\varkappa}\frac{e^{-\vartheta_{nm}x_3}}{\vartheta_{nm}} e^{i(\alpha_n x_1 + \beta_m x_2)}, \quad \vartheta_{nm} = \sqrt{\alpha_n^2 + \beta_m^2 + i\eta}. \tag{2.11}$$

Replacing $i\eta$ by $-k_1^2$ in (2.11), we obtain the function

$$V = \frac{Q_0}{2\varkappa}\frac{e^{-\varepsilon_{nm}x_3}}{\varepsilon_{nm}} e^{i(\alpha_n x_1 + \beta_m x_2)}, \quad \varepsilon_{nm} = \sqrt{\alpha_n^2 + \beta_m^2 - k_1^2}. \tag{2.12}$$

Substituting (2.11) and (2.12) into (1.22), we find

$$\Phi = \frac{Q_0 m e^{i(\omega t + \alpha_n x_1 + \beta_m x_2)}}{2\varkappa(i\eta + k_1^2)}\left(\frac{e^{-\vartheta_{nm}x_3}}{\vartheta_{nm}} - \frac{e^{-\varepsilon_{nm}x_3}}{\varepsilon_{nm}}\right). \tag{2.13}$$

The stresses will be found from (1.9).

Let the heat source vary harmonically in t and x_3

$$Q(x_r, t) = Q_0 \delta(x_1)\delta(x_2) e^{i(\omega t + \gamma_n x_3)}, \quad \gamma_n = \frac{n\pi}{c}, \tag{2.14}$$

$$n = 0, \pm 1, \pm 2, \ldots.$$

A solution of the equation

$$(\nabla^2 - i\eta)\, U = -\frac{Q_0}{\varkappa} e^{i\gamma_n x_3}\, \delta(x_1)\, \delta(x_2), \tag{2.15}$$

is the function

$$U = \frac{Q_0}{2\varkappa} e^{i\gamma_n x_3}\, K_0\left(r\sqrt{\gamma_n^2 + i\eta}\right), \tag{2.16}$$

where $K_0(z)$ is a modified Bessel function of the third kind and zero order. Bearing in mind (2.16), we obtain the following expression for the function V.

$$V = \frac{Q_0}{2\varkappa} e^{i\gamma_n x_3}\, K_0\left(r\sqrt{\gamma_n^2 - k_1^2}\right). \tag{2.17}$$

Thus, in agreement with (1.22), we have

$$\Phi = e^{i\omega t}\hat{\Phi} = \frac{Q_0 m e^{i(\omega t + \gamma x_3)}}{2\varkappa(i\eta + k_1^2)} \left[K_0\left(r\sqrt{\gamma_n^2 + i\eta}\right) - K_0\left(r\sqrt{\gamma_n^2 - k_1^2}\right)\right]. \tag{2.18}$$

Let us consider finally an elastic layer of thickness $2h$ heated on the bounding surfaces $x_3 = \pm h$, according to the equation:

$$T(x_1, x_2, \pm h, t) = e^{i\omega t} \sum_{n=-\infty}^{n=\infty} \sum_{m=-\infty}^{m=\infty} T_{nm}\, e^{i(\alpha_n x_1 + \beta_m x_2)}, \tag{2.19}$$

$$\alpha_n = \frac{n\pi}{a}, \qquad \beta_m = \frac{m\pi}{b}.$$

The solution of the equation $(\nabla^2 - i\eta)\, U = 0$ has the form

$$U = \sum_{n=-\infty}^{n=\infty} \sum_{m=-\infty}^{m=\infty} T_{nm} \frac{\operatorname{ch} \gamma_{nm} x_3}{\operatorname{ch} \gamma_{nm} h} e^{i(\alpha_n x_1 + \beta_m x_2)}, \tag{2.20}$$

$$\gamma_{nm} = \sqrt{\alpha_n^2 + \beta_m^2 + i\eta}.$$

The temperature T is an even function in the x_3-direction. The function V being obtained from U by changing $i\eta$ into $-k_1^2$, therefore

$$\hat{\Phi} = \frac{m}{i\eta + k_1^2} \sum_{n=-\infty}^{n=\infty} \sum_{m=-\infty}^{m=\infty} T_{nm} \left(\frac{\operatorname{ch} \gamma_{nm} x_3}{\operatorname{ch} \gamma_{nm} h} - \frac{\operatorname{ch} \vartheta_{nm} x_3}{\operatorname{ch} \vartheta_{nm} h}\right) e^{i(\alpha_n x_1 + \beta_m x_2)}, \tag{2.21}$$

$$\vartheta_{nm} = \sqrt{\alpha_n^2 + \beta_m^2 - k_1^2}.$$

We find easily that for $x_3 = \pm h$ we have $\Phi = 0$, therefore the stresses σ_{33}^0 are also zero in planes bounding the layer. Since the stresses σ_{13}^0 and σ_{23}^0 do not vanish in the planes $x_3 = \pm h$, we should make use of the general solution of Eq. (1.4) and choose the constants appearing there in such a manner that the free surface conditions be satisfied in the planes $x_3 = \pm h$.

Choosing a single function φ_i, that is $\varphi_3 = \varphi_1$, and denoting $\varphi = \chi_1 + \chi_2$, we must satisfy the equation

$$\Box_1^2 \chi_1 = 0, \quad \Box_2^2 \chi_2 = 0. \tag{2.22}$$

Introducing the symbols $\varphi = e^{i\omega t}\hat{\varphi}$, $\chi_i = e^{i\omega t}\hat{\chi}_i$, $i = 1, 2$, we must solve the following system of two equations

$$(\nabla^2+k_1^2)\,\hat{\chi}_1 = 0, \quad (\nabla^2+k_2^2)\,\hat{\chi}_2 = 0.$$

Let us assume the solution of these equations in the form

$$\text{(2.23)} \quad \begin{cases} \hat{\chi}_1 = \sum\limits_{n=-\infty}^{n=\infty} \sum\limits_{m=-\infty}^{m=\infty} a_{nm} \operatorname{sh} \vartheta_{nm} x_3\, e^{i(\alpha_n x_1+\beta_m x_2)}, \\ \hat{\chi}_2 = \sum\limits_{n=-\infty}^{n=\infty} \sum\limits_{m=-\infty}^{m=\infty} b_{nm} \operatorname{sh} \varepsilon_{nm} x_3\, e^{i(\alpha_n x_1+\beta_m x_2)}, \end{cases}$$

$$\varepsilon_{nm} = \sqrt{\alpha_n^2+\beta_m^2-k_2^2}.$$

The functions χ_1 and χ_2 have been assumed to be odd in relation to the plane $x_3 = 0$ because the stress σ_{33}' should be even which is a consequence of the form of the expression (1.14) for stress σ_{33}'. The following boundary conditions should be satisfied in the plane $x_3 = h$.

$$\text{(2.24)} \quad \sigma_{33}' = 0, \quad \sigma_{13}^0+\sigma_{13}' = 0, \quad \sigma_{23}^0+\sigma_{23}' = 0,$$

or

$$\text{(2.25)} \quad \begin{cases} \partial_3\,[-2\,(\lambda+\mu)\,\partial_3^2+(\lambda+2\mu)\,\Box_2^2+2\mu a\nabla^2]\,\varphi = 0, \\ \partial_1\,\{[-2\,(\lambda+\mu)\,\partial_3^2+(\Box_2^2+a\nabla^2)\,\mu]\,\varphi+2\mu\Phi_{,3}\} = 0, \\ \partial_1\,\{[-2\,(\lambda+\mu)\,\partial_3^2+(\Box_2^2+a\nabla^2)\,\mu]\,\varphi+2\mu\Phi_{,3}\} = 0. \end{cases}$$

The second and third boundary conditions (2.25) lead to the same relation between the functions a_{nm} and b_{nm}.

Making use of these boundary conditions, we obtain the following system of equations from which we can determine the constants a_{nm} and b_{nm}.

$$\text{(2.26)} \quad b_{nm} = -a_{nm}\frac{[(\lambda+2\mu)\,\vartheta_{nm}^2+(\lambda+2\mu)\,k_2^2-(3\lambda+2\mu)\,\Delta_{nm}]\,\vartheta_{nm}\operatorname{ch}\vartheta_{nm}h}{[(\lambda+2\mu)\,\varepsilon_{nm}^2+(\lambda+2\mu)\,k_2^2-(3\lambda+2\mu)\,\Delta_{nm}]\,\varepsilon_{nm}\operatorname{ch}\varepsilon_{nm}h},$$

$$a_{nm}\,[(2\mu+\lambda)\,(k_1^2-\Delta_{nm})-\lambda\vartheta_{nm}^2]\operatorname{sh}\vartheta_{nm}h+ \\ +b_{nm}\,[(2\mu+\lambda)\,(k_1^2-\Delta_{nm})-\lambda\varepsilon_{nm}^2]\operatorname{sh}\varepsilon_{nm}h+ \\ +\frac{2\mu m\,T_{nm}}{i\eta+k_1^2}(\gamma_{nm}\operatorname{th}\gamma_{nm}h-\vartheta_{nm}\operatorname{th}\vartheta_{nm}h) = 0, \quad \Delta_{nm} = \alpha_n^2+\beta_m^2.$$

The knowledge of a_{nm} and b_{nm} enables us to find φ.

Knowing Φ and φ, the stress can be found from (1.9) and (1.27). The final values of stresses are obtained by superposition, $\sigma_{ij} = \mathring{\sigma}_{ij}+\sigma_{ij}'$.

DEPARTMENT OF MECHANICS OF CONTINUOUS MEDIA, INSTITUTE OF FUNDAMENTAL TECHNICAL PROBLEMS, POLISH ACADEMY OF SCIENCES

(ZAKŁAD MECHANIKI OŚRODKÓW CIĄGŁYCH, INSTYTUT PODSTAWOWYCH PROBLEMÓW TECHNIKI, PAN)

REFERENCES

[1] W. Nowacki, *Dynamical problem of thermoelasticity*, Arch. Mech. Stos., **9** (1957), 3.

[2] S. Kaliski, *Pewne problemy brzegowe dynamicznej teorii sprężystości i ciał niesprężystych*, Warsaw, 1957.

[3] W. Nowacki, *Zagadnienia termosprężystości*, Warsaw, 1960.

[4] R. A. Eubanks, E. Sternberg, *On stress functions for elastokinetics and the integration of the repeated wave equation*, Quart. Appl. Math., **15** (1957).

[5] H. Parcus, *Instationäre Wärmespannungen*, Vienna, 1959.