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APPLIED MECHANICS

Dynamic Distortion Problem

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1. Solution of displacement equations

The equations between the stresses σ_{ij}, strains ε_{ij} and distortions $\overset{0}{\varepsilon}_{ij}$ are of the form

$$\sigma_{ij} = 2\mu(\varepsilon_{ij} - \overset{0}{\varepsilon}_{ij}) + \lambda\delta_{ij}(\varepsilon_{kk} - \overset{0}{\varepsilon}_{kk}), \qquad i, j, k = 1, 2, 3. \tag{1.1}$$

Herein, μ and λ denote the Lamé constants, and δ_{ij} is Kronecker's symbol.

We now introduce the stresses into the equations of motion:

$$\sigma_{ij,j} + X_i = \varrho\ddot{u}_i, \qquad i, j = 1, 2, 3 \qquad \ddot{u}_i = \frac{\partial^2 u_i}{\partial t^2} = \partial_t^2 u_i, \tag{1.2}$$

wherein X_i denotes the components of the mass force, and ϱ — the density.

With the relations

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}), \tag{1.3}$$

the following set of displacement equations is obtained:

$$\mu u_{i,jj} + (\lambda+\mu) u_{j,ji} + X_i - \Gamma_i = \varrho\ddot{u}_i, \qquad i, j = 1, 2, 3, \tag{1.4}$$

wherein

$$\Gamma_i = 2\mu\overset{0}{\varepsilon}_{ij,j} + \lambda\overset{0}{\varepsilon}_{jj,i}.$$

Eqs. (1.4) should be supplemented by the boundary conditions

$$\sigma_{ij} n_j = p_i, \tag{1.5}$$

wherein p_i are the components of the forces acting on the surface bounding the body, and by the initial conditions at $t = 0$

$$u_i(x, 0) = 0, \quad \dot{u}_i(x, 0) = 0. \tag{1.6}$$

Obviously, by (1.4), (1.6) and (1.1), Eqs. (1.4) are solved in the distortions by replacing the mass force in the displacement equations by $X_i - \Gamma_i$ and by substituting $p_i + (2\mu\overset{0}{\varepsilon}_{ij} + \lambda\delta_{ij}\overset{0}{\varepsilon}_{kk}) n_j$ for the surface tractions in the boundary conditions.

Thus, formally, the elastodynamic problem with prescribed surface forces and the dynamic distortional problem become identical.

This analogy between the mass forces and distortions will be put to use in further solutions.

Let us represent the solution of the set of Eqs. (1.4) by means of Green's function $U_i^{(k)}(x, \xi)$, $i, k = 1, 2, 3$. Let $U_i^{(k)}(x, \xi)$ denote the displacement of the point (x) of the body along the axis x_i as resulting from the force concentrated at the point (ξ) and directed along the axis x_k.

It is required that the functions $U_i^{(k)}$ fulfill the following set of equations:

$$\mu U_{i,jj}^{(k)} + (\lambda+\mu) U_{j,ji}^{(k)} + \delta(x-\xi)\delta_{ik} = 0, \qquad i, j, k = 1, 2, 3, \tag{1.7}$$

wherein

$$\delta(x-\xi) = \delta(x_1-\xi_1)\,\delta(x_2-\xi_2)\,\delta(x_3-\xi_3),$$

δ being the Dirac function.

It should be noted that the function $U_i^{(k)}(x, \xi)$ is the solution of the static problem. The set (1.7) is solved on assuming an identical form for the body and support as for the system considered.

On solving the set of Eqs. (1.4) with the boundary conditions (1.5), making use of the analogy between the mass forces and distortions, and applying the functions $U_i^{(k)}$, we have

$$u_i = -\varrho \int_V \ddot{u}_k U_i^{(k)} dV + \int_V (X_k - \Gamma_k) U_i^{(k)} dV + \int_\sigma (p_k + p_k^*) U_i^{(k)} d\sigma \tag{1.8}$$

with

$$\Gamma_k = 2\mu \overset{0}{\varepsilon}_{kj,j} + \lambda \overset{0}{\varepsilon}_{jj,k}, \qquad p^* = (2\mu \overset{0}{\varepsilon}_{kj} + \delta_{kj} \overset{0}{\varepsilon}_{rr}) n_j.$$

Applying the integral transformation

$$\int_\sigma n_j F d\sigma = \int_V F_{,j} dV,$$

to the surface integral, and taking into account Maxwell's theorem on the reciprocity of displacements:

$$U_i^{(k)}(x, \xi) = U_k^{(i)}(\xi, x),$$

and the relation

$$\frac{\partial U_i^{(k)}(x, \xi)}{\partial x_k} = \frac{\partial U_k^{(i)}(\xi, x)}{\partial \xi_k},$$

we obtain Eq. (1.8) in the form

$$\begin{aligned} u_i(x, t) = -\varrho \int_V \ddot{u}_k(\xi, t) U_k^{(i)}(\xi, x) dV + \int_V X_k(\xi, t) U_k^{(i)}(\xi, x) dV + \\ + \int_\sigma p_k(\xi, t) U_k^{(i)}(\xi, x) d\sigma + 2\mu \int_V \overset{0}{\varepsilon}_{kj}(\xi, t) \bar{\varepsilon}_{kj}^{(i)}(\xi, x) dV + \\ + \lambda \int_V \overset{0}{\varepsilon}_{kk}(\xi, t) \bar{\varepsilon}_{kk}^{(i)}(\xi, x) dV, \qquad i, k = 1, 2, 3. \end{aligned} \tag{1.9}$$

Thus, we have a set of Fredholm's equations of the second kind. Herein, the following notation has been used:

$$\bar{\varepsilon}_{kj}^{(i)} = \frac{1}{2}\left(U_{k,j}^{(i)} + U_{j,k}^{(i)}\right), \qquad \bar{\varepsilon}_{kk}^{(i)} = U_{k,k}^{(i)}, \tag{1.10}$$

representing the strains and sum of axial strains at (ξ), due to the effect of a concentrated force at the point (x) directed along the axis x_i.

If neither mass forces within the body $(X_i = 0)$ no surface tractions $(p_i = 0)$ on σ are present, Eq. (1.9) assumes the form

$$u_i = -\varrho \int_V \ddot{u}_k U_k^{(i)} dV + 2\mu \int_V \varepsilon_{kj}^0 \bar{\varepsilon}_{kj}^{(i)} dV + \lambda \int_V \varepsilon_{kk}^0 \bar{\varepsilon}_{kk}^{(i)} dV, \tag{1.11}$$

or

$$u_i + \varrho \int_V \ddot{u}_k U_k^{(i)} dV = \int_V \varepsilon_{kj}^0 \bar{\sigma}_{kj}^{(i)} dV, \qquad i, k = 1, 2, 3. \tag{1.11'}$$

Herein, $\bar{\sigma}_{ij}^{(i)}$ denote the stress at (ξ), due to the concentrated force at (x) and directed along x_i.

In the special case of thermal distortions, we have $\varepsilon_{kj}^0 = \alpha_t \delta_{kj} T$, with T denoting a function of the non-stationary temperature field.

Eq. (1.11′) now assumes the form [1]

$$u_i + \varrho \int_V \ddot{u}_k U_k^{(i)} dV = \alpha_t \int_V T \bar{\sigma}_{kk}^{(i)} dV, \qquad i = 1, 2, 3. \tag{1.11''}$$

Let the distortion field vary harmonically with the time:

$$\varepsilon_{kj}^0 (x, t) = e^{ipt} \varepsilon_{kj}^* (x).$$

The set of equations (1.11′) is now solved by the function

$$u_i (x, t) = u_i^* (x) e^{ipt} + \sum_{n=1}^{\infty} u_i^{**} (x) e^{i\omega_n t}, \tag{1.12}$$

with u_i^* fulfilling the set of equations

$$u_i^* - \varrho p^2 \int_V u_i^* U_k^{(i)} dV = \int_V \varepsilon_{kj}^* \bar{\sigma}_{kj}^{(i)} dV, \tag{1.13}$$

and u_i^{**} — the set of homogeneous equations

$$u_i^{**} - \varrho \omega^2 \int_V u_i^{**} U_k^{(i)} dV = 0. \tag{1.14}$$

Eq. (1.13) yields the amplitudes of the displacements resulting from the distortions ε_{kj}^0, and Eq. (1.14) — the frequencies $\omega_1, \omega_2, \ldots$ of the proper oscillations of the system. The set of equations (1.13) has no solutions if $p = \omega_n$, i.e. if there is resonance.

If the distortions ε_{ij}^0 vary slowly with the time, the problem can be considered to be quasi-static. The inertial terms in Eqs. (1.9)—(1.11″) should now be neglected.

In order to determine the displacements, it is sufficient to carry out integration over the volume of the body according to the formulae

$$u_i(x, t) = \int_V \overset{0}{\varepsilon}_{kj}(\xi, t)\,\bar{\sigma}^{(i)}_{kj}(\xi, x)\, dV. \tag{1.11'''}$$

Let us now assume the functions $G^{(k)}_i(x, \xi, t)$, $i, k = 1, 2, 3$, fulfilling the set of equations

$$\mu G^{(k)}_{i,jj} + (\lambda+\mu)\, G^{(k)}_{j,ji} + \delta(x-\xi)\,\delta_{ik}\,\delta(t) = \varrho \ddot{G}^{(k)}_i, \qquad i, j, k = 1, 2, 3, \tag{1.15}$$

as Green's functions. Herein, the $G^{(k)}_i(x, \xi, t)$ denote the displacements of the point (x) of the body in the direction of the axis x_i as due to the effect of a momentary force concentrated at (ξ) and directed along x_i.

The solution of Eqs. (1.4) will be represented in the form

$$u_i(x, t) = \int_0^t d\tau \int_V [X_k(\xi, \tau) - \Gamma_k(\xi, \tau)]\, G^{(i)}_k(\xi, x, t-\tau)\, dV + \\ + \int_0^t d\tau \int_\sigma [p_k(\xi, \tau) + p^*_k(\xi, \tau)]\, G^{(i)}_k(\xi, x, t-\tau)\, d\sigma, \tag{1.16}$$

or

$$u_i(x, t) = \int_0^t d\tau \int_V X_k(\xi, \tau)\, G^{(i)}_k(\xi, x, t-\tau)\, dV + \int_0^t d\tau \int_V \overset{0}{\varepsilon}_{ij}(\xi, \tau)\, \bar{\bar{\sigma}}^{(i)}_{kj}(\xi, x, t-\tau)\, dV. \tag{1.16'}$$

Herein, $\bar{\bar{\sigma}}^{(i)}_{kj}$ is the stress resulting from the momentary concentrated force acting at the point (x) and directed along the axis x_i.

An alternative method of solving the set of Eqs. (1.4) consists in introducing three displacement functions $\varphi_i\,(i = 1, 2, 3)$.

Let us represent the set of Eqs. (1.4) (at $X_i = 0$) in the form

$$L_{ij}(u_j) = \frac{1}{\mu}\Gamma_i, \qquad i, j = 1, 2, 3, \tag{1.17}$$

with

$$L_{ij} = \delta_{ij}\left(\nabla^2 - \frac{1}{c_2^2}\partial_t^2\right) + \beta^2 \partial_i \partial_j, \qquad \beta^2 = \frac{\lambda+\mu}{\mu}, \qquad i, j = 1, 2, 3.$$

Let us express the displacements u_i by means of the functions φ_i as follows:

$$u_i = [\Box_2^2 \delta_{ij} + \beta^2(\nabla^2 \delta_{ij} - \partial_i \partial_j)]\,\varphi_j, \qquad \Box_2^2 = \nabla^2 - \frac{1}{c_2^2}\partial_t^2. \tag{1.18}$$

On substituting (1.18) into Eq. (1.17), we obtain the following set of equations:

$$\Box_1^2 \Box_2^2 \varphi_i = \frac{1}{\mu}\Gamma_i, \qquad i = 1, 2, 3, \tag{1.19}$$

wherein

$$\Box_1^2 = \nabla^2 - \frac{1}{c_1^2}\partial_t^2, \qquad c_1^2 = \frac{\lambda+2\mu}{\varrho}.$$

The solution of (1.19) consists of the particular integral $\varphi_i^{(0)}$ and the particular integrals φ_i' of the homogeneous equations (1.19). The method of solving proposed here is a generalization of the function of B. G. Galerkin to the dynamic distortion problem.

2. Theorem of reciprocity of displacements

By applying the mass force analogy, the theorem of the reciprocity of displacements, already variously derived by different methods [2], [3], [1], can be readily proved. In the absence of distortions, E. Bettie's theorem assumes the form

$$\int_V (X_i - \varrho \ddot{u}_i) u_i' \, dV + \int_\sigma p_i u_i' \, d\sigma = \int_V (X_i' - \varrho \ddot{u}_i') u_i \, dV + \int_\sigma p_i' u_i \, d\sigma. \tag{2.1}$$

In this equation, the quantities X_i, p_i, u_i are related to the first system of loads and X_i', p_i', u_i' — to the second system of loads acting on the body.

In the presence of distortions ε_{ij}^0 in the first and — of distortions $\varepsilon_{ij}'^0$ in the second system, Eq. (2.1) should be enlarged to account for them, as follows:

$$\int_V (X_i - \Gamma_i - \varrho \ddot{u}_i) u_i' \, dV + \int_\sigma (p_i + p_i^*) u_i' \, d\sigma = \int_V (X_i' - \Gamma_i' - \varrho \ddot{u}_i') u_i \, dV + \int_\sigma (p_i' + p_i'^*) u_i \, d\sigma, \tag{2.2}$$

wherein

$$\Gamma_i = 2\mu\varepsilon_{ij,j}^0 + \lambda\varepsilon_{jj,i}^0, \qquad p_i = 2\mu\varepsilon_{ij}^0 + \lambda\delta_{ij}\varepsilon_{kk}^0,$$
$$\Gamma_i' = 2\mu\varepsilon_{ij,j}'^0 + \lambda\varepsilon_{jj,i}'^0, \qquad p_i^* = 2\mu\varepsilon_{ij}'^0 + \lambda\delta_{ij}\varepsilon_{kk}'^0.$$

Carrying out the integral transformation on the surface integrals, we finally obtain the theorem of reciprocity in a form taking into account the effect of distortion:

$$\int_V (X_i - \varrho \ddot{u}_i) u_i' \, dV + \int_\sigma p_i u_i' \, d\sigma + \int_V \varepsilon_{ij}^0 \bar{\sigma}_{ij}' \, dV = \int_V (X_i' - \varrho \ddot{u}_i') u_i \, dV + \int_\sigma p_i' u_i \, d\sigma + \int_V \varepsilon_{ij}'^0 \bar{\sigma}_{ij} \, dV. \tag{2.3}$$

Eq. (2.3) holds also in the case of isotropic non-homogeneous bodies, and in those of anisotropic homogeneous and non-homogeneous bodies, as is readily verified by generalizing the mass force analogy to such bodies.

In an anisotropic non-homogeneous body, the equations between the stresses and strains are of the form

$$\sigma_{ij} = C_{ijkl}(\varepsilon_{kl} - \overset{0}{\varepsilon}_{kl}), \qquad \varepsilon_{ij} - \overset{0}{\varepsilon}_{ij} = C_{ijkl}^* \sigma_{kl}. \tag{2.4)]}$$

with C_{ijkl} denoting quantities characteristic of the elastic properties of the anisotropic inhomogeneous medium, and being functions of the position.

Substituting (2.4) into the equations of motion (1.2), and with respect to (1.3), the following displacement equations are obtained:

$$\frac{1}{2}[C_{ijkl}(u_{k,l} + u_{l,k})]_{,j} + X_i - \Gamma_i = \varrho \ddot{u}_i, \tag{2.5}$$

with

$$\Gamma_i = (C_{ijkl}\varepsilon_{kl}^0)_{,j}.$$

It is readily verified that

$$p_i^* = C_{ijkl}\varepsilon_{kl}^0. \tag{2.6}$$

On substituting p_i^* and Γ_i in Eq. (2.1) and on transforming the surface integrals, we obtain Eq. (2.3).

From the reciprocity theorem, a number of conclusions can be drawn. In particular, if we assume the second state of loads to consist only of a static force concentrated at point (x) and directed along the axis x_i, the set of Eqs. (1.9) is obtained.

Let us consider a singly connected body, free of tractions at its surface and free from mass forces within. Assume the stresses in the body to be due to the distortions ε_{ij}^0 alone.

On assuming the second state of loads to be one tension in all directions, we have $\bar{\sigma}'_{ij}(\xi, x) = \delta_{ij}$ and $p'_i = n_i$ for a homogeneous body of arbitrary anisotropic structure.

In this case, Eq. (2.3) for the quasi-static problem yields

$$\int_\sigma n_i u_i \, d\sigma = \int_V \varepsilon_{ij}^0 \delta_{ij} \, dV.$$

The left-hand side of this equation represents the increase in volume of the body. Thus,

$$\Delta V = \int_V \varepsilon_{kk}^0 \, dV. \tag{2.7}$$

However, $\Delta V = \int_V \varepsilon_{kk} \, dV$. With respect to the relations (2.4), we now have

$$\int_V (\varepsilon_{kk} - \varepsilon_{kk}^0) \, dV = \int_V C_{ijkl}^* \sigma_{kl} \, dV = 0. \tag{2.8}$$

In the special case of an isotropic body, $\int_V \sigma_{kk} \, dV = 0$, in accordance with M. Hiecke [4]. In that of an anisotropic body presenting rectilinear anisotropy, Eq. (2.7) leads to the following relation [5] for the case of thermal distortion $(\varepsilon_{ji}^0 = \alpha_{ij} T)$:

$$\Delta V = (\alpha_{11} + \alpha_{22} + \alpha_{33}) \int_V T dV, \tag{2.9}$$

whereas for a homogeneous isotropic body $(\varepsilon_{ij} = \alpha_t \delta_{ij} T)$ we have

$$\Delta V = 3\alpha_t \int_V T dV. \tag{2.10}$$

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