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Some Problems of Rectangular Plates

by

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In this paper we consider statics, free and forced vibrations and stability of a rectangular plate simply supported at all edges. Particularly, we are interested in rectangular plates having one or two opposite edges clamped. We intend to obtain the solution in the form of a double trigonometrical series by means of an appropriate system of orthogonal functions.

1. System of orthogonal functions

First of all we solve the auxiliary problem, namely we seek the solution of the following partial differential equation

$$(\nabla^4 - \lambda^4) f(x, y) = 0 \tag{1.1}$$

in the rectangular region of sides a and b. We assume that the function $f(x, y)$ satisfies on the boundary of the rectangle the following homogeneous boundary conditions

$$f(x, 0) = f(x, b) = \nabla^2 f(x, 0) = \nabla^2 f(x, b) = 0, \tag{1.2}$$

$$f(0, y) = f(a, y) = \frac{\partial f(a, y)}{\partial x} = \nabla^2 f(0, y) = 0. \tag{1.3}$$

The solution Eq. (1.1) can be expressed in the form of the ordinary series

$$f(x,y) = \sum_{m=1}^{\infty} X_m(x)\, \psi_m(y), \tag{1.4}$$

where the functions

$$\psi_m(y) = \sqrt{\frac{2}{b}} \sin \beta_m y, \qquad \beta_m = \frac{m\pi}{b}.$$

satisfy the boundary conditions (1.2) and constitute a complete system of orthogonal and normed functions.

Substituting (1.4) into Eq. (1.1), we obtain the ordinary differential equation

$$\frac{d^4 X_m}{dx^4} - 2\beta_m^2 \frac{d^2 X_m}{dx^2} + (\beta_m^4 - \lambda^4) X_m = 0. \tag{1.5}$$

The functions $X_m(x)$ should satisfy, according to the relations (1.3), the boundary conditions

$$(1.6)\quad X_m(0) = X_m(a) = X_m''(0) = X_m'(a) = 0, \qquad X_m' = \frac{dX_m}{dx}, \qquad X_m'' = \frac{d^2 X_m}{dx^2}.$$

The solution of Eq.(1.5) takes the form

$$(1.7)\qquad X_m(x) = C_m\left(\frac{\operatorname{sh}\delta_m x}{\operatorname{sh}\delta_m a} - \frac{\sin\varepsilon_m x}{\sin\varepsilon_m a}\right),$$

where

$$\delta_m = \sqrt{\lambda^2+\beta_m^2}, \qquad \varepsilon_m = \sqrt{\lambda^2-\beta_m^2}, \qquad \lambda_m > \beta_m.$$

Moreover, the transcendental equation written below, from which we can determine the quantities;

$$(1.8)\quad \pi\sqrt{\zeta^2-m^2\nu^2}\operatorname{ctg}\pi\sqrt{\zeta^2-m^2\nu^2} = \pi\sqrt{\zeta^2+m^2\nu^2}\operatorname{ctgh}\pi\sqrt{\zeta^2+m^2\zeta^2},$$

$$(1.9)\qquad \zeta^2 = \frac{\lambda^2 a^2}{\pi^2}, \qquad \nu = \frac{a}{b}$$

should be satisfied.

Let us observe that for a given m we get an infinite number of roots which can be ordered with respect to their increasing values

$$(1.10)\qquad \lambda_{m1}, \lambda_{m2}, \ldots, \lambda_{mn}, \ldots.$$

Then the solution of Eq. (1.5) can be represented in the form

$$(1.11)\qquad X_m(x) = \sum_{n=1}^{\infty} A_n \varphi_n^{(m)}(x),$$

where the functions $\varphi_n^{(m)}(x)$ are given by the formulae

$$(1.12)\qquad \varphi_n^{(m)}(x) = C_n^{(m)}\left(\frac{\operatorname{sh}\delta_{mn} x}{\operatorname{sh}\delta_{mn} a} - \frac{\sin\varepsilon_{nm} x}{\sin\varepsilon_{nm} a}\right), \qquad n = 1, 2, \ldots, \infty$$

We can easily prove that the functions $\varphi_n^{(m)}(x)$ are orthogonal for the assumed boundary, and their normalization leads to the relation

$$(1.13)\quad \int_0^a [\varphi_n^{(m)}(x)]^2\,dx = \frac{a}{2}[C_n^{(m)}]^2\left[\frac{1}{\sin^2\varepsilon_{nm} a} + \frac{1}{\operatorname{sh}^2\delta_{nm} a} - \left(\frac{\operatorname{ctgh}\delta_{nm} a}{\delta_{nm} a} + \frac{\operatorname{ctg}\varepsilon_{nm} a}{\varepsilon_{nm} a}\right)\right] = 1, \qquad \int_0^a \varphi_n^{(m)}\varphi_{n'}^{(m)}\,dx = \delta_{nm}'.$$

Afterwards, we shall make use of the system of orthogonal functions $\varphi_n^{(m)}(x)$ when expanding the plate deflection $w(x, y)$ in the rectangular region in the double series of orthogonal functions $\varphi_n^{(m)}(x)\ \psi_m(y)$

$$(1.14)\qquad w(x, y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{nm}\,\varphi_n^{(m)}(x)\,\psi_m(y).$$

Let us transfer parallelly the system of co-ordinates in such a way that the origin of co-ordinates is at the point $(a/2, 0)$. We consider Eq. (1.5) with the boundary conditions

$$X_m\left(\frac{a}{2}\right)=X_m\left(-\frac{a}{2}\right)=X'_m\left(\frac{a}{2}\right)=X'_m\left(-\frac{a}{2}\right)=0. \tag{1.15}$$

As a result we get the family of orthogonal functions symmetric with respect to the axis y:

$$\hat{\varphi}_n^{(m)}(x)=C_n^{(m)}\left(\frac{\operatorname{ch}\delta_{nm}x}{\operatorname{ch}\dfrac{\delta_{nm}a}{2}}-\frac{\cos\varepsilon_{nm}x}{\cos\dfrac{\varepsilon_{nm}a}{2}}\right). \tag{1.16}$$

The quantities $\hat{\lambda}_{nm}$ can be determined from the transcendental equation

$$\frac{\pi}{2}\sqrt{\zeta^2-m^2\nu^2}\operatorname{tg}\frac{\pi}{2}\sqrt{\zeta^2-m^2\nu^2}+\frac{\pi}{2}\sqrt{\zeta^2+m^2\nu^2}\operatorname{tgh}\frac{\pi}{2}\sqrt{\zeta^2+m^2\nu^2}=0, \tag{1.17}$$

$$\zeta^2=\frac{\hat{\lambda}_{nm}^2 a^2}{\pi^2},\qquad \nu=\frac{a}{b};$$

we also obtain the family of orthogonal functions skew-symmetric with respect to the axis y:

$$\hat{\varphi}_n^{(m)}(x)=D_n^{(m)}\left(\frac{\operatorname{sh}\delta_{nm}x}{\operatorname{sh}\dfrac{\delta_{nm}a}{2}}-\frac{\sin\varepsilon_{nm}x}{\sin\dfrac{\varepsilon_{nm}a}{2}}\right). \tag{1.18}$$

Now the quantities $\hat{\lambda}_{nm}$ should satisfy the transcendental equation

$$\frac{\pi}{2}\sqrt{\zeta^2-m^2\nu^2}\operatorname{ctg}\frac{\pi}{2}\sqrt{\zeta^2-m^2\nu^2}-\frac{\pi}{2}\sqrt{\zeta^2+m^2\nu^2}\operatorname{ctgh}\frac{\pi}{2}\sqrt{\zeta^2+m^2\nu^2}=0. \tag{1.19}$$

For the function $\hat{\varphi}_n^{(m)}(x)$ to be normed, the following relations must be satisfied

$$[C_n^{(m)}]^2\frac{a}{2}\left[\frac{1}{\operatorname{ch}^2\mu_{nm}}+\frac{1}{\cos^2\bar{\mu}_{nm}}+\left(\frac{\operatorname{tgh}\mu_{nm}}{\mu_{nm}}+\frac{\operatorname{tg}\bar{\mu}_{nm}}{\bar{\mu}_{nm}}\right)\right]=1; \tag{1.20}$$

$$[D_n^{(m)}]^2\frac{a}{2}\left[\frac{1}{\operatorname{sh}^2\mu_{nm}}+\frac{1}{\sin^2\bar{\mu}_{nm}}-\left(\frac{\operatorname{ctgh}\mu_{nm}}{\mu_{nm}}+\frac{\operatorname{tg}\bar{\mu}_{nm}}{\bar{\mu}_{nm}}\right)\right]=1,\ \mu_{nm}=\delta_{nm}a,\ \bar{\mu}_{nm}=\varepsilon_{nm}a. \tag{1.21}$$

Hence, for a plate clamped on the edges $x=0$, $x=a$ and simply supported at the edges $y=0$, $y=b$ the deflection can be expressed by the series

$$w(x,y)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}A_{nm}\hat{\varphi}_n^{(m)}(x)\psi_m(y), \tag{1.22}$$

where for the symmetric form of the deflection of a plate $\hat{\varphi}_n^{(m)}$ with respect to axis y the functions (1.16) and for the skew-symmetric form of the deflection the functions (1.18) must be taken. In order to solve the series, appearing in subsequent problems of statics, vibrations and stability of plates, it is necessary to find from Eqs. (1.8), (1.17) and (1.19) the roots λ_{nm}, $\hat{\lambda}_{nm}$ corresponding to functions $\varphi_n^{(m)}(x)$ and functions $\hat{\varphi}_n^{(m)}(x)$.

2. Free and forced vibrations of a plate

The differential equation of the deflection of a plate performing forced vibrations takes the form

$$N\nabla^4 \bar{w} + \mu \frac{\partial^2 \bar{w}}{\partial t^2} = \bar{q}, \tag{2.1}$$

here N is the flexural rigidity of the plate, μ — the mass of the plate per unit area of the middle surface of the plate.

In the case of the harmonic forced vibrations we have

$$\bar{w}(x, y, t) = e^{i\omega t} w(x, y), \qquad \bar{q}(x, y, t) = e^{i\omega t} q(x, y), \tag{2.2}$$

where ω is the frequency forcing the vibrations.

Eq. (2.1) takes the form

$$N\nabla^4 w - \gamma^2 w = q, \qquad \gamma^2 = \mu\omega^2. \tag{2.3}$$

In the particular case $q = 0$ and for the homogeneous boundary conditions we have free vibrations of the plate. For $\omega \to 0$, $\bar{q} \to q$ we deal with a statical problem.

Let us assume the solution (2.3) in the form of the double series

$$w(x, y) = \sum_{n, m}^{\infty} A_{nm} \varphi_n^{(m)}(x) \psi_m(y). \tag{2.4}$$

We have tacitly assumed that the plate is simply supported at its edges $x = 0$, $y = 0$, $y = b$ and on the edge $x = a$ clamped. Expressing also the loading q by means of the double series

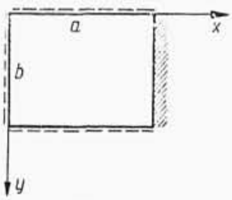

Fig. 1

$$q(x, y) = \sum_{n, m}^{\infty} q_{nm} \varphi_n^{(m)}(x) \psi_m(y), \tag{2.5}$$

where

$$q_{nm} = \int_0^a \int_0^b q(x, y) \varphi_n^{(m)}(x) \psi_m(y), \tag{2.6}$$

and taking into account that

$$\nabla^4 w(x, y) = \sum_{n, m}^{\infty} A_{nm} \lambda_{nm}^4 \varphi_n^{(m)}(x) \psi_m(y), \tag{2.7}$$

we obtain from Eq. (2.3) the following equation, from which we can find the coefficients A_{nm}

$$A_{nm}(N\lambda_{nm}^4 - \gamma^2) = q_{nm}, \tag{2.8}$$

hence

$$w(x, y) = \sum_{m, n}^{\infty} \frac{q_{nm}}{\Delta_{nm}} \varphi_n^{(m)}(x) \psi_m(y), \Delta_{nm} = N\lambda_{nm}^4 - \gamma^2. \tag{2.9}$$

Then, the deflection of the plate can be represented in the form of the double series, similarly as in the case of a plate simply supported at all the edges (Navier's method). The analogous solution can also be obtained in the case of plate clamped on the edges $x=0$, $x=a$ and simply supported on the edges $y=0$, $y=b$. Instead of $\varphi_n^{(m)}(x)$ we must only take the function $\hat{\varphi}_n^{(m)}(x)$, and instead of λ_{nm} the values $\hat{\lambda}_{nm}$. Transferring parallelly the system of co-ordinates from the corner of the plate to the point $(a/2, 0)$ we obtain for the symmetric loading with respect to y axis

$$w(x,y)=\sum_{n,m}^{\infty}\frac{q_{nm}^{(s)}}{\hat{\Delta}_{nm}}\hat{\varphi}_n^{(m)}(x)\,\psi_m(y),\quad \hat{\Delta}_{nm}=N\hat{\lambda}_{nm}^4-\gamma^2, \tag{2.10}$$

$$n=1,2,\ldots,\infty,\qquad m=1,3,5,\ldots,\infty$$

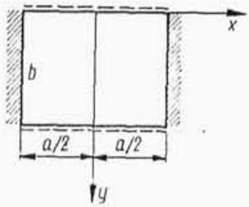

Fig. 2

and for the skew-symmetric $q^{(a)}(x,y)$ with respect to y axis

$$w(x,y)=\sum_{n,m}^{\infty}\frac{q_{nm}^{(a)}}{\hat{\Delta}_{nm}}\hat{\varphi}_n^{(m)}(x)\,\psi_m(y) \tag{2.11}$$

$$n=2,4,6,\ldots,\infty,\qquad m=1,2,\ldots,\infty$$

Let us return to the preceding case of the rectangular plate (Fig. 1) and consider the concentrated loading $\bar{q}$ of unit intensity. Then

$$q(x,y)=\delta(x-\xi)\,\delta(y-\eta). \tag{2.12}$$

In this case we obtain from Eq. (2.6)

$$q_{nm}=\varphi_n^{(m)}(\xi)\,\psi_m(\eta); \tag{2.13}$$

hence,

$$w(x,y)=\sum_{n,m}^{\infty}\frac{\varphi_n^{(m)}(\xi)\,\psi_m(\eta)}{\Delta_{nm}}\varphi_n^{(m)}(x)\,\psi_m(y). \tag{2.14}$$

This is the Green function for the amplitude of the deflection of a plate.

Let the loading $Y(y)\,e^{i\omega t}$ act on the plate along the straight line $x=\xi$. Then

$$q_{nm}=Y_m\,\varphi_n^{(m)}(\xi),\qquad Y_m=\int_0^b Y(y)\,\psi_m(y)\,dy.$$

In this case for the amplitude of the deflection we obtain the following equation

$$w(x,y)=\sum_{n,m}^{\infty}\frac{Y_m\,\varphi_n^{(m)}(\xi)}{\Delta_{nm}}\varphi_n^{(m)}(x)\,\psi_m(y). \tag{2.15}$$

The above results can be used in more complicated cases. Let the loading $\bar{q}(x,y,t)=e^{i\omega t}q(x,y)$ and the linear loading $Y(y)\,\delta(x-\xi)$ act on the plate. Then the total amplitude of the deflection is given by the equation

$$w(x,y)=\sum_{n,m}^{\infty}\frac{\varphi_n^{(m)}(x)\,\psi_m(y)}{\Delta_{nm}}\left[q_{nm}+Y_m\,\varphi_n^{(m)}(\xi)\right]. \tag{2.16}$$

At present we require that the deflection of a plate along the straight line $x = \xi$ be zero. From this condition we get the equation

$$Y_m \sum_{n=1}^{\infty} \frac{[\varphi_n^{(m)}(\xi)]^2}{\Delta_{nm}} + \sum_{n=1}^{\infty} \frac{p_{nm}}{\Delta_{nm}} \varphi_n^{(m)}(\xi) = 0, \tag{2.17}$$

from which the coefficient Y_m can be determined. The knowledge of the value of Y_m allows us to determine the function $w(x, y)$ from Eq. (2.16). Thus we have established the deflection of a two-span plate simply supported on its periphery and additionally along the straight line $x = \xi$. Consider an infinite strip simply supported at the edge $x = 0$ and clamped on the edge $x = a$. Let the loading $\bar{q}(x, y, t)$ be composed of concentrated forces acting at the points $(\xi \pm 2\,bk)$ $k = 1, 2, ..., \infty$, the distance between them being equal to $2\,b$.

$$\bar{q}(x, y, t) = e^{i\omega t} \delta(x - \xi) \sum_{k=-\infty}^{k=+\infty} \delta(y + 2\,bk) \tag{2.18}$$

The amplitude of the loading q can be expressed by the double series

$$q(x, y) = \frac{1}{2b} \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \delta_m \varphi_n^{(m)}(\xi) \varphi_n^{(m)}(x) \cos \beta_m y \tag{2.19}$$

where

$$\delta_m = \begin{cases} 1 \text{ for } m = 0, \\ 2 \text{ for } m = 1, 2, ..., \infty. \end{cases}$$

Expressing also the amplitude of the deflection by the double series

$$w(x, y) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} A_{nm} \varphi_n^{(m)}(x) \delta_m \cos \beta_m y, \tag{2.20}$$

we obtain the solution of Eq. (2.3) in the form

$$w(x, y) = \frac{1}{2b} \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \frac{\varphi_n^{(m)}(\xi) \varphi_n^{(m)}(x)}{\Delta_{nm}} \delta_m \cos \beta_m y, \quad -b \leqslant y \leqslant b. \tag{2.21}$$

Let the loading $\bar{q}(x, y, t) = e^{i\omega t} q(x)$ and the system of forces $Xe^{i\omega t}$ applied at the points $(\xi, \pm 2\,bk)$ act on the plate strip. The total amplitude of the deflection due to these forces can be expressed by the equation

$$w(x, y) = w_p(x) + \frac{X}{2b} \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \frac{\varphi_n^{(m)}(\xi) \varphi_n^{(m)}(x)}{\Delta_{nm}} \delta_m \cos \beta_m y. \tag{2.22}$$

The deflection $w_p(x)$ can easily be obtained from the solution of the equation

$$N \frac{d^4 w_p}{dx^4} - \mu\omega^2 w_p = q(x), \; w_p(0) = w_p''(0) = w_p(a) = w_p'(a) = 0. \tag{2.23}$$

From the condition that the value of the deflection at the points $(\xi, \pm 2\,bk)$ is equal to zero we obtain the relation

$$(2.24) \qquad w_p(\xi) + \frac{X}{2b} \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \frac{[\varphi_n^{(m)}(\xi)]^2}{\Delta_{nm}} \delta_m = 0 .$$

From the equation written above we can determine the values of the support reactions X, and from Eq. (2.22) the plate deflection. Let us observe that along the straight lines $y = \pm 2\,bk$, $k = 1, 2, ..., \infty$ we have $\partial w/\partial y = 0$.

If the loading producing the vibrations vanishes $(\bar{q} = 0)$, then the equation

$$(2.25) \qquad \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \frac{[\varphi_n^{(m)}(x)]^2}{\Delta_{nm}} \delta_m = 0$$

is the condition of free vibrations of a plate strip, additionally supported at the points $(\xi, \pm 2\,bk)$.

3. The free and forced vibrations of a rectangular plate in presence of forces acting within the surface of the plate. Buckling of a rectangular plate

The differential equation of the deflection of a plate, performing forced vibrations and simultaneously being subject to the action of forces independent of time in the plane of the plate, takes the form

$$(3.1) \qquad N\nabla^4 \bar{w} + \mu \frac{\partial^2 \bar{w}}{\partial t^2} + p \frac{\partial^2 \bar{w}}{\partial x^2} + r \frac{\partial^2 \bar{w}}{\partial y^2} + s \frac{\partial^2 \bar{w}}{\partial x\, \partial y} = \bar{q}.$$

Here, $\bar{q}$, $\bar{w}$ are functions of x, y, t, while p, r, s are functions of position only.

Confining ourselves to harmonic vibrations and assuming that

$$\bar{q}(x, y, t) = e^{i\omega t} q(x, y), \qquad \bar{w}(x, y, t) = e^{i\omega t} w(x, y),$$

we obtain the equation of the amplitude of the deflection in the form

$$(3.2) \qquad N\nabla^4 w - \gamma^2 w + p\, w_{,xx} + r w_{,yy} + s w_{,xy} = q.$$

Let us confine our considerations to the rectangular plate simply supported at the edges $x = 0$, $y = 0$, $y = b$, the edge $x = a$ being clamped.

We introduce the Green function $G(x, y; \xi, \eta)$ satisfying the differential equation

$$(3.3) \qquad N\nabla^4 G - \gamma^2 G = \delta(x - \xi)\, \delta(y - \eta),$$

where we have assumed that G satisfies the same boundary conditions as the function w.

Using the Green function, the solution of differential equation (3.2) can be represented in the form

$$(3.4) \qquad w(x, y) = -\int_0^a \int_0^b G(\xi, \eta; x, y) \left[p(\xi, \eta) \frac{\partial^2 w(\xi, \eta)}{\partial \xi^2} + r(\xi, \eta) \frac{\partial^2 w(\xi, \eta)}{\partial \eta^2} + s(\xi, \eta) \frac{\partial^2 w(\xi, \eta)}{\partial \xi\, \partial \eta} \right] d\xi\, d\eta + \int_0^a \int_0^b q(\xi, x)\, G(\xi, \eta; x, y)\, d\xi\, d\eta .$$

Applying the Green identity on the plane and making use of the boundary conditions for the functions w and G, Eq. (3.4) can be represented in the form of the Fredholm integral equations of second kind [1], [2]

$$(3.5)\quad w(x,y) = -\int_0^a\int_0^b w(\xi,\eta)\left[p(\xi\,\eta)\frac{\partial^2 G}{\partial\xi^2}+r(\xi\,\eta)\frac{\partial^2 G}{\partial\eta^2}+ +s(\xi,\eta)\frac{\partial^2 G}{\partial\xi\,\partial\eta}\right]d\xi\,d\eta+\int_0^a\int_0^b q(\xi,\eta)\,G(\xi,\eta;x,y)\,d\xi\,d\eta.$$

Function G as the solution of Eq. (3.3) can be represented by the double series (compare Eq. (2.14))

$$(3.6)\quad G(x,y,\xi,\eta) = \sum_{m,n}^{\infty}\frac{\varphi_n^{(m)}(\xi)\,\psi_m(\eta)}{\Delta_{nm}}\varphi_n^{(m)}(x)\,\psi_m(y), \qquad \Delta_{nm} = N\lambda_{mn}^4-\mu\omega^2.$$

Expressing the function $w(x,y)$ appearing in Eq. (3.5) by the series

$$(3.7)\qquad w(x,y) = \sum_{n,m}^{\infty} A_{nm}\,\varphi_n^{(m)}(x)\,\psi_m(y)$$

and substituting (3.6) into this equation, we get after a simple manipulation the following system of equations

$$(3.8)\qquad A_{nm} = -\frac{1}{\Delta_{nm}}\sum_{i,k}^{\infty} A_{ik}\,G_{nmik}+\frac{q_{nm}}{\Delta_{nm}},$$

where

$$(3.9)\quad G_{nmik} = \int_0^a\int_0^b \varphi_i^{(k)}(\xi)\,\psi_k(\eta)\,[p(\xi,\eta)\,\varphi_n^{\prime\prime(m)}(\xi)\,\psi_m(\eta)+ +r(\xi,\eta)\,\varphi_n^{(m)}(\xi)\,\psi_m^{\prime\prime}(\eta)+s(\xi,\eta)\,\varphi_n^{\prime(m)}(\xi)\,\psi_m^{\prime}(\eta)]\,d\xi\,d\eta.$$

If the loading producing vibrations vanishes ($\bar{q}=0$), then we obtain the system of homogeneous equations

$$(3.8')\qquad A_{nm} = -\frac{1}{\Delta_{nm}}\sum_{i,k}^{\infty} A_{ik}\,G_{nmik}, \qquad n,m,i,k = 1,2,\ldots,\infty.$$

From the condition that the determinant of the system of Eq. (3.8′) is equal to zero, we get the equation of vibration

$$(3.10)\qquad |\Delta_{ik}\,\delta_{in}\,\delta_{km}+G_{nmik}| = 0 \qquad n,m,i,k = 1,2,\ldots,\infty.$$

In the particular case ($\omega=0$) we have to do with the problem of the buckling of a plate.

The system of Eq. (3.8) can be considerably simplified for a number of particular cases.

a) If we assume that $r=s=0$ and that the loading p depends only on the variable x, then from Eq. (3.8) we get

$$G_{nmik} = \delta_{km}\int_b^a q(\xi)\,\varphi_i^{(k)}(\xi)\,\varphi_n^{\prime\prime(m)}(\xi)\,d\xi = \delta_{km}\,a_{in}.$$

The system of Eq. (3.8) can be reduced to the form

$$A_{nm} = -\frac{1}{\Delta_{nm}} \sum_{i=1}^{\infty} A_{im} a_{in}.$$

b) If $p = s = 0$ and $r = r(y)$, then

$$G_{nmik} = -\beta_m^2 \delta_{in} \int_0^b r(\eta) \psi_n(\eta) \psi_m(\eta) d\eta = -\beta_m^2 \delta_{in} b_{km}.$$

The latter system of equations can be simplified to the form

$$A_{nm} = \frac{\beta_m^2}{\Delta_{nm}} \sum_{k=1}^{\infty} A_{nk} b_{km}.$$

In the particular case $r = r_0 = \text{const}$ we have $b_{km} = r_0 \delta_{km}$. Then

$$A_{nm} = \frac{\beta_m^2 r_0}{\Delta_{nm}} A_{nm},$$

whence

$$\omega_{nm}^2 = \left(1 - \frac{r_0 \beta_m^2}{N \lambda_{nm}^4}\right) \frac{N \lambda_{nm}^4}{\mu}. \tag{3.11}$$

The following form can be given to the last equation

$$\omega_{nm}^2 = \bar{\omega}_{nm}^2 \left(1 - \frac{r_0}{\bar{r}_0}\right). \tag{3.12}$$

Here $\bar{r}_0 = \dfrac{N\lambda_{nm}^4}{\beta_m^2}$ is the force which can be obtained from Eq. (3.11), therefore if we assume that $\omega = 0$, it is the critical force. On the other hand, $\bar{\omega}_{nm}^2 = \dfrac{N\lambda_{nm}^4}{\mu}$ is the frequency of the vibrations of a plate under the assumption that $r = 0$. We see from relation (3.12) that, if the critical force r_0 increases (here r_0 is a compressive force), the frequency of the vibrations of the plate will tend to zero. If r_0 is a tensile force, the increase of the force will cause the increase of the frequency of free vibrations of a plate. Relation (3.12) is also true for a plate clamped on the edges $x = 0$, $x = a$ and simply supported at the edges $y = 0$, $y = b$. We must only substitute in Eq. (3.11) the values $\hat{\lambda}_{nm}$ instead of λ_{nm}.

c) Now, let us assume that $p = s = 0$, and $r = r_0 \delta(x - \xi_0)$. Here the plate is compressed by concentrated forces of intensity r_0 acting along the straight line $x = \xi$. In this case we obtain

$$G_{nmik} = -r_0 \beta_m^2 \delta_{mk} \varphi_n^{(m)}(\xi_0) \varphi_i^{(k)}(\xi_0).$$

The system of Eq. (3.8) takes the form

$$\mathbf{A}_{nm} = \frac{r_0 \beta_m^2 \varphi_n^{(m)}(\xi_0)}{\Delta_{nm}} \sum_{i=1}^{\infty} A_{im} \varphi_i^{(m)}(\xi_0).$$

We multiply both sides of equation by $\varphi_n^{(m)}(\xi_0)$ and sum with respect to n. As the result we obtain

$$\sum_{n=1}^{\infty} A_{nm}\varphi_n^{(m)}(\xi_0) = r_0\beta_m^2 \sum_{n=1}^{\infty} \frac{[\varphi_n^{(m)}(\xi_0)]^2}{\Delta_{nm}} \sum_{i=1}^{\infty} A_{im}\varphi_i^{(k)}(\xi_0), \; k = m,$$

whence

$$r_0\beta_m^2 \sum_{n=1}^{\infty} \frac{[\varphi_n^{(m)}(\xi_0)]^2}{N\lambda_{nm}^4 - \mu\omega^2} = 1. \tag{3.13}$$

For the fixed r_0 we get, from this equation, the successive values of frequency ω.

For the particular case $\omega = 0$ we obtain the value of the critical force from the equation

$$r_0 = \frac{1}{\sum\limits_{n=1}^{\infty} \frac{[\varphi_n^{(m)}(\xi_0)]^2}{N\lambda_{nm}^4}}. \tag{3.14}$$

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