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APPLIED MECHANICS

On the Treatment of the Two-dimensional Coupled Thermoelastic Problems in Terms of Stresses

by

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The two-dimensional coupled problem of thermoelasticity was up till now formulated only in terms of displacements. Two methods were applied: generalization of the W. G. Galerkin's displacement function [1], and decomposition of the displacement vector into two parts, the potential and the rotational [1]—[3]. If the boundary conditions are formulated in stresses, it is advantageous to introduce the dynamical stress functions, which in the stationary case reduce to the Airy stress functions. It has been shown that the solution of the plane coupled thermoelastic problem can be achieved in the general case, involving body forces, with the aid of three stress functions φ_i, $i = 1, 2, 3$. In the case when the body forces are absent one stress function will be sufficient, which in the isothermal case reduces to the stress function by Radok [4].

1. Stress functions

Let us consider the plane state of strain of an elastic medium, acted on by surface and body forces and by temperature, the field of deformation being coupled with the field of temperature.

The stress-strain relations are

$$\sigma_{ij} = 2\mu\varepsilon_{ij}+(\lambda\varepsilon_{kk}-\gamma T)\,\delta_{ij}, \quad i,j = 1, 2, \tag{1.1}$$

with the notations: σ_{ij} — stresses, ε_{ij} — strains, μ, λ — Lamé constants, T — temperature, $\gamma = (3\lambda+2\mu)\,\alpha_t$, α_t — coefficient of thermal expansion, δ_{ij} — Kronecker's symbol.

Observe that

$$\sigma_{kk} = 2\,(\lambda+\mu)\,\varepsilon_{kk}-2\gamma T. \tag{1.2}$$

The deformations are expressed by displacements as u_i

$$\varepsilon_{ij} = \frac{1}{2}\,(u_{i,j}+u_{j,i}), \quad i,j = 1, 2. \tag{1.3}$$

The stresses, deformations, displacements and temperature depend both on the time and the variables x_1, x_2.

The clamping between the field of temperature and deformation is characterized by the following set of equations: the equation of motion

$$\sigma_{ij,j}+F_i=\varrho \ddot{u}_i, \qquad i,j=1,2, \tag{1.4}$$

and the equation of heat conduction

$$\Box_3^2 T-\eta \dot{\varepsilon}_{kk}=-Q/\varkappa, \qquad \Box_3^2=\nabla^2-\frac{1}{\varkappa}\partial_t, \tag{1.5}$$

with the notations: F_i — components of body forces, ϱ — density, $\varkappa$ — coefficient of heat conduction, η — a magnitude which clamps Eqs. (1.4) and (1.5), Q — distribution of heat sources.

Let us differentiate the first of Eqs. (1.4) and Eq. (1.5) with respect to x_1, the second one with respect to x_2. Summing and subtracting the results, and taking into account Eq. (1.3), we arrive at two equations

$$\sigma_{11,11}+\sigma_{22,22}+2\sigma_{12,12}+F_{1,1}+F_{2,2}=\varrho \ddot{\varepsilon}_{kk}, \tag{1.6}$$

$$\left(\partial_1^2-\frac{1}{2c_2^2}\partial_t^2\right)\sigma_{11}-\left(\partial_2^2-\frac{1}{2c_2^2}\partial_t^2\right)\sigma_{22}+F_{1,1}-F_{2,2}=0, \qquad c_2^2=\mu/\varrho. \tag{1.7}$$

Now let us differentiate the first Eqs. (1.4) with respect to x_1 and the second one with respect to x_2; adding the results we obtain:

$$\sigma_{kk,12}+\Box_2^2\sigma_{12}+F_{1,2}+F_{2,1}=0, \qquad \Box_2^2=\nabla^2-\frac{1}{c_2^2}\partial_t^2. \tag{1.8}$$

Inserting the stresses (1.1) into Eq. (1.6) and eliminating σ_{12} by means of the compatibility equation

$$\varepsilon_{11,22}+\varepsilon_{22,11}=2\varepsilon_{12,12} \tag{1.9}$$

we obtain, using Eq. (1.2), the equation

$$\Box_1^2\sigma_{kk}+2\mu m\,\Box_2^2 T+\beta(F_{1,1}+F_{2,2})=0, \tag{1.10}$$

with the notations

$$m=\frac{\gamma}{\lambda+2\mu}, \qquad \beta=\frac{2(\lambda+\mu)}{\lambda+2\mu}, \qquad c_1^2=\frac{\lambda+2\mu}{\varrho}, \qquad \Box_1^2=\nabla^2-\frac{1}{c_1^2}\partial_t^2.$$

Eqs. (1.7), (1.8), (1.10) give the relations between the three stresses (σ_{11}, σ_{12}, σ_{22}), temperature T and the components of body forces F_1, F_2. Let us introduce now three stress functions φ_1, φ_2, φ_3

$$\sigma_{11}=D_2^2\varphi_1+\Box_1^2\varphi_2, \qquad \sigma_{22}=D_1^2\varphi_1-\Box_1^2\varphi_2, \qquad \sigma_{12}=-\partial_1\partial_2\varphi_1+\Box_1^2\varphi_3, \tag{1.11}$$

$$\sigma_{kk}=(D_1^2+D_2^2)\varphi_1=\Box_2^2\varphi_1.$$

Here

$$D_i^2=\partial_i^2-\frac{1}{2c_2^2}\partial_t^2, \qquad i=1,2.$$

Inserting (1.11) into (1.10), (1.7) and (1.8), we obtain three independent equations

$$\Box_1^2 \Box_2^2 \varphi_i = A_i + B_1 \delta_{1i}, \qquad i = 1, 2, 3, \tag{1.12}$$

with the notation

$$A_1 = -\beta(F_{1,1} + F_{2,2}), \qquad A_2 = -(F_{1,1} - F_{2,2}),$$

$$A_3 = -(F_{1,2} + F_{2,1}), \qquad B_1 = -2\mu m \Box_2^2 T.$$

Into the set of Eqs. (1.12) the equation of heat conduction (1.5) should be involved. Expressing ε_{kk} by σ_{kk} (1.2) and then by φ_1, we obtain

$$\Box_4^2 T - \varepsilon \Box_2^2 \dot{\varphi}_1 = -Q/\varkappa, \tag{1.13}$$

where

$$\Box_4^2 = \nabla^2 - \frac{1}{\varkappa_0}\partial_t, \qquad \frac{1}{\varkappa_0} = \frac{1}{\varkappa} + 2\varepsilon\gamma, \qquad \varepsilon = \frac{\eta}{2(\lambda+\mu)}.$$

Eliminating the temperature T from the first Eq. (1.12) and from (1.13), we obtain the following equation for φ_1

$$\Box_2^2 \left\{ [\Box_1^2 \Box_4^2 + 2\mu m \varepsilon\, \partial_t \Box_2^2] \varphi_1 - \frac{2\mu m Q}{\varkappa} \right\} = \Box_4^2 A_1. \tag{1.14}$$

To determine the function φ_i $(i = 1, 2, 3)$ we can use the relations

$$\Box_2^2 \left\{ [\Box_1^2 \Box_3^2 - m\eta\, \partial_t \nabla^2] \varphi_1 - \frac{2\mu m Q}{\varkappa} \right\} = \Box_4^2 A_1, \tag{1.15}$$

$$\Box_1^2 \Box_2^2 \varphi_2 = A_2, \tag{1.16}$$

$$\Box_1^2 \Box_2^2 \varphi_3 = A_3. \tag{1.17}$$

If the functions φ_i are known, the stresses can be found from Eqs. (1.11). The temperature must be determined with the aid of Eq. (1.13).

A considerable simplification can be achieved in the case when body forces are absent ($\vec{F} = 0$); then one stress function $\varphi = \varphi_1$ will be sufficient. Eqs. (1.11) can be written in a simple form

$$\sigma_{ij} = -\varphi_{,ij} + \delta_{ij}\left(\nabla^2 - \frac{1}{2c_2^2}\partial_t^2\right)\varphi, \qquad i, j = 1, 2. \tag{1.18}$$

It can be easily seen that these expressions with $\vec{F} = 0$ satisfy Eqs. (1.7), (1.8) identically. Instead of the set of equations (1.15), (1.17) we have here

$$\Box_2^2 \left\{ [\Box_1^2 \Box_3^2 - m\eta\, \partial_t \nabla^2] \varphi - \frac{2\mu m Q}{\varkappa} \right\} = 0. \tag{1.19}$$

If the influence of coupling is disregarded (i.e. assuming $\varepsilon = 0$, $\eta = 0$), the equation of heat conduction (1.5) simplifies to

$$\Box_3^2 T = -Q/\varkappa. \tag{1.20}$$

The temperature can be determined from (1.20), B_1 in (1.12) being treated as a known function. It is obvious that the set of Eqs. (1.12) can be split into two independent equations

$$\Box_1^2 \Box_2^2 \bar{\varphi}_i = A_i, \quad i = 1, 2, 3, \tag{1.21}$$

$$\Box_1^2 \Box_2^2 \bar{\bar{\varphi}}_1 = B_1. \tag{1.22}$$

From (1.21) the functions $\bar{\varphi}_i$, corresponding to the action of body forces in the isothermal case $T = 0$, can be determined. Eq. (1.22) gives the function $\bar{\bar{\varphi}}$, which describes the state of thermal stresses.

In our approximate solution, neglecting the coupling between the fields of temperature and deformations, the stresses caused by body forces are given by

$$\bar{\sigma}_{11} = D_2^2 \bar{\varphi}_1 + \Box_1^2 \varphi_2, \quad \sigma_{22} = D_1^2 \bar{\varphi}_1 - \Box_1^2 \bar{\varphi}_2, \quad \sigma_{12} = -\partial_1 \partial_2 \bar{\varphi}_1 + \Box_1^2 \varphi_3, \tag{1.23}$$

and the thermal stresses by

$$\bar{\bar{\sigma}}_{ij} = -\bar{\bar{\varphi}}_{1,ij} + \delta_{ij} \left(\nabla^2 - \frac{1}{2c_2^2} \partial_t^2 \right) \bar{\bar{\varphi}}_1, \quad i, j = 1, 2. \tag{1.24}$$

If the fluctuations of temperature and body forces are sufficiently slow, the coupled problem can be treated as a quasi-steady one, the inertia terms in the equation of motion being neglected. In such a case

$$\sigma_{11} = \partial_2^2 \varphi_1 + \nabla^2 \varphi_2, \quad \sigma_{22} = \partial_1^2 \varphi_1 - \nabla^2 \varphi_2, \quad \sigma_{12} = -\partial_1 \partial_2 \varphi_1 + \nabla^2 \varphi_3. \tag{1.25}$$

Eqs. (1.12) simplify to

$$\nabla^4 \varphi_i = A_i + \partial_{1i} B_1 \quad (i = 1, 2, 3), \tag{1.26}$$

with

$$B_1 = -2\mu m \nabla^2 T.$$

Eliminating the temperature from the first Eq. (1.26) and from (1.5), we obtain

$$\nabla^2 \left\{ \nabla^2 (\nabla^2 - \varepsilon_0 \partial_t) \varphi_1 - \frac{2\mu m Q}{\varkappa} \right\} = \Box_4^2 A_1, \quad \varepsilon_0 = \frac{1}{\varkappa} + \eta m. \tag{1.27}$$

If the body forces are absent ($A_i = 0$, $i = 1, 2, 3$) the function $\varphi_1 = \varphi$ will be sufficient to determine the particular stress functions; it can be found from the equation

$$\nabla^2 \left[\nabla^2 (\nabla^2 - \varepsilon_0 \partial_t) \varphi - \frac{2\mu m Q}{\varkappa} \right] = 0. \tag{1.28}$$

Finally in a stationary case we have the set of uncoupled equations

$$\nabla^4 \varphi_i = A_i + B_1 \delta_{1i}, \quad B_1 = -2\mu m \nabla^2 T, \tag{1.29}$$

$$\nabla^2 T = -Q/\varkappa. \tag{1.30}$$

Let us return to the dynamic problem described in the general case by Eqs. (1.15), (1.17) and begin with the problem of propagation of elastic waves in an unlimited body. Here the particular integral of Eqs. (1.15), (1.17) will solve the problem.

We observe that, with body forces absent ($\vec{F}=0$), it will be sufficient to find the particular integral of the equation

$$(1.31)\qquad (\Box_1^2\,\Box_3^2 - m\eta\,\partial_t\,\nabla^2)\,\varphi = -\frac{2\mu m Q}{\varkappa},$$

as only longitudinal waves will appear.

If the body is limited and the surface values of temperature T and displacements or stresses are known, the solution can be combined of two parts

$$(1.32)\qquad \varphi_i = \varphi_i' + \varphi_i'', \qquad i = 1, 2, 3.$$

The first part φ_i' is a particular integral of Eqs. (1.15), (1.17) (e.g. the solution for an infinite space), the second one φ_i'' constitutes the general solution of the set of equations

$$(1.33)\qquad \Box_1^2\,\sigma_{kk}'' + 2\mu m\,\Box_2^2\,T'' = 0,$$

$$(1.34)\qquad D_1^2\,\sigma_{11}'' - D_2^2\,\sigma_{22}'' = 0,$$

$$(1.35)\qquad \sigma_{kk,12}'' + \Box_2^2\,\sigma_{12}'' = 0,$$

$$(1.36)\qquad \Box_4^2\,T'' - \varepsilon\,\dot{\sigma}_{kk}'' = -Q/\varkappa.$$

No body forces interfere in these equations; the temperature T'' results from the action of surface temperature and loads, applied in order to fulfil the boundary conditions. In many cases one stress function $\varphi_1'' = \varphi''$ will be sufficient to satisfy Eqs. (1.33), (1.36) and to determine the stresses σ_{ij}''; it can be found from the differential equation

$$(1.36)\qquad \Box_2^2\,[\Box_1^2\,\Box_2^2 - m\eta\,\partial_t\,\nabla^2]\,\varphi'' = 0.$$

Using the function φ'' the stresses σ_{ij}'' can be calculated from

$$(1.37)\qquad \sigma_{ij}'' = -\varphi_{,ij}'' + \delta_{ij}\left(\nabla^2 - \frac{1}{2c_2^2}\,\partial_t^2\right)\varphi''.$$

Similarly in the quasi-static case only one stress function φ'' will be sufficient for the determination of stresses

$$(1.38)\qquad \sigma_{ij}'' = (\delta_{ij}\,\nabla^2 - \partial_i\,\partial_j)\,\varphi'',$$

and φ'', T'' can be found from the set of equations

$$(1.39)\qquad \Box_4^2\,T - \varepsilon\,\nabla^2\,\dot{\varphi}'' = 0, \qquad \nabla^4\,\varphi'' + 2\mu m\,\nabla^2\,T'' = 0.$$

The procedure shown above has some advantages. Namely the stresses are found in the form of a combination of derivatives of the stress functions φ_i. It is possible to pass from the general, dynamic and coupled problem to quasi-steady, and then to static ones, coupled or separated. It is also easy to arrive at the dynamic, quasi-static and static one-dimensional problem.

If there is no coupling ($\eta = 0$), no temperature and no body forces act on the body ($\vec{F} = 0$), ($T = 0$), then Eqs. (1.7), (1.8), (1.10) reduce to the equation

$$(1.40)\qquad \Box_1^2\,\Box_2^2\,\varphi = 0,$$

and

$$\sigma_{ij} = -\varphi_{,ij} + \delta_{ij}\left(\nabla^2 - \frac{1}{2c_2^2}\partial_t^2\right)\varphi, \qquad i,j = 1,2 \tag{1.41}$$

in accordance with the result obtained by J. R. M. Radok [4].

Finally, we have to discuss the relations between stresses and displacement functions

$$\vec{u} = \operatorname{grad}\Phi + \operatorname{rot}\vec{\psi}. \tag{1.42}$$

Inserting (1.42) into the equations for displacements

$$\mu\nabla^2\vec{u} + (\lambda+\mu)\operatorname{grad}\operatorname{div}\vec{u} - \gamma\operatorname{grad}T + \vec{F} = \varrho\ddot{\vec{u}}, \tag{1.43}$$

and assuming that the body forces can be written in the form

$$\vec{F} = \operatorname{grad}\chi + \operatorname{rot}\vec{\zeta}, \tag{1.44}$$

Eq. (1.43) will be transformed to

$$\Box_1^2\Phi - mT = -n\chi, \tag{1.45}$$

$$\Box_2^2\vec{\psi} = -\frac{1}{\mu}\vec{\zeta}, \qquad n = \frac{1}{\lambda+2\mu}. \tag{1.46}$$

Here the equation of heat conduction must be added. Because of the relation (1.42) it takes the form

$$\Box_3^2 T - \eta\nabla^2\dot{\Phi} = -Q/\varkappa. \tag{1.47}$$

Eliminating the temperature T from Eqs. (1.45) and (1.47), the following set of equations for displacements will be obtained (cf. [1])

$$(\Box_3^2\Box_1^2 - \eta m\,\partial_t)\Phi = -\frac{mQ}{\varkappa} - n\Box_3^2\chi, \tag{1.48}$$

$$\Box_2^2\vec{\psi} = -\frac{1}{\mu}\vec{\zeta}. \tag{1.49}$$

The knowledge of Φ enables us to determine the temperature T from Eq. (1.47).

It can be easily shown that in an infinite space body forces $\vec{F} = \operatorname{rot}\vec{\zeta}$, $\chi = 0$ do not induce any longitudinal waves or temperature T; only transversal waves arise in such a case. Yet the forces $\vec{F} = \operatorname{grad}\chi$, $\vec{\zeta} = 0$ give rise to the longitudinal waves only, and to temperature T.

In the case when $\vec{F} = 0$ ($\vec{\zeta} = 0$, $\chi = 0$) and heat sources act in the body, we obtain only longitudinal waves. We have then

$$u_i = \Phi_{,i}, \qquad \sigma_{ij} = 2\mu(\Phi_{,ij} - \delta_{ij}\Phi_{,kk}) + \varrho\delta_{ij}\Phi'', \qquad \vec{\psi} = 0. \tag{1.50}$$

Eq. (1.48) with $\chi = 0$ coincides in this case with Eq. (1.31) — with the assumption that $\Phi = -\frac{1}{2\mu}\varphi$. The same holds for Eqs. (1.50) and (1.18).

In a limited body, in general, longitudinal and transversal waves arise (except for the one-dimensional case). The solution consists of two parts: the particular integral $\vec{u}'$ and the integral $\vec{u}''$ satisfying the equations

$$\begin{cases} (\Box_3^2\Box_1^2 - \eta m\,\partial_t\,\nabla^2)\,\Phi'' = 0, \\ \Box_2^2\vec{\psi} = 0, \qquad \Box_1^2\Phi'' - mT'' = 0. \end{cases} \tag{1.51}$$

The functions Φ'', $\vec{\psi}''$, T are to be chosen to satisfy all boundary conditions.

Are the boundary conditions expressed in terms of stresses, the following procedure will be convenient. We determine the particular integral of Eqs. (1.48), (1.49) arriving at the solution $\vec{u}'$. Instead of solving Eqs. (1.51) the function φ'' discussed before can be used; it satisfies the equation

$$\Box_2^2\,[\Box_1^2\,\Box_3^2 - m\eta\,\partial_t\,\nabla^2]\,\varphi'' = 0 \tag{1.52}$$

and the prescribed boundary conditions. With the aid of φ'' we find the stresses

$$\sigma_{ij}'' = -\varphi''_{,ij} + \delta_{ij}\left(\nabla^2 - \frac{1}{2c_2^2}\,\partial_t^2\right)\varphi'', \qquad i,j = 1,2. \tag{1.53}$$

It is the so-called mixed procedure.

In the case of a prescribed field of body forces (1.44) we determine the particular integral of simple equations (1.48), (1.49) instead of the complicated ones (1.15), (1.17). The additional solution $\vec{u}''$ can be derived from Eq. (1.52) with less difficulty than from (1.51).

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