BULLETIN
DE
L'ACADÉMIE POLONAISE DES SCIENCES

SÉRIE DES SCIENCES TECHNIQUES

VOLUME VII
NUMÉROS 7-8

VARSOVIE 1959

BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Série des sciences techniques
Volume VII, Nos. 7-8 — 1959

APPLIED MECHANICS

Thermal Stress Propagation in Visco-elastic Bodies. II.

by

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Presented on May 14, 1959

1. Introduction

In Ref. [1] relationships between stress, strain and temperature, and the wave equation for longitudinal thermo-elastic waves were given. Finally, the state of stress was determined due to an instantaneous concentrated heat source in an infinite visco-elastic space. We shall now concern ourselves with the propagation of thermal stress due to a linear or a plane source of heat. Visco-elastic bodies will be considered, here also, in which the stress-strain-temperature relations are given by the equations ([2] and [3]):

$$\sigma_{ij}^{(1)}(x_r, t) = 2\int_0^t \mu(t-\tau)\frac{\partial \varepsilon_{ij}^{(1)}(x_r,\tau)}{\partial \tau}d\tau + \delta_{ij}\int_0^t \left\{\lambda(t-\tau)\frac{\partial \Theta^{(1)}(x_r,\tau)}{\partial \tau} - [3\lambda(t-\tau)+2\mu(t-\tau)]\,\alpha_t\frac{\partial T(x_r,\tau)}{\partial \tau}\right\}d\tau, \tag{1.1}$$

$$P_1(D)P_3(D)\sigma_{ij}^{(2)}(x_r,t) = P_2(D)P_3(D)\varepsilon_{ij}^{(2)}(x_r,t) + \delta_{ij}\left\{\tfrac{1}{3}[P_1(D)P_4(D)-P_2(D)P_3(D)]\,\Theta^{(2)}(x_r,t) - P_1(D)P_4(D)\alpha_t T\right\}, \tag{1.2}$$

$$i = 1, 2, 3.$$

The relations (1.1) were derived by M. A. Biot [2] and generalized by D. S. Berry [4] to three-dimensional problems. $\lambda(t)$, $\mu(t)$ are relaxation functions which reduce, in the case of a perfectly elastic body, to Lamé's constants. The operators $P_i(D)$ $(i = 1, 2, 3, 4)$ in (1.2) are represented by the equations, [3]:

$$P_i(D) = \sum_{n=0}^{N_i} a_i^{(n)} D^n, \quad a_i^{(N_i)} \neq 0, \quad D = \frac{\partial}{dt}. \tag{1.3}$$

If the stresses σ_{ij} are substituted in the equations of motion, the strains are expressed in terms of displacements, and the potential of thermoelastic displacement Φ is introduced, where

$$u_i = \frac{\partial \Phi}{\partial x_i}, \quad i = 1, 2, 3, \tag{1.4}$$

we obtain the following wave equations

$$\int_0^t [2\mu(t-\tau) + \lambda(t-\tau)] \frac{\partial}{\partial \tau} \nabla^2 \Phi^{(1)} d\tau - \varrho \frac{\partial^2 \Phi^{(1)}}{\partial t^2} = \\ = \alpha_t \int_0^t [3\lambda(t-\tau) + 2\mu(t-\tau)] \frac{\partial T}{\partial \tau} d\tau; \tag{1.5}$$

$$\frac{1}{3} [2P_2(D)P_3(D) + P_4(D)P_1(D)] \nabla^2 \Phi^{(2)} - \\ - P_1(D)P_3(D) \varrho \frac{\partial^2 \Phi^{(2)}}{\partial t^2} = P_1(D)P_4(D) \alpha_t T. \tag{1.6}$$

Expressing the strains in terms of Φ and using the Eqs. (1.5), (1.6), we obtain the following equations for stresses

$$\sigma_{ij}^{(1)} = \int_0^t 2\mu(t-\tau) \frac{\partial}{\partial \tau} \left[\frac{\partial^2}{\partial x_i \partial x_j} - \delta_{ij} \nabla^2 \right] \Phi^{(1)} d\tau + \varrho \frac{\partial^2 \Phi^{(1)}}{\partial t^2}, \tag{1.7}$$

$$P_1(D)P_3(D)\sigma_{ij}^{(2)} = P_2(D)P_3(D) \left(\frac{\partial^2}{\partial x_i \partial x_j} - \delta_{ij} \nabla^2 \right) \Phi^{(2)} + \\ + P_1(D)P_3(D) \varrho \frac{\partial^2 \Phi^{(2)}}{\partial t^2}. \tag{1.8}$$

Performing on the Eqs. (1.5), (1.6) and (1.7), (1.8) the Laplace transformation and assuming that the body is free from stresses at the initial time, we obtain the following equations

$$\nabla^2 \bar{\Phi}(x_r, p) - \sigma^2(p) p^2 \bar{\Phi}(x_r, p) = \vartheta(p) \bar{T}(x_r, p); \tag{1.9}$$

$$\bar{\sigma}_{ij}(x_r, p) = 2G(p) \left(\frac{\partial^2}{\partial x_i \partial x_j} - \delta_{ij} \nabla^2 \right) \bar{\Phi}(x_r, p) + \delta_{ij} p^2 \varrho \bar{\Phi}(x_r, p), \tag{1.10}$$

where

$$\bar{K}(x_r, p) = \int_0^\infty e^{-pt} K(x_r, t)\, dt, \quad K = (\Phi, T, \sigma_{ij}). \tag{1.11}$$

The following notations

$$\begin{cases} \sigma^2(p) = \dfrac{\varrho}{p[2\bar{\mu}(p) + \bar{\lambda}(p)]}, \\ \vartheta(p) = \dfrac{3\bar{\lambda}(p) + 2\bar{\mu}(p)}{\bar{\lambda}(p) + 2\bar{\mu}(p)} \alpha_t, \quad G(p) = p\bar{\mu}(p) \end{cases} \tag{1.12}$$

are introduced for a visco-elastic body, where the stress-strain relations are given by the Eqs. (1.1) and

$$
(1.13)\qquad \begin{cases} \sigma^2(p) = \dfrac{3P_1(p)\,P_3(p)\,\varrho}{2P_2(p)\,P_3(p) + P_1(p)\,P_4(p)}, \\[2ex] \vartheta(p) = \dfrac{3P_1(p)\,P_4(p)}{2P_2(p)\,P_3(p) + P_1(p)\,P_4(p)}\,\alpha_t, \\[2ex] G(p) = \dfrac{P_2(p)}{2P_1(p)}, \end{cases}
$$

for a visco-elastic body, where the Eqs. (1.2) are valid. Introducing the function $\bar{\Psi}(x_r,p) = G(p)\bar{\Phi}(x_r,p)$, we can express the relations (1.10) in a somewhat different manner

$$
(1.14)\qquad \sigma_{ij}(x_r,p) = 2\left(\frac{\partial^2}{\partial x_i\,\partial x_j} - \delta_{ij}\nabla^2\right)\bar{\Psi}(x_r,p) + \varrho p^2\bar{\Phi}(x_r,p)\,\delta_{ij}.
$$

2. The action of a linear source of heat

Let a linear instantaneous heat source act in an infinite visco-elastic body. The temperature field is described by the following equation (cf. [5]):

$$
(2.1)\qquad T = \frac{Q}{4\pi\varkappa t}\,e^{-\frac{x^2}{4\varkappa t}}.
$$

Performing the Laplace transformation on the function T, we obtain

$$
(2.2)\qquad T = \frac{Q}{2\pi\varkappa}\,K_0(r)\sqrt{p/\varkappa},
$$

or

$$
T = \frac{Q}{2\pi\varkappa}\int_0^\infty \frac{aJ_0(ar)\,da}{(a^2 + p/\varkappa)},
$$

in this equation $\varkappa = \lambda'/c\varrho$ where λ' is the coefficient of heat conduction, ϱ — density, c — specific heat, and $Q = W/\varrho c$, where W is the quantity of heat emitted by the heat source per unit time and volume.

The solution of the Eq. (1.1) may be written in the form

$$
\bar{\Phi} = -\frac{Q\vartheta(p)}{2\pi\varkappa}\int_0^\infty \frac{aJ_0(ar)\,da}{(a^2 + p/\varkappa)\,[a^2 + p^2\sigma^2(p)]},
$$

or

$$
(2.3)\qquad \Phi = -\frac{Q\vartheta(p)}{2\pi\varkappa\,[p^2\sigma^2(p) - p\varkappa^{-1}]}\left[K_0\left(r\sqrt{\frac{p}{\varkappa}}\right) - K_0(rp\sigma(p))\right],
$$

where $K_0(z)$ is a modified Bessel function of the third kind.

Let us consider in detail a visco-elastic body, where the Eqs. (1.1) are valid. We assume that the functions λ, μ are expressed by the simple exponential relation and have the same relaxation time ε^{-1}

$$\lambda(t) = \lambda_0 e^{-\varepsilon t}, \quad \mu(t) = \mu_0 e^{-\varepsilon t}.$$

Therefore,

$$\vartheta(p) = \frac{3\lambda_0 + 2\mu_0}{\lambda_0 + 2\mu_0} \alpha_t = \vartheta_0 = \text{const.} \qquad \sigma^2(p) = \sigma_0 \frac{p+\varepsilon}{p},$$

$$\sigma_0 = \frac{\varrho}{\lambda_0 + 2\mu_0}, \quad G(p) = \mu_0 \frac{p}{p+\varepsilon},$$

and the function $\bar{\Phi}$ takes the form

$$(2.4) \qquad \begin{cases} \bar{\Phi} = \dfrac{Q\vartheta_0}{2\pi\varkappa\sigma_0^2\beta}\left(\dfrac{1}{p} - \dfrac{1}{p-\beta}\right)\left[K_0\left(r\sqrt{\dfrac{p}{\varkappa}}\right) - K_0\left(r\sigma_0\sqrt{p(p+\varepsilon)}\right)\right], \\ \beta = \dfrac{1}{\varkappa\sigma_0^2} - \varepsilon. \end{cases}$$

Performing the inverse Laplace transformation, we find that

$$(2.5) \qquad \Phi(r,t) = \frac{Q\vartheta_0}{2\pi\varkappa\sigma_0^2\beta}\left[F_1(r,t) - F_2(r,t)\right],$$

where

$$(2.6) \qquad F_1(r,t) = \int_0^t \left(1 - e^{-\beta(t-\tau)}\right)\zeta(r,\tau)\, d\tau,$$

and

$$\zeta(r,t) = \frac{1}{2t}\exp\left(-\frac{r^2}{4\varkappa t}\right);$$

$$(2.7) \qquad F_2(r,t) = \int_0^t \left(1 - e^{-\beta(t-\tau)}\right)\gamma(r,\tau)\, d\tau,$$

and

$$\gamma(r,t) = \frac{1}{2}\sqrt{\pi\varepsilon}\, e^{-\frac{\varepsilon t}{2}} I_{-1/2}\left(\frac{\varepsilon}{2}\sqrt{t^2 - r^2\sigma_0^2}\right) H(t - r\sigma_0),$$

H denoting the Heaviside function.

For a linear source with intensity $Q(t)$, we obtain

$$(2.8) \qquad \Phi(r,t) = \frac{Q\vartheta_0}{2\pi\varkappa\sigma_0^2\beta}\int_0^t Q(t-\tau)\left[F_1(r,\tau) - F_2(r,\tau)\right] d\tau.$$

Let us find now the function $\Psi(r, t)$ and observe that

$$\bar{\Psi}(r,p) = G(p)\bar{\Phi}(r,p) = \mu_0 \frac{p}{p+\varepsilon}\bar{\Phi}(r,p).$$

For an instantaneous linear source of heat we obtain

$$(2.9)\qquad \bar{\Psi}(r,p) = -\frac{\mu_0\vartheta_0 Q}{2\pi\varkappa\sigma_0^2(\varepsilon+\beta)}\left(\frac{1}{p-\beta}-\frac{1}{p+\varepsilon}\right)\left[K_0\left(r\sqrt{\frac{p}{\varkappa}}\right)-\right.$$
$$\left.-K_0\left(r\sigma_0\sqrt{p(p+\varepsilon)}\right)\right].$$

Performing the inverse Laplace transformation on the function $\bar{\Psi}$, we obtain

$$(2.10)\qquad \Psi(r,t) = -\frac{\mu_0\vartheta_0 Q}{2\pi\varkappa\sigma_0^2(\varepsilon+\beta)}\left[\chi(r,t;\beta)-\chi(r,t;-\varepsilon)-\right.$$
$$\left.-\omega(r,t;\beta)+\omega(r,t;-\varepsilon)\right],$$

where

$$(2.11)\qquad \begin{cases} \chi(r,t;\nu) = \int\limits_0^t e^{\nu(t-\tau)}\zeta(r,\tau)\,d\tau, \\ \omega(r,t;\nu) = \int\limits_0^t e^{\nu(t-\tau)}\gamma(r,\tau)\,d\tau. \end{cases}$$

The functions $\zeta(r,t)$ and $\gamma(r,t)$ are given by the Eqs. (2.6) and (2.7). For time-variable source intensity, or, in other words, for $Q=Q(t)$, we shall determine the function Ψ from the equation

$$(2.12)\qquad \Psi(r,t) = -\frac{\mu_0\vartheta_0}{2\pi\varkappa\sigma_0^2(\varepsilon+\beta)}\int\limits_0^t Q(t-t)\left[\chi(r,\tau;\beta)-\chi(r,\tau;-\varepsilon)-\right.$$
$$\left.-\omega(r,\tau;\beta)+\omega(r,\tau;-\varepsilon)\right]d\tau.$$

The knowledge of the function $\Psi(r,t)$ enables (on the basis of the Eqs. (1.10)) the determination of the stress components. Performing the inverse transformation on the Eqs. (1.10) and bearing in mind the axially-symmetric character of the stress, we obtain

$$(2.13)\qquad \sigma_{rr} = -\frac{2}{r}\frac{\partial\Psi}{\partial r}+\varrho\frac{\partial^2\Phi}{\partial t^2},\qquad \sigma_{\varphi\varphi} = -2\frac{\partial^2\Psi}{\partial r^2}+\varrho\frac{\partial^2\Phi}{\partial t^2},\qquad \sigma_{r\varphi}=0.$$

3. The action of a plane source of heat

Let a plane continuous source of heat of intensity Q act in the $x=0$ plane of the infinite visco-elastic space. The solution of the heat equation

$$(3.1)\qquad \frac{\partial^2 T}{\partial x^2}-\frac{1}{\varkappa}\frac{\partial T}{\partial x} = -\frac{Q}{\varkappa}H(t)\,\delta(x),$$

with the initial condition $T(x,0)=0$ and with the condition $T(\infty,t)=0$ is

$$(3.2)\qquad T = \frac{Q}{2\sqrt{\varkappa}}\left[2\sqrt{\frac{t}{\pi}}\exp\left(\frac{-x^2}{4\varkappa t}\right)-\frac{x}{\sqrt{\varkappa}}\operatorname{erfc}\left(\frac{x}{\sqrt{4\varkappa t}}\right)\right].$$

Performing the Laplace transformation on the function T, we obtain

$$\bar{T}=\frac{Q}{2p\sqrt{\varkappa p}}e^{-x\sqrt{p/\varkappa}}=\frac{Q}{p\varkappa\pi}\int\limits_0^\infty\frac{\cos\alpha x\,d\alpha}{\alpha^2+p/\varkappa},\quad x>0. \tag{3.3}$$

The solution of the wave equation (1.9) may be represented in the form

$$\bar{\Phi}=-\frac{Q\vartheta(p)}{p\pi\varkappa}\int\limits_0^\infty\frac{\cos\alpha x\,d\alpha}{(\alpha^2+p/\varkappa)\,[\alpha^2+p^2\sigma^2(p)]} \tag{3.4}$$

or, after integration,

$$\bar{\Phi}=-\frac{Q\vartheta(p)}{2\varkappa p\,[p^2\sigma^2(p)-p\varkappa^{-1}]}\left[\frac{e^{-x\sqrt{p/\varkappa}}}{\sqrt{p/\varkappa}}-\frac{e^{-xp\sigma(p)}}{p\sigma(p)}\right]. \tag{3.5}$$

Let us consider, as was done in the preceding section, a visco-elastic body where the relations (1.1) are valid. In this particular case we have

$$p^2\bar{\Phi}=-\frac{\vartheta_0 Q}{2\varkappa\sigma_0^2(p-\beta)}\left\{\sqrt{\frac{\varkappa}{p}}\,e^{-x\sqrt{p/\varkappa}}-\frac{1}{\sigma_0\sqrt{p(p+\varepsilon)}}\,e^{-x\sigma_0\sqrt{p(p+\varepsilon)}}\right\}. \tag{3.6}$$

Let us perform the inverse transformation on the function $p^2\bar{\Phi}$. We have

$$\frac{\partial^2\Phi}{\partial t^2}=-\frac{\vartheta_0 Q}{2\sigma_0^2\sqrt{\varkappa}}\left[f_1(x,t;\beta)-\frac{1}{\sigma_0\sqrt{\varkappa}}f_2(x,t;\beta)\right], \tag{3.7}$$

where

$$f_1(x,t;\beta)=\frac{e^{\beta t}}{2\sqrt{\beta}}\left[\exp\left(-x\sqrt{\frac{\beta}{\varkappa}}\right)\operatorname{erfc}\left(\frac{x}{\sqrt{4\varkappa t}}-\sqrt{\beta t}\right)-\right.$$
$$\left.-\exp\left(x\sqrt{\frac{\beta}{\varkappa}}\right)\operatorname{erfc}\left(\frac{x}{\sqrt{4\varkappa t}}+\sqrt{\beta t}\right)\right], \tag{3.8}$$

$$f_2(x,t;\beta)=\int\limits_0^t e^{\beta(t-\tau)}\varphi(x,\tau)\,d\tau,$$

and

$$\varphi(x,t)=e^{-\frac{\varepsilon t}{2}}I_0\left(\frac{\varepsilon}{2}\sqrt{t^2-x^2\sigma_0^2}\right)H(t-x\sigma_0). \tag{3.9}$$

If the source of intensity varies in time, we have

$$\frac{\partial^2\Phi}{\partial t^2}=-\frac{\vartheta_0}{2\sigma_0^2\sqrt{\varkappa}}\frac{\partial}{\partial t}\int\limits_0^t Q(t-\tau)\left[f_1(x,\tau;\beta)-\frac{1}{\sigma_0\sqrt{\varkappa}}f_2(x,\tau;\beta)\right]d\tau=$$
$$=-\frac{\vartheta_0}{2\sigma_0^2\sqrt{\varkappa}}\left\{\int\limits_0^t\frac{dQ(t-\tau)}{dt}\left[f_1(x,\tau;\beta)-\frac{1}{\sigma_0\sqrt{\varkappa}}f_2(x,\tau;\beta)\right]d\tau+\right.$$
$$\left.+Q_0\left[f_1(x,t;\beta)-\frac{1}{\sigma_0\sqrt{\varkappa}}f_2(x,t;\beta)\right]\right\}, \tag{3.10}$$

where Q_0 denotes the initial value of the function $Q(t)$.

Bearing in mind that $\bar{\Psi} = \frac{\mu_0 p}{p+\varepsilon} \bar{\Phi}$ we find that, for a continuous source of heat, we have

$$(3.11) \quad \bar{\Psi} = \frac{\vartheta_0 Q \mu_0}{2\varkappa\sigma_0^2(\varepsilon+\beta)} \left\{ \frac{\varepsilon+\beta}{\varepsilon\beta} \frac{1}{p} - \frac{1}{\beta(p-\beta)} - \frac{1}{\varepsilon(p+\varepsilon)} \right\} \left(\frac{e^{-x\sqrt{p/\varkappa}}}{\sqrt{p/\varkappa}} - \frac{e^{-x\sigma_0\sqrt{p(p+\varepsilon)}}}{\sigma_0\sqrt{p(p+\varepsilon)}} \right).$$

Performing the inverse Laplace transformation, we have

$$(3.12) \quad \Psi(r,t) = \frac{\vartheta_0 Q \mu_0}{2\varkappa\sigma_0^2(\beta+\varepsilon)} \left\{ \frac{\varepsilon+\beta}{\varepsilon\beta} \left[f_1(x,\tau;0) - \frac{1}{\sigma_0\sqrt{\varkappa}} f_2(x,\tau;0) \right] + \right.$$
$$\left. - \frac{1}{\beta} \left[f_1(x,\tau;\beta) - \frac{1}{\sigma_0\sqrt{\varkappa}} f_2(x,\tau;\beta) \right] - \frac{1}{\varepsilon} \left[f_1(x,\tau;-\varepsilon) - \frac{1}{\sigma_0\sqrt{\varkappa}} f_2(x,\tau;-\varepsilon) \right] \right\},$$

where

$$f_1(x,\tau;0) = 2\sqrt{\frac{t}{\pi}} \exp\left(\frac{-x^2}{4\varkappa t}\right) - \frac{x}{\sqrt{\varkappa}} \operatorname{erfc}\left(\frac{x}{\sqrt{4\varkappa t}}\right).$$

The stresses σ_{ij} will be found from the Eqs. (1.10)

$$(3.13) \quad \sigma_{xx} = \varrho \frac{\partial^2 \Phi}{\partial t^2}, \qquad \sigma_{yy} = \sigma_{zz} = -2\frac{\partial^2 \Psi}{\partial x^2} + \varrho \frac{\partial^2 \Phi}{\partial t^2}, \qquad \sigma_{xy} = \sigma_{yz} = \sigma_{zx} = 0.$$

4. Sudden heating of a visco-elastic semi-space

The problem of propagation of stresses, due to sudden heating of the surface bounding the semi-space, was solved for a perfectly elastic medium by V. I. Danilovskaya [6]. Taking into consideration the boundary condition $T(0,t) = T_0 H(t)$ and the initial condition $T(x,0) = 0$, the temperature field is expressed by the function

$$(4.1) \quad T(x,t) = T_0 \operatorname{erfc} \frac{x}{\sqrt{4\varkappa t}}.$$

Performing on the function T the Laplace transformation, we have

$$\bar{T} = \frac{T_0}{p} \exp(-x\sqrt{p/\varkappa}), \quad x > 0$$

or

$$(4.2) \quad \bar{T} = \frac{2T_0}{\pi p} \int_0^\infty \frac{a \sin ax \, da}{a^2 + p/\varkappa}.$$

The solution of the wave equation (1.9) may be represented in the form

$$(4.3) \quad \bar{\Phi}^* = -\frac{2T_0\vartheta(p)}{\pi p} \int_0^\infty \frac{a \sin ax \, da}{(a^2+p/\varkappa)[a^2+p^2\sigma^2(p)]}$$

or after integration

$$\overline{\Phi}^* = -\frac{T_0 \vartheta(p)}{p\,[p^2\sigma^2(p) - p\varkappa^{-1}]}\,[e^{-x\sqrt{p/\varkappa}} - e^{-xp\sigma(p)}]. \tag{4.4}$$

For $x = 0$ we have $\overline{\Phi}^* = 0$; therefore, in virtue of the Eq. (1.10), the stress $\overline{\sigma}_{xx} = \varrho p^2 \overline{\Phi}$ is equal to zero for $x = 0$, as it should be. For a visco-elastic body, where the Eqs. (1.1) hold, the function Φ^* is expressed by the equation

$$\overline{\Phi}^* = -\frac{T_0 \vartheta_0}{\sigma_0^2 (p - \beta)}\,(e^{-x\sqrt{p/\varkappa}} - e^{-p\sigma_0\sqrt{p(p+\varepsilon)}}). \tag{4.5}$$

From a comparison of the function $\overline{\Phi}^*$ and the function $\overline{\Phi}$ of Sec. 3. Eq. (3.6) it follows that

$$\overline{\Phi}^* = -2\varkappa \frac{T_0}{Q}\frac{d\overline{\Phi}}{dx}. \tag{4.6}$$

Therefore,

$$\frac{\partial^2 \Phi^*}{\partial t^2} = \frac{T_0 \vartheta_0 \sqrt{\varkappa}}{\sigma_0^2}\left[\eta_1(x, t; \beta) - \frac{1}{\sigma_0\sqrt{\varkappa}}\eta_2(x, t; \beta)\right], \tag{4.7}$$

where

$$\eta_1 = \frac{df_1}{dx}, \qquad \eta_2 = \frac{df_2}{dx}.$$

It can easily be found that

$$\eta_1(x, t; \beta) = \frac{e^{\beta t}}{2}\left[e^{x\sqrt{\beta/\varkappa}} \operatorname{erfc}\left(\frac{x}{\sqrt{4\varkappa t}} + \sqrt{\beta t}\right) + \right. \\ \left. + \exp\left(-x\sqrt{\frac{\beta}{\varkappa}}\right)\operatorname{erfc}\left(\frac{x}{\sqrt{4\varkappa t}} - \sqrt{\beta t}\right)\right]. \tag{4.8}$$

Let us observe that

$$\eta_2 = \int_0^t e^{\beta(t-\tau)}\frac{d\varphi(x,\tau)}{dx}\,d\tau,$$

where the function φ is given by the Eq. (3.9). But

$$\frac{d\varphi}{dx} = \sigma_0 e^{-\frac{\varepsilon t}{2}}\left[\left(\frac{\varepsilon}{2}\right)^2 x\sigma_0 \frac{I_1\left(\frac{\varepsilon}{2}\sqrt{\tau^2 - x^2\sigma_0^2}\right)}{\frac{\varepsilon}{2}\sqrt{\tau^2 - x^2\sigma_0^2}} H(\tau - x\sigma_0) + \right. \\ \left. + I_0\left(\frac{\varepsilon}{2}\sqrt{\tau^2 - x^2\sigma_0^2}\right)\delta(\tau - x_0\sigma_0)\right],$$

and, finally, bearing in mind that

$$\int_0^t f(\tau)\,\delta(\tau - t_0)\,d\tau = f(t_0)\,H(t - t_0), \qquad t_0 > 0,$$

we obtain

$$(4.9) \qquad \eta_2(x,t;\beta) = \sigma_0 e^{\beta t}\left[e^{-x\sigma_0\left(\beta+\frac{\varepsilon}{2}\right)} H(t - x\sigma_0) + \right.$$

$$\left. + \sigma_0 x \left(\frac{\varepsilon}{2}\right)^2 \int_0^t e^{-\tau\left(\beta+\frac{\varepsilon}{2}\right)} \frac{I_1\left(\frac{\varepsilon}{2}\sqrt{\tau^2 - x^2\sigma_0^2}\right)}{\frac{\varepsilon}{2}\sqrt{\tau^2 - x^2\sigma_0^2}} H(\tau - x\sigma_0)\, d\tau \right].$$

Since $\sigma_{xx} = \varrho \dfrac{\partial^2 \Phi}{\partial t^2}$, it is seen that, for $t < x\sigma_0$ the time-variability of stress is characterized by the function $\eta_1(x,t;\beta)$. For $t = x\sigma_0$ we obtain a "jump" of the stress σ_{xx}, for $t \to \infty$ the stress tends to zero. In the case of a perfectly elastic body we should assume that $\varepsilon = 0$, therefore $\beta = 1/\varkappa\sigma_0^2$. The stress σ_{xx} is given by the equation

$$(4.10) \qquad \sigma_{xx} = \frac{T_0 \vartheta_0 \varrho \sqrt{\varkappa}}{\sigma_0^2}\left[\eta_1\left(x, t; \frac{1}{\varkappa\sigma_0^2}\right) - \frac{1}{\sqrt{\varkappa}} \exp\left(\frac{t - x\sigma_0}{\varkappa\sigma_0^2}\right) H(t - x\sigma_0)\right],$$

according to the result obtained by V. I. Danilovskaya.

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